Exponential domination in function spaces

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Abstract. Given a Tychonoff space X and an infinite cardinal κ, we prove that exponential κ-domination in X is equivalent to exponential κ-cofinality of $C_p(X)$. On the other hand, exponential κ-cofinality of X is equivalent to exponential κ-domination in $C_p(X)$. We show that every exponentially κ-cofinal space X has a κ^+-small diagonal; besides, if X is κ-stable, then $nw(X) \le \kappa$. In particular, any compact exponentially κ-cofinal space has weight not exceeding κ. We also establish that any exponentially κ-cofinal space X with $l(X) \le \kappa$ and $t(X) \le \kappa$ has i-weight not exceeding κ while for any cardinal κ, there exists an exponentially ω-cofinal space X such that $l(X) \ge \kappa$.

Keywords: exponential κ-domination; exponential κ-cofinality; κ-stable space; i-weight; function space; duality; κ^+-small diagonal

Classification: 54C35, 54C05, 54G20

1. Introduction

It was proved in the paper [6] that a regular space X must have density not exceeding κ if any subset of X of cardinality $(2^\kappa)^+$ is κ-dominated, i.e., contained in the closure of a set $B \subset X$ such that $|B| \le \kappa$. This makes it natural to study *spaces X with exponential κ-domination,* in which every set $A \subset X$ with $|A| \le 2^\kappa$ is κ-dominated. It was established in [6] that spaces with exponential κ-domination have nice categorical properties: they are preserved by continuous images, products with 2^κ-many factors and κ-unions. Any Čech-complete space with exponential κ-domination turns out to have density less than or equal to κ as well as any space X with $\chi(X) \le \kappa$. It was also shown in [6] that there exist non-separable spaces of countable pseudocharacter with exponential ω-domination.

In this paper we introduce a new property called *exponential κ-cofinality* for any infinite cardinal κ. A space X is exponentially κ-cofinal if for any continuous onto map $f\colon X \to Y$ such that $w(Y) \le 2^\kappa$, there exist continuous surjective maps $g\colon X \to Z$ and $h\colon Z \to Y$ such that $iw(Z) \le \kappa$ and $h \circ g = f$. Exponentially κ-cofinal spaces generalize the class of spaces of i-weight not exceeding κ and give global information about a space via its small continuous images. The importance of this class stems from the fact that it is bidual to the class of spaces with

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exponential κ-domination: we will prove that a space X features exponential κ-domination if and only if the function space $C_p(X)$ is exponentially κ-cofinal and X is exponentially κ-cofinal if and only if $C_p(X)$ is a space with exponential κ-domination.

The class of exponentially κ-cofinal spaces turned out to have nontrivial relationships with some classical properties which makes it interesting in itself. We will show that the diagonal of any exponentially κ-cofinal space is κ^+-small. The concept of λ-small diagonal was introduced in [8] (under a different name) where it was proved, among other things, that, under CH, a compact space of countable tightness is metrizable whenever it has an ω_1-small diagonal. Later it was proved in [9] that the requirement of countable tightness can be omitted in some models of ZFC (Zermelo–Fraenkel set theory). We will show that exponential ω-cofinality of a compact space X implies metrizability of X so this property is strictly stronger than having a small diagonal.

More generally, if X is an exponentially κ-cofinal space which is κ-stable, then $nw(X) \le \kappa$. It is worth noting that the concept of κ-stability was introduced and studied in [1]. Last, but not least, we show that an exponentially κ-cofinal space X with $l(X) \le \kappa$ and $t(X) \le \kappa$ must have i-weight not exceeding κ while exponentially ω-cofinal spaces can have Lindelöf number as large as we wish.

2. Notation and terminology

In this paper all spaces are assumed to be Tychonoff. If X is a space, then $\tau(X)$ is its topology and $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any point $x \in X$. If κ is a cardinal, then $[X]^{\le\kappa} = \{A \subset X : |A| \le \kappa\}$. The set of reals with its usual topology is denoted by $\mathbb{R}$ and $\mathbb{I} = [0, 1] \subset \mathbb{R}$. A set $B \subset X$ is said to dominate a set $A \subset X$ if $A \subset \overline{B}$. The space X *features exponential κ-domination* if for any set $A \in [X]^{\le 2^\kappa}$, there exists $B \in [X]^{\le\kappa}$ that dominates A.

A family $\mathcal{N}$ of subsets of a space X is a *network* in X if every open subset of X is the union of a subfamily of $\mathcal{N}$. The *network weight* $nw(X)$ of a space X is the minimal cardinality of a network in X and the density $d(X)$ of the space X is the minimal cardinality of a dense subset of X. The cardinal $w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\}$ is the *weight of* X. Let $s(X) = \sup\{|D| : D \text{ is a discrete subspace of } X\}$ and $\mathrm{ext}(X) = \sup\{|D| : D \text{ is a closed discrete subspace of } X\}$. The cardinals $s(X)$ and $\mathrm{ext}(X)$ are the *spread and extent* of X, respectively. Given spaces X and Y and a continuous map $f : X \to Y$ say that f is a *condensation* if it is a bijection.

The cardinal $iw(X) = \min\{\kappa :$ there is a condensation of X onto a space of weight less than or equal to $\kappa\}$ is called the *i-weight* of the space X. If $x \in X$, then the cardinal $\psi(x, X) = \min\left\{|\mathcal{U}| : \mathcal{U} \subset \tau(X) \text{ and } \bigcap\mathcal{U} = \{x\}\right\}$ is called the *pseudocharacter of x in X* and $\psi(X) = \sup\{\psi(x,X)\colon x \in X\}$ is the *pseudocharacter of X*. Furthermore, a space X is *κ-stable* if $nw(Y) \le \kappa$ for any continuous image Y of the space X such that $iw(Y) \le \kappa$. If μ is a cardinal, then X is said to be a *P_μ-space* if $\bigcap \mathcal{U} \in \tau(X)$ for any $\mathcal{U} \subset [\tau(X)]^{\le\mu}$.

For any space X, let $\Delta_X = \{(x,x)\colon x \in X\} \subset X \times X$ be the diagonal of the space X. The space X has a *κ^+-small diagonal* if for any set $A \subset (X \times X)\backslash\Delta_X$ such that $|A| = \kappa^+$, there exists $B \subset A$ such that $|B| = \kappa^+$ and $\overline{B} \cap \Delta_X = \emptyset$. The cardinal $t(X) = \min\{\kappa\colon \bar{A} = \bigcup\{\overline{B}\colon B \in [A]^{\le\kappa}\}$ for every $A \subset X\}$ is the *tightness* of the space X. A space X is *κ-monolithic* if $nw(\bar{A}) \le \kappa$ for each $A \in [X]^{\le\kappa}$. We say that X is *a Lindelöf Σ-space* if there exists a space Y that maps continuously onto X and perfectly onto a second countable space. A set A is *concentrated* around a set $F \subset X$ if $|A\backslash U| < |A|$ whenever $F \subset U \in \tau(X)$; the set A is concentrated around a point $x \in X$ if it is concentrated around $\{x\}$.

The expression $C(X,Y)$ denotes the set of all continuous maps from a space X to a space Y. We follow the usual practice to write $C(X)$ instead of $C(X,\mathbb{R})$. The space $C_p(X)$ is the set $C(X)$ endowed with the pointwise convergence topology. Let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for each natural number n. Given spaces X and Y, if $\varphi\colon X \to Y$ is a continuous onto map then its dual map $\varphi^*\colon C_p(Y) \to C_p(X)$ is defined by $\varphi^*(f) = f \circ \varphi$ for every $f \in C_p(Y)$. For any $A \subset X$, the restriction map $\pi_A\colon C_p(X) \to C_p(A)$ is defined by $\pi_A(f) = f|A$ for each $f \in C_p(X)$. Given a set $F \subset C_p(X)$ let $e_F(x)(f) = f(x)$ for any $x \in X$ and $f \in F$; then $e_F\colon X \to C_p(F)$ is the *reflection map* which coincides with the diagonal product ΔF of the functions from F.

The rest of our notation is standard and follows the book [5]. All relevant information on cardinal invariants can be found in the paper of Hodel [7]. The books [12], [13], [14] contain all necessary facts and notions of C_p-theory.

3. A dual property for exponential κ-domination

We will present a topological property, called exponential κ-cofinality, that is dual to exponential κ-domination with respect to function spaces in the sense that X features exponential κ-domination if and only if $C_p(X)$ is exponentially κ-cofinal. Our purpose is to show that the class of exponentially κ-cofinal spaces is interesting in itself.

Definition 3.1. Given an infinite cardinal κ, we will say that a space X is *exponentially κ-cofinal* if for any continuous onto map $f\colon X \to Y$ such that $w(Y) \le 2^\kappa$, there exist continuous surjective maps $g\colon X \to Z$ and $h\colon Z \to Y$ such that $iw(Z) \le \kappa$ and $h \circ g = f$.

The proof of the following proposition is straightforward from the definition. It shows that exponential κ-cofinality is a weakening of the property of having i-weight not exceeding κ. Therefore an important line of study of exponential κ-cofinality is to find nice classes of spaces in which it coincides with $iw \le \kappa$.

Proposition 3.2. *If $iw(X) \le \kappa$, then X is an exponentially κ-cofinal space. In particular, any space X with $nw(X) \le \kappa$ is exponentially κ-cofinal.*

Proposition 3.3. *Assume that X is an exponentially κ-cofinal space and Y is a continuous image of X with $iw(Y) \le 2^\kappa$. Then $|Y| \le 2^\kappa$. In particular, if $nw(Y) \le 2^\kappa$, then $|Y| \le 2^\kappa$.*

Proof: Let $f\colon X \to Y$ be a continuous onto map. Take a space M and a condensation $u\colon Y \to M$ such that $w(M) \le 2^\kappa$ and choose continuous onto maps $v\colon X \to Z$ and $w\colon Z \to M$ such that $iw(Z) \le \kappa$ and $w \circ v = u \circ f$. It follows from $|Z| \le 2^{iw(Z)} \le 2^\kappa$ that $|M| \le 2^\kappa$ and hence $|Y| = |M| \le 2^\kappa$. □

Corollary 3.4. *Any discrete exponentially κ-cofinal space has cardinality not exceeding 2^κ.*

Proof: If X is a discrete exponentially κ-cofinal space and $|X| > 2^\kappa$, then there exists a surjective map $f\colon X \to Y \subset \mathbb{I}^{2^\kappa}$ such that $|Y| > 2^\kappa$. Since the map f is continuous and $w(Y) \le 2^\kappa$, we have a contradiction with Proposition 3.3. □

Proposition 3.5. *If X is an exponentially κ-cofinal space and a set $Y \subset X$ is C^*-embedded in X, then Y is exponentially κ-cofinal. In particular, if X is a normal exponentially κ-cofinal space, then every closed subspace of X is exponentially κ-cofinal.*

Proof: Take any continuous onto map $f\colon Y \to Y'$ such that $w(Y') \le \mu = 2^\kappa$. There is no loss of generality to consider that $Y' \subset \mathbb{I}^\mu$ and there is a family $\{f_\alpha\colon \alpha < \mu\} \subset C_p(X, \mathbb{I})$ such that $f = \Delta\{f_\alpha\colon \alpha < \mu\}$. Let $u_\alpha\colon X \to \mathbb{I}$ be a continuous extension of f_α for every $\alpha < \mu$. Then the diagonal product $u = \Delta\{u_\alpha\colon \alpha < \mu\}$ maps X continuously onto a space $X' \subset \mathbb{I}^\mu$ and $u|Y = f$. By exponential κ-cofinality of X, we can find continuous onto maps $v\colon X \to Z$ and $w\colon Z \to X'$ such that $w \circ v = u$ and $iw(Z) \le \kappa$. Let $g = v|Y$ and $h = w|v(Y)$. Then $h \circ g = f$ and the i-weight of the space $Z' = v(Y)$ does not exceed $iw(Z) \le \kappa$ so the maps g and h witness that Y is exponentially κ-cofinal. □

Corollary 3.6. *Assume that X is an exponentially κ-cofinal space and D is a discrete subset of X. If D is C^*-embedded in X, then $|D| \le 2^\kappa$. In particular, if X is a normal exponentially κ-cofinal space, then $\mathrm{ext}(X) \le 2^\kappa$.*

Proof: Just apply Proposition 3.5 and Corollary 3.4. □

Corollary 3.7. *If κ is an infinite cardinal and X is an exponentially κ-cofinal space, then any discrete family of nonempty open subsets of X has cardinality not exceeding 2^κ.*

Proof: If $\mathcal{U}$ is a discrete family of nonempty open subsets of the space X and $|\mathcal{U}| > 2^\kappa$, then pick a point $x_U \in U$ for every $U \in \mathcal{U}$. The set $D = \{x_U : U \in \mathcal{U}\}$ is closed, discrete and $|D| = |\mathcal{U}| > 2^\kappa$. Since the set D is C-embedded in X by [13, Fact 5 of Theorem 132], we have a contradiction with Corollary 3.6. □

Theorem 3.8. *Given an exponentially κ-cofinal space X and a set $A \subset X$ with $nw(A) \leq 2^\kappa$, there exists a continuous map $\varphi\colon X \to M$ such that $w(M) \leq \kappa$ and $\varphi|A$ is an injection. In particular, $iw(A) \leq \kappa$ and $|A| \leq 2^\kappa$.*

Proof: The restriction mapping $\pi_A\colon C_p(X) \to C_p(A)$ is continuous and the set $E = \pi_A(C_p(X))$ is dense in $C_p(A)$, see [3, Proposition 0.4.1]. Observe that $d(E) \leq nw(E) \leq nw(C_p(A)) = nw(A) \leq 2^\kappa$ so we can find a dense subset $D \subset E$ with $|D| \leq 2^\kappa$. The set D is also dense in $C_p(A)$ and hence it separates the points of A. If $h = \Delta(D)$ is the diagonal product of the functions from D, then $h\colon X \to \mathbb{R}^D$; let $Y = h(X)$. The map $h\colon X \to Y$ is continuous and surjective. Since $w(Y) \leq 2^\kappa$, there exists a space Z and continuous onto maps $p\colon X \to Z$ and $q\colon Z \to Y$ such that $q \circ p = h$ and $iw(Z) \leq \kappa$. Observe that the map $h|A$ is injective and hence so is $p|A$. There exists a condensation $r\colon Z \to M$ for some space M such that $w(M) \leq \kappa$. It is immediate that the map $\varphi = r \circ p$ is as promised. □

Corollary 3.9. *For any exponentially κ-cofinal space X, if $A \subset X$ and $|A| \leq \kappa$, then $|\bar{A}| \leq 2^\kappa$ and $iw(\bar{A}) \leq \kappa$.*

Proof: Just note that $w(\bar{A}) \leq 2^\kappa$ and apply Theorem 3.8. □

Corollary 3.10. *Assume that X is a metrizable exponentially κ-cofinal space. Then $w(X) \leq 2^\kappa$ and hence $iw(X) \leq \kappa$.*

Proof: It follows from Corollary 3.6 that $w(X) = \mathrm{ext}(X) \leq 2^\kappa$ and we can apply Theorem 3.8 to conclude that $iw(X) \leq \kappa$. □

Corollary 3.11. *If X is an exponentially κ-cofinal space and $s(X) \leq \kappa$, then $iw(X) \leq \kappa$.*

Proof: Observe that $nw(X) \leq 2^{s(X)} \leq 2^\kappa$, see [7, Theorem 5.3], and apply Theorem 3.8. □

If $iw(X) \leq \kappa$, then the diagonal of X is easily seen to be a G_κ-set. This is not necessarily the case for an exponentially κ-cofinal space X but we still have an important weaker property in X.

Proposition 3.12. *If X is an exponentially κ-cofinal space, then the diagonal of X is κ^+-small. In particular, any exponentially ω-cofinal space has a small diagonal.*

PROOF: Take any faithfully indexed set $A = \{z_\alpha = (x_\alpha, y_\alpha)\colon \alpha < \kappa^+\}$ contained in $(X \times X)\backslash\Delta_X$; then the cardinality of the set $Y = \{x_\alpha, y_\alpha\colon \alpha < \kappa^+\}$ does not exceed $\kappa^+ \le 2^\kappa$ and therefore there exists a continuous map $\varphi\colon X \to M$ such that $w(M) \le \kappa$ and $\varphi|Y$ is injective, see Theorem 3.8. This implies that the set $\{(\varphi(x_\alpha), \varphi(y_\alpha))\colon \alpha < \kappa^+\}$ is contained in $(M \times M)\backslash\Delta_M$ which, together with $w(M) \le \kappa$, guarantees that there is a set $E \subset \kappa^+$ such that $|E| = \kappa^+$ and the closure of the set $\{(\varphi(x_\alpha), \varphi(y_\alpha))\colon \alpha \in E\}$ in $M \times M$ does not meet Δ_M. Then the closure of the set $B = \{z_\alpha\colon \alpha \in E\} \subset A$ in $X \times X$ does not meet Δ_X and $|B| = \kappa^+$, i.e., the set B witnesses that X has a κ^+-small diagonal. □

Proposition 3.13. *Given an infinite cardinal κ, if X is a P_{2^κ}-space with* $\operatorname{ext}(X) \le 2^\kappa$, *then X is exponentially κ-cofinal.*

PROOF: Let $f\colon X \to Y$ be a continuous onto map such that $w(Y) \le 2^\kappa$. Then $f^{-1}(y)$ is open in X being a G_{2^κ}-set for every $y \in Y$. Therefore the partition $\mathcal{P} = \{f^{-1}(y)\colon y \in Y\}$ is a discrete family of open subsets of X so it follows from $\operatorname{ext}(X) \le 2^\kappa$ that $|\mathcal{P}| \le 2^\kappa$ and hence $|Y| \le 2^\kappa$. If Z is the set Y with the discrete topology and $g(x) = f(x)$ for any $x \in X$, then the map $g\colon X \to Z$ is continuous and for the identity map $h\colon Z \to Y$, we have $h \circ g = f$. Since there exists an injection of Z into $\mathbb{I}^\kappa$, which is automatically continuous, we conclude that the maps g and h witness exponential κ-cofinality of X. □

Example 3.14. For any cardinal $\kappa > \mathfrak{c}$, consider the set $X = \kappa \cup \{p\}$ where $p \notin \kappa$. All points of κ are isolated in X and a set $U \subset X$ with $p \in U$ is open if and only if $\kappa\backslash U \le \mathfrak{c}$. It is immediate that X is a $P_\mathfrak{c}$-space and $l(X) = \operatorname{ext}(X) = \mathfrak{c}$ so X is exponentially ω-cofinal by Proposition 3.13. Therefore the Souslin number of an exponentially ω-cofinal space can be arbitrarily large. This result should be compared with Corollary 3.7.

Recall that we have the equalities $iw(X) = d(C_p(X))$ and $d(X) = iw(C_p(X))$ for any space X, see [10], i.e., the density and i-weight are bidual with respect to the functor C_p. Since exponential κ-domination and exponential κ-cofinality are their respective weakenings, it is natural to expect them to be bidual as well. We will show next that this is, indeed, the case.

Theorem 3.15. *A space X is exponentially κ-cofinal if and only if $C_p(X)$ is a space with exponential κ-domination.*

PROOF: Suppose that X is exponentially κ-cofinal and take a set $A \subset C_p(X)$ with $|A| \le 2^\kappa$. If $u = \Delta A$ is the diagonal product of the family A, then $u\colon X \to \mathbb{R}^A$

and hence the space $Y = u(X)$ has weight not exceeding 2^κ. We can consider that $u\colon X \to Y$ and hence the dual map $u^*\colon C_p(Y) \to C_p(X)$ is an embedding. Let $Q = u^*(C_p(Y))$ and observe that $A \subset Q$, see [14, Fact 5 of U.086].

There exists a space Z together with continuous onto maps $v\colon X \to Z$ and $w\colon Z \to Y$ such that $iw(Z) \le \kappa$ and $w \circ v = u$. The space $C_p(Z)$ has density not exceeding κ, see [10]; since $E = v^*(C_p(Z))$ is homeomorphic to $C_p(Z)$, we can fix a set $B \subset E$ such that $|B| \le \kappa$ and $E \subset \overline{B}$. It follows from [12, Problem 163] that $Q \subset E$ whence $A \subset Q \subset E \subset \overline{B}$ so the set B witnesses exponential κ-domination in $C_p(X)$.

To prove sufficiency, assume that $C_p(X)$ features exponential κ-domination and take a continuous onto map $u\colon X \to Y$ for some space Y of weight not exceeding 2^κ. The dual map $u^*\colon C_p(Y) \to C_p(X)$ is an embedding so the density of the set $Q = u^*(C_p(Y))$ is the same as the density of $C_p(Y)$ while $d(C_p(Y)) \le nw(C_p(Y)) = nw(Y) \le 2^\kappa$. This makes it possible to take a set $B \subset C_p(X)$ such that $|B| \le \kappa$ and $Q \subset \overline{B}$. The reflection map $e_{\overline{B}}\colon X \to C_p(\overline{B})$ is continuous and the i-weight of space $Z = e_{\overline{B}}(X) \subset C_p(\overline{B})$ does not exceed $iw(C_p(\overline{B})) \le |B| \le \kappa$. The dual map $\varphi = e^*_{\overline{B}}\colon C_p(Z) \to C_p(X)$ is an embedding and $\overline{B} \subset \varphi(C_p(Z))$ by [14, Fact 5 of U.086]. Therefore $Q \subset \overline{B} \subset \varphi(C_p(Z))$ so we can apply [3, Proposition 0.4.7] to see that there exist continuous onto maps $v\colon X \to Z$ and $w\colon Z \to Y$ such that $u = w \circ v$. Since $iw(Z) \le \kappa$, the space Z witnesses that X is exponentially κ-cofinal. $\square$

Theorem 3.16. *Let κ be an infinite cardinal. A space X features exponential κ-domination if and only if $C_p(X)$ is exponentially κ-cofinal.*

Proof: Assume first that X is a space with exponential κ-domination and we have a continuous onto map $u\colon C_p(X) \to Y$ for some space Y with $w(Y) \le 2^\kappa$. There exists a set $A \subset X$ and a continuous onto map $\varphi\colon \pi_A(C_p(X)) \to Y$ such that $|A| \le 2^\kappa$ and $\varphi \circ \pi_A = u$, see [2, Theorem 1]. By exponential κ-domination of X there exists a set $B \subset X$ such that $|B| \le \kappa$ and $A \subset \overline{B}$. It is standard that the restriction map $\pi_A\colon C_p(X) \to \pi_A(C_p(X))$ factorizes through $\pi_{\overline{B}}(C_p(X))$ so there exists a continuous onto map $w\colon Z = \pi_{\overline{B}}(C_p(X)) \to Y$ such that $u = w \circ \pi_{\overline{B}}$. Since $iw(Z) \le C_p(\overline{B}) \le |B| \le \kappa$, we conclude that Z witnesses exponential κ-cofinality of $C_p(X)$.

Now assume that the space $C_p(X)$ is exponentially κ-cofinal and $A \subset X$ is a set of cardinality less than or equal to 2^κ. By Theorem 3.15 the space $C_pC_p(X)$ features exponential κ-domination so there exists a set $E \subset C_pC_p(X)$ such that $|E| \le \kappa$ and $A \subset \overline{E}$. Here we identify the space X with its canonical copy in $C_pC_p(X)$, see [3, Corollary 0.5.5]. Every continuous real-valued function on $C_p(X)$ depends on countably many coordinates, see [2, Theorem 1], so we can choose for each $u \in E$ a countable set $B_u \subset X$ such that $u(f) = u(g)$ whenever $f, g \in C_p(X)$ and $f|B_u = g|B_u$.

The set $B = \bigcup\{B_u : u \in E\}$ has cardinality not exceeding κ; assume that $p \in A \backslash \overline{B}$. There exists a function $f \in C_p(X)$ such that $f(p) = 1$ and $f(\overline{B}) \subset \{0\}$; let $g(x) = 0$ for any $x \in X$. It follows from $p \in \overline{E}$ that there is a function $u \in E$ such that $u(f) > \frac{1}{2}$ and $u(g) < \frac{1}{2}$. However, $f|B = g|B$ and hence $f|B_u = g|B_u$ which implies that $u(f) = u(g)$. This contradiction shows that $A \subset \overline{B}$, i.e., the set B witnesses that X is a space with exponential κ-domination. □

Corollary 3.17. *Given a cardinal $\kappa \geq \omega$, if a space X features exponential κ-domination, then $C_{p,2n}(X)$ is a space with exponential κ-domination and $C_{p,2n+1}(X)$ is exponentially κ-cofinal for all $n \in \omega$.*

PROOF: It follows from Theorem 3.16 that the space $C_p(X)$ must be exponentially κ-cofinal and hence Theorem 3.15 can be applied to see that $C_pC_p(X)$ is a space with exponential κ-domination. Proceeding by induction assume that $C_{p,2n}(X)$ is a space with exponential κ-domination. Then $C_{p,2n+1}(X) = C_p(C_{p,2n}(X))$ is an exponentially κ-cofinal space by Theorem 3.16 which makes it possible to apply Theorem 3.15 again to conclude that

$$C_{p,2n+2}(X) = C_p(C_{p,2n+1}(X))$$

features exponential κ-domination. □

Corollary 3.18. *If X is an exponentially κ-cofinal space, then $C_{p,2n+1}(X)$ is a space with exponential κ-domination and $C_{p,2n}(X)$ is exponentially κ-cofinal for all $n \in \omega$.*

PROOF: By Theorem 3.15, the space $Y = C_p(X)$ has exponential κ-domination; apply Corollary 3.17 to convince ourselves that $C_{p,2n+1}(X) = C_{p,2n}(Y)$ is a space with exponential κ-domination and $C_{p,2n}(X) = C_{p,2n-1}(Y)$ is an exponentially κ-cofinal space. □

Theorem 3.19. *Suppose that X is an exponentially κ-cofinal space such that $l(X) \leq \kappa$ and $t(X) \leq \kappa$. Then $iw(X) \leq \kappa$ and hence $|X| \leq 2^\kappa$.*

PROOF: We will prove first that $\psi(X) \leq \kappa$. Striving for contradiction, assume that $p \in X$ and $\psi(p, X) > \kappa$. Take any $x_0 \in X \backslash \{p\}$ and let $G_0 = X$. Proceeding by induction, assume that $\beta < \kappa^+$ and we have a set $\{x_\alpha : \alpha < \beta\} \subset X \backslash \{p\}$ and a family $\{G_\alpha : \alpha < \beta\}$ of closed G_κ-subsets of X with the following properties:

(1) $\{p, x_\alpha\} \subset G_\alpha$ for all $\alpha < \beta$;
(2) $G_\alpha \subset G_\gamma$ whenever $\gamma < \alpha < \beta$;
(3) $\overline{\{x_\gamma : \gamma < \alpha\}} \cap G_\alpha \subset \{p\}$ for every $\alpha < \beta$.

The set $B_\beta = \{p\} \cup \overline{\{x_\alpha : \alpha < \beta\}}$ has cardinality less than or equal to 2^κ by Corollary 3.9 which implies, by Theorem 3.8, that $\psi(p, B_\beta) \le \kappa$ and hence we can choose a closed G_κ-set Q in the space X such that $Q \cap B_\beta = \{p\}$. Let $G_\beta = \bigcap\{G_\alpha : \alpha < \beta\} \cap Q$; since $\psi(p, X) > \kappa$, we can pick a point $x_\beta \in G_\beta \backslash \{p\}$ completing our inductive construction. Observe that it follows from the properties (1) and (3) that the set $A = \{x_\alpha : \alpha < \kappa^+\}$ is faithfully indexed and hence $|A| = \kappa^+$.

Let $F_\alpha = \overline{\{x_\beta : \alpha \le \beta < \kappa^+\}} \subset G_\alpha$ for any $\alpha < \kappa^+$; it follows from $l(X) \le \kappa$ that $\emptyset \ne F = \bigcap\{F_\alpha : \alpha < \kappa^+\}$. If $x \ne p$ and $x \in F$, then $x \in \bar{A}$; since $t(X) \le \kappa$, there exists $\beta < \kappa^+$ such that $x \in \overline{\{x_\alpha : \alpha < \beta\}}$. Since also $x \in F_\beta \subset G_\beta$, we obtained a contradiction with the property (3). Thus, $F = \{p\}$ and it is standard to deduce from $l(X) \le \kappa$ that

(4) the family $\{F_\alpha : \alpha < \kappa^+\}$ is a network at p, i.e., for any set $U \in \tau(p, X)$ there exists $\alpha < \kappa^+$ such that $F_\alpha \subset U$ and hence $\{x_\beta : \alpha < \beta\} \subset U$.

It is an immediate consequence of the property (4) that A is concentrated around the point p and hence the set $D = \{(p, x_\alpha) : \alpha < \kappa^+\} \subset (X \times X) \backslash \Delta_X$ is concentrated around the diagonal Δ_X. Since $|D| = \kappa^+$, the diagonal of X is not κ^+-small; this contradiction with Proposition 3.12 shows that $\psi(X) \le \kappa$. Arhangel'skii's inequality $|X| \le 2^{l(X) \cdot \psi(X) \cdot t(X)} \le 2^\kappa$ shows that $|X| \le 2^\kappa$ and therefore $iw(X) \le \kappa$ by Theorem 3.8. □

Corollary 3.20. *If X is a Lindelöf exponentially ω-cofinal space and $t(X) \le \omega$, then $iw(X) \le \omega$.*

Corollary 3.21. *Suppose that X is a space for which $C_p(X)$ features exponential κ-domination while $l(C_p(X)) \le \kappa$ and $t(C_p(X)) \le \kappa$. Then $d(C_p(X)) \le \kappa$.*

Proof: The space X is exponentially κ-cofinal by Theorem 3.15. Next, observe that $t(X) \le l(C_p(X)) \le \kappa$ by [4] and $l(X) \le t(C_p(X)) \le \kappa$ by [11] which shows that Theorem 3.19 can be applied to see that $iw(X) \le \kappa$ whence $d(C_p(X)) = iw(X) \le \kappa$, see [10]. □

Proposition 3.22. *Assume that X is a κ-stable exponentially κ-cofinal space. Then $nw(X) \le \kappa$.*

Proof: If $d(C_p(X)) > \kappa$, then there exists a left-separated set $L \subset C_p(X)$ with $|L| = \kappa^+$. Next, observe that $C_p(X)$ is a space with exponential κ-domination by Theorem 3.15 and hence there exists a set $A \subset C_p(X)$ such that $|A| \le \kappa$ and $L \subset \bar{A}$. The space $C_p(X)$ is κ-monolithic by [1] so $nw(\bar{A}) \le \kappa$. Therefore $\kappa^+ \le hd(L) \le nw(L) \le nw(\bar{A}) \le \kappa$; this contradiction shows that $d(C_p(X)) \le \kappa$ and therefore $iw(X) = d(C_p(X)) \le \kappa$. Finally, apply κ-stability of X to conclude that $nw(X) \le \kappa$. □

Corollary 3.23. *Any exponentially ω-cofinal Lindelöf Σ-space has a countable network.*

PROOF: Observe that any Lindelöf Σ-space is ω-stable by [3, Theorem II.6.21]; Proposition 3.22 does the rest. □

Corollary 3.24. *Any exponentially ω-cofinal pseudocompact space is compact and metrizable.*

PROOF: Observe that it follows from Corollary II.6.34 of [3] that any pseudocompact space is ω-stable and apply Proposition 3.22. □

Example 3.25. For any cardinal $\kappa > \mathfrak{c}$, there exists an exponentially ω-cofinal space Y such that $l(Y) \geq \kappa$. To see it, consider the exponentially ω-cofinal space $X = \kappa \cup \{p\}$ from Example 3.14. All points of κ are isolated in X and a set $U \subset X$ with $p \in U$ is open if and only if $\kappa \backslash U \leq \mathfrak{c}$. It is standard to see that $C_p(X)$ is homeomorphic to the $\Sigma_{\mathfrak{c}}$-product $S = \{x \in \mathbb{R}^\kappa\colon |x^{-1}(\mathbb{R}\backslash\{0\})| \leq \mathfrak{c}\}$ in the space $\mathbb{R}^\kappa$ and therefore S is a space with exponential ω-domination by Theorem 3.15. If we let $u(\alpha) = 1$ for any $\alpha < \kappa$, then we obtain a point $u \in \mathbb{R}^\kappa$ such that $u \notin \bar{A}$ for any $A \subset S$ with $|A| < \kappa$. This shows that the space $Z = S \cup \{u\}$ has tightness equal to κ. The union of countably many spaces with exponential ω-domination is easily seen to have exponential ω-domination so Z features exponential ω-domination. It was proved in [4] that $l(C_p(Z)) \geq t(Z) = \kappa$ and therefore $l(C_p(Z)) \geq \kappa$. Finally, observe that $C_p(Z)$ is exponentially ω-cofinal by Theorem 3.16 and hence the space $Y = C_p(Z)$ is as promised.

4. Open questions

The author hopes that this work demonstrates that exponentially κ-cofinal spaces form a class interesting in itself. This class encompasses important new information about function spaces and a new metrization theorem for compact spaces. The most intriguing open question in this topic is whether Lindelöf exponentially ω-cofinal spaces must have countable i-weight.

Question 4.1. Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $iw(X) \leq \omega$?

Question 4.2. Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $\psi(X) \leq \omega$?

Question 4.3. Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $|X| \leq 2^{\mathfrak{c}}$?

Question 4.4. Suppose that X is an exponentially ω-cofinal space of countable character. Is it true that $iw(X) \leq \omega$?

Question 4.5. Suppose that X is an exponentially ω-cofinal Fréchet–Urysohn space. Is it true that $iw(X) \leq \omega$?

Question 4.6. Let X be an exponentially ω-cofinal space with $\psi(X) \leq \omega$. Is it true that $iw(X) \leq \omega$?

Question 4.7. Let X be an exponentially ω-cofinal space with a G_δ-diagonal. Is it true that $iw(X) \leq \omega$?

Question 4.8. Suppose that X is an exponentially ω-cofinal space. Is it true that $\mathrm{ext}(X) \leq \mathfrak{c}$?

Question 4.9. Suppose that X is a space with exponential ω-domination such that $t(X) = l(X) = \omega$. Must X be separable?

Question 4.10. Suppose that X is a Lindelöf Fréchet–Urysohn space featuring exponential ω-domination. Must X be separable?

Question 4.11. Suppose that $C_p(X)$ is a Fréchet–Urysohn space that features exponential ω-domination. Must $C_p(X)$ be separable?

Question 4.12. Suppose that $C_p(X)$ is an exponentially ω-cofinal Fréchet–Urysohn space. Must X be separable?

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