Inequalities of DVT-type – the one-dimensional case

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Abstract. In this note, particular inequalities of DVT-type in real and integer numbers are investigated.

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Let Q be a finite quasigroup of order n. The associativity index $a(Q)$ is the number of associative triples, i.e., $a(Q) = |\{(a,b,c) \in Q^3 \colon a(bc) = (ab)c\}|$. Of course, $a(Q) \le n^3$ and $a(Q) = n^3$ if and only if Q is a group. On the other hand, to find lower bounds for $a(Q)$ is rather complicated. The problem of finding $a(Q)$ has been investigated since 1983, see [4]. Recently, it was discovered that quasigroups with small associative index may have applications in cryptography, see [2].

In [1], A. Drápal and V. Valent proved that $a(Q) \ge 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q, i.e., $i(Q) = |\{x \in Q \colon xx = x\}|$, $\delta_1 = |\{z \in Q \colon zx \ne x \text{ for all } x \in Q\}|$ and $\delta_2 = |\{z \in Q \colon xz \ne x \text{ for all } x \in Q\}|$ (Theorem 2.5). This important result is an easy consequence of the inequality

$$\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^{k}(a_i + b_i) \ge 3n - 2k + (r + s),$$

where $n \ge k \ge 0$, $a_1, \dots, a_n$, $b_1 \dots, b_n$ are nonnegative integers such that $\sum a_i = n = \sum b_i$, $a_i \ge 1$ and $b_i \ge 1$ for $1 \le i \le k$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$ (Proposition 2.4 (ii)). The lengthy and complicated proof of this DVT-inequality (inequality of Drápal–Valent type) in [1] is based on highly semantically involved insight.

In [3], a short and elementary arithmetical proof of a more general inequality of this type was found (unfortunately on expenses of brutalist syntax). This inequality is two-dimensional in the sense that it works with two n-tuples of integers. The approach in [3] opens a road to investigation of similar inequalities of

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DVT-type which could be useful in further investigations of estimates in nonassociative algebra and they are also of independent interest. Hence they deserve a thorough examination, however the research is only at its beginning. In this note, the one-dimensional case working with one n-tuple of real numbers is investigated. Inequalities derived for different properties of the n-tuple $a_1, \dots, a_n$ in case of real numbers and integers together with some examples are summarized in Section 6.

1. First concepts

1.1. Let $n \geq 1$ and let $\alpha = (a_1, \dots, a_n)$ be an ordered n-tuple of real numbers. We put

(1) $z(\alpha, a) = |\{i\colon 1 \leq i \leq n,\ a_i = a\}|$ for every $a \in \mathbb{R}$;
(2) $z(\alpha) = z(\alpha, 0)$;
(3) $z(\alpha, +) = \sum_{a>0} z(\alpha, a)$;
(4) $z(\alpha, -) = \sum_{a<0} z(\alpha, a)$;
(5) $\max(\alpha) = \max(a_1, \dots, a_n)$;
(6) $\min(\alpha) = \min(a_1, \dots, a_n)$;
(7) $z(\alpha, \max) = z(\alpha, \max(\alpha))$;
(8) $z(\alpha, \min) = z(\alpha, \min(\alpha))$;
(9) $s(\alpha) = \sum_{i=1}^n a_i$;
(10) $r(\alpha) = \sum_{i=1}^n a_i^2$;
(11) $q(\alpha) = r(\alpha) - s(\alpha)$;
(12) $t(\alpha) = q(\alpha) - z(\alpha)$.

It is immediately clear that

(13) $n = \sum_{\alpha \in \mathbb{R}} z(\alpha, a)$;
(14) $n = z(\alpha, +) + z(\alpha, -) + z(\alpha)$;
(15) $q(\alpha) = \sum_{i=1}^n a_i(a_i - 1)$;
(16) $r(\alpha) + s(\alpha) = \sum_{i=1}^n a_i(a_i + 1)$.

For every $a \in \mathbb{R}$, let $Z_a(\alpha) = \{i\colon 1 \leq i \leq n,\ a_i = a\}$. We have

(17) $q(\alpha) = \sum_{i \in V(\alpha)} (a_i^2 - a_i)$, where $V(\alpha) = \{1, \dots, n\} \setminus (Z_0(\alpha) \cup Z_1(\alpha))$;
(18) $t(\alpha) = \sum_{i \in V(\alpha)} (a_i^2 - a_i) - |Z_0(\alpha)|$.

We put $|\alpha| = (|a_1|, \dots, |a_n|)$, $\alpha + a = (a_1 + a, \dots, a_n + a)$ and $a\alpha = (aa_1, \dots, aa_n)$ for every $a \in \mathbb{R}$. The following two lemmas are obvious.

Lemma 1.2.

(i) *$q(\alpha) \geq q(|\alpha|)$.*
(ii) *$q(\alpha) = q(|\alpha|)$ if and only if $a_i \geq 0$ for every $i = 1, \dots, n$.*
(iii) *$t(\alpha) \geq t(|\alpha|)$.*
(iv) *$t(\alpha) = t(|\alpha|)$ if and only if $a_i \geq 0$ for every $i = 1, \dots, n$.*

Lemma 1.3. *Assume that $|a_j| \geq 1$ for every $j \notin Z_0(\alpha)$. Then:*

(i) $r(\alpha) \geq n - z(\alpha) = z(\alpha,+) + z(\alpha,-) \geq 0$.
(ii) $r(\alpha) = n - z(\alpha)$ *if and only if* $a_i \in \{0,1,-1\}$ *for every* $i = 1,\dots,n$.
(iii) $r(\alpha) = z(\alpha,+)$ *if and only if* $a_i \in \{0,1\}$ *for every* $i = 1,\dots,n$.
(iv) $r(\alpha) = z(\alpha,-)$ *if and only if* $a_i \in \{0,-1\}$ *for every* $i = 1,\dots,n$.

Lemma 1.4. *Assume that $|a_j| \geq 1$ for every $j \notin Z_0(\alpha)$. Then:*

(i) $r(\alpha) \geq s(\alpha) + 2z(\alpha,-)$.
(ii) $r(\alpha) = s(\alpha) + 2z(\alpha,-)$ *if and only if* $a_i \in \{0,1,-1\}$ *for every* $i = 1,\dots,n$.

Proof: We have $a^2 > a + 2\cdot 0$ for $a > 1$, $a^2 = a + 2\cdot 0$ for $a = 1, 0$, $a^2 = a + 2\cdot 1$ for $a = -1$ and $a^2 > a + 2\cdot 1$ for $a < -1$. The rest is clear. □

Lemma 1.5. *Assume that $|a_j| \geq 1$ for every $j \notin Z_0(\alpha)$ and that $s(\alpha) \geq 0$. Then:*

(i) $r(\alpha) \geq 2z(\alpha,-)$.
(ii) $r(\alpha) = 2z(\alpha,-)$ *if and only if* $s(\alpha) = 0$ *and* $a_i \in \{0,1,-1\}$ *for every* $i = 1,\dots,n$.

Proof: This follows immediately from Lemma 1.4. □

Lemma 1.6. *Assume that $|a_j| \geq 1$ for every $j \notin Z_0(\alpha)$. Then:*

(i) $r(\alpha) \geq 2z(\alpha,+) - s(\alpha)$.
(ii) $r(\alpha) = 2z(\alpha,+) - s(\alpha)$ *if and only if* $a_i \in \{0,1,-1\}$ *for every* $i = 1,\dots,n$.

Proof: This follows from Lemma 1.4 via $\alpha \leftrightarrow -\alpha$. □

Lemma 1.7. *Assume that $|a_j| \geq 1$ for every $j \notin Z_0(\alpha)$ and that $s(\alpha) \leq 0$. Then:*

(i) $r(\alpha) \geq 2z(\alpha,+)$.
(ii) $r(\alpha) = 2z(\alpha,+)$ *if and only if* $s(\alpha) = 0$ *and* $a_i \in \{0,1,-1\}$ *for every* $i = 1,\dots,n$.

Proof: This follows immediately from Lemma 1.6. □

Example 1.8. Put $\alpha = (\frac{1}{2}, \frac{1}{2}, -1)$. Then $s(\alpha) = 0$, $r(\alpha) = \frac{3}{2} < 2 = 2z(\alpha,-)$ and $r(\alpha) = \frac{3}{2} < 4 = 2z(\alpha,+)$.

Example 1.9. Assume that $-1 \leq s(\alpha) < 0$ and $r(\alpha) \leq 2z(\alpha,-)$. If $n = 1$ then $\alpha = (a_1)$, $-1 \leq a_1 < 0$ and $r(\alpha) = a_1^2 \leq 1 < 2 = 2z(\alpha,-)$. Assume, therefore, that $n \geq 2$, $a_n = \min(\alpha)$ and put $\beta = (a_1,\dots,a_{n-1})$. We have $a_n < 0$, $s(\beta) = s(\alpha) - a_n \geq -1 - a_n$, $r(\beta) = r(\alpha) - a_n^2$, $z(\beta,-) = z(\alpha,-) - 1$ and $2z(\beta,-) = 2z(\alpha,-) - 2 \geq r(\alpha) - 2 = r(\beta) + a_n^2 - 2 > r(\alpha) - 3$.

If $s(\beta) \geq 0$ and $|a_j| \geq 1$ for every $j \notin Z_0(\alpha)$, $j \neq n$, then $r(\beta) \geq 2z(\beta,-)$ by Lemma 1.4, and so $a_n \geq -\sqrt{2}$.

If $a_n \leq -1$ then $s(\beta) \geq s(\alpha) + 1 \geq 0$.

Example 1.10. Assume that $s(\alpha) = -1$ and $r(\alpha) = 2z(\alpha, -)$. If $n = 1$ then $\alpha = (-1)$ and $r(\alpha) = 1 < 2 = 2z(\alpha, -)$, so that $n \geq 2$. Assume again that $a_n = \min(\alpha)$ and put $\beta = (a_1, \dots, a_{n-1})$. We have $a_n < 0$, $s(\beta) = s(\alpha) - a_n = -1 - a_n$, $r(\beta) = r(\alpha) - a_n^2 = 2z(a, -) - a_n^2 = 2z(\beta, -) + 2 - a_n^2$. Consequently, $r(\beta) \geq 2z(\beta, -)$ if and only if $a_n \geq -\sqrt{2}$ and $r(\beta) = 2z(\beta, -)$ if and only if $a_n = -\sqrt{2}$ (then $s(\beta) = -1 + \sqrt{2} > 0$).

Lemma 1.11. *Put $\beta = \alpha - 1$. Then:*

(i) $s(\beta) = s(\alpha) - n$.
(ii) $r(\beta) = r(\alpha) - 2s(\alpha) + n$.
(iii) $z(\alpha) = z(\beta, -1)$.
(iv) $q(\beta) = r(\alpha) - 3s(\alpha) + 2n$.
(v) $t(\beta) = r(\alpha) - 3s(\alpha) - z(\alpha, 1) + 2n$.

Proof: It follows directly from the definition of the respective numbers. ☐

Lemma 1.12. *Let all the numbers $a_1, \dots, a_n$ be nonnegative. Then:*

(i) $t(\alpha) = r(\alpha - 1) + s(\alpha - 1) - z(\alpha - 1, -)$.
(ii) $t(\alpha) - z(\alpha) = r(\alpha - 1) + s(\alpha - 1) - 2z(\alpha - 1, -)$.

Proof: We have $r(\alpha - 1) + s(\alpha - 1) - z(\alpha - 1, -) = r(\alpha) - 2s(\alpha) + n + s(\alpha) - n - z(\alpha - 1, -1) = r(\alpha) - s(\alpha) - z(\alpha) = t(\alpha)$ by (11), (12) and Lemma 1.11. ☐

Example 1.13. Put $\alpha = (1, 1, 1, -1, -1)$ ($n = 5$). Then $s(\alpha) = 1$, $r(\alpha) = 5$, $z(\alpha, -) = 2$ and $r(\alpha) < az(\alpha, -)$ for every $a \in \mathbb{R}$, $a > \frac{5}{2}$. Furthermore, $\beta = \alpha - 1 = (0, 0, 0, -2, -2)$, $s(\beta) = -4$, $r(\beta) = 8$, $z(\beta, -) = 2$, $z(\beta) = 3$ and $r(\beta) + s(\beta) > z(\beta)$.

Remark 1.14. Put $W(\alpha) = \{1, \dots, n\} \setminus Z_0(\alpha)$. Clearly, $t(\alpha) = -|Z_0(\alpha)| + \sum_{i \in W(\alpha)} (a_i^2 - a_i) = \sum_{i \in W(\alpha)} (a_i - 1)^2 + \sum_{i \in W(\alpha)} a_i - |W(\alpha)| - n + |W(\alpha)| = \sum_{i \in V(\alpha)} (a_i - 1)^2 + s(\alpha) - n$. Therefore, $t(\alpha) \geq 0$ if and only if $s(a) + \sum_{i \in V(\alpha)} (a_i - 1)^2 \geq n$ (in particular, $t(\alpha) \geq 0$, provided that $s(\alpha) \geq n$). We also have $t(\alpha) = \sum_{i \in V(\alpha)} (a_i - 1)^2 + \sum_{i \in V(\alpha)} a_i + |Z_1(\alpha)| - n$ and $t(\alpha) - z(\alpha) = \sum_{i \in V(\alpha)} (a_i - 1)^2 + s(\alpha) + |W(\alpha)| - 2n$. (In particular, $t(\alpha) \geq z(\alpha)$, provided that $s(\alpha) + |W(\alpha)| \geq 2n$. That is, $s(\alpha) \geq n + z(\alpha)$.)

Lemma 1.15. *Assume that for every $i = 1, \dots, n$ we have either $a_i \leq 1$ or $a_i \geq 2$. Then:*

(i) $r(\alpha) - 3s(\alpha) + 2n \geq 0$.
(ii) $r(\alpha) - 3s(\alpha) + 2n = 0$ *if and only if* $a_i \in \{1, 2\}$ *for every* $i = 1, \dots, n$.

Proof: It is enough to observe that $a^2 - 3a + 2 > 0$ for $a < 1$ or $a > 2$ and that $a^2 - 3a + 2 = 0$ just for $a = 1, 2$. ☐

Lemma 1.16. *Assume that for every* $i = 1, \dots, n$ *we have either* $a_i \leq 1$ *or* $a_i \geq 2$. *Then:*

(i) $r(\alpha) - 3s(\alpha) + 2n - 2z(\alpha) \geq 0$.

(ii) $r(\alpha) - 3s(\alpha) + 2n - 2z(a) = 0$ *if and only if* $a_i \in \{0, 1, 2\}$ *for every* $i = 1, \dots, n$.

Proof: We can assume that $a_1, \dots, a_m$ are nonzero and $a_{m+1} = \dots = a_n = 0$. Then $z(\alpha) = n - m$. If $m = n$ then the result is settled down by Lemma 1.15. If $m = 0$ then the result is clear. Assume, therefore, that $1 \leq m < n$. Put $\beta = (a_1, \dots, a_m)$. By Lemma 1.15 $r(\alpha) - 3s(\alpha) + 2n - 2z(\alpha) = r(\beta) - 3s(\beta) + 2m \geq 0$. The rest is clear. $\square$

2. First bunch of technical results

Throughout this section, let $n \geq 2$, $\alpha = (a_1, \dots, a_n)$ be an ordered n-tuple of integers, $1 \leq j, k \leq n$, $j \neq k$, $b_i = a_i$ for $1 \leq i \leq n$, $i \neq j, k$, $b_j = a_j - 1$, $b_k = a_k + 1$ and $\beta = (b_1, \dots, b_n)$.

The following six assertions are easy.

Lemma 2.1. $z(\beta) \in \{z(\alpha) - 2, z(\alpha) - 1, z(\alpha), z(\alpha) + 1, z(\alpha) + 2\}$.

Lemma 2.2. $z(\beta) = z(\alpha)$ *if and only if at least (and then just) one of the following three cases takes place:*

(1) $a_j = 0,\ a_k = -1$;

(2) $a_j = 1,\ a_k = 0$;

(3) $a_j \neq 0, 1$ *and* $a_k \neq -1, 0$.

Lemma 2.3. $z(\beta) = z(\alpha) + 2$ *if and only if* $a_j = 1,\ a_k = -1$.

Lemma 2.4. $z(\beta) = z(\alpha) + 1$ *if and only if at least (and then just) one of the following two cases takes place:*

(1) $a_j = 1,\ a_k \neq -1, 0$;

(2) $a_j \neq 0, 1,\ a_k = -1$.

Lemma 2.5. $z(\beta) = z(\alpha) - 1$ *if and only if at least (and then just) one of the following two cases takes place:*

(1) $a_j = 0,\ a_k \neq -1, 0$;

(2) $a_j \neq 0, 1,\ a_k = 0$.

Lemma 2.6. $z(\beta) = z(\alpha) - 2$ *if and only if* $a_j = 0 = a_k$.

Lemma 2.7. $s(\beta) = s(\alpha)$.

Proof: We have $s(\beta) = \sum_{i-1, i\neq j,k}^{n} a_i + a_j - 1 + a_k + 1 = \sum_{i=1}^{n} a_i = s(\alpha)$. $\square$

Lemma 2.8. *$r(\alpha) - r(\beta) = 2(a_j - a_k - 1)$.*

Proof: We have $r(\alpha) - r(\beta) = a_j^2 + a_k^2 + \sum_{i=1, i\neq j,k}^{n} a_i^2 - \sum_{i=1, i\neq j,k}^{n} a_i^2 - (a_j - 1)^2 - (a_k + 1)^2 = 2a_j - 1 - 2a_k - 1$. □

Lemma 2.9. *$t(\alpha) - t(\beta) = 2(a_j - a_k - 1) + z(\beta) - z(\alpha)$.*

Proof: Use Lemma 2.7 and Lemma 2.8. □

Lemma 2.10. *If $a_j \geq a_k + 2$ then $t(\alpha) > t(\beta)$.*

Proof: By Lemma 2.9 $t(\alpha) - t(\beta) = 2(a_j - a_k - 1) + z(\beta) - z(\alpha) \geq 0$ (use Lemma 2.1). If $t(\alpha) = t(\beta)$ then $z(\beta) = z(\alpha) - 2$ and Lemma 2.6 yields $a_j = a_k = 0$, a contradiction. □

Lemma 2.11. *If $a_j = a_k + 1$ then $t(\alpha) = t(\beta)$.*

Proof: First, it follows from Lemmas 2.1, 2.3, 2.4, 2.5 and 2.6 that $z(\beta) = z(\alpha)$. Now it remains to use Lemma 2.9. □

Lemma 2.12. *If $a_j \leq a_k$ then $t(\alpha) < t(\beta)$.*

Proof: First, it follows from Lemmas 2.1 and 2.3 that $z(\beta) - z(\alpha) \leq 1$. Now it remains to use Lemma 2.9. □

The following three lemmas are easy.

Lemma 2.13. *Let $a_k \neq \max(\alpha), \max(\alpha) - 1$.*

(i) *If $a_j \neq \max(\alpha)$ then $\max(\beta) = \max(\alpha)$ and $z(\beta, \max) = z(\alpha, \max)$.*
(ii) *If $a_j = \max(\alpha)$ and $z(\alpha, \max) \geq 2$ then $\max(\beta) = \max(\alpha)$ and $z(\beta, \max) = z(\alpha, \max) - 1$.*
(iii) *If $a_j = \max(\alpha)$ and $z(\alpha, \max) = 1$ then $\max(\beta) < \max(\alpha)$.*

Lemma 2.14. *Let $a_k = \max(\alpha)$. Then $\max(\beta) = \max(\alpha) + 1$, $z(\beta, \max) = 1$.*

Lemma 2.15. *Let $a_k = \max(\alpha) - 1$.*

(i) *If $a_j \neq \max(\alpha)$ then $\max(\beta) = \max(\alpha)$ and $z(\beta, \max) = z(\alpha, \max) + 1$.*
(ii) *If $a_j = \max(\alpha)$ then $\max(\beta) = \max(\alpha)$ and $z(\beta, \max) = z(\alpha, \max)$.*

Lemma 2.16. *$\max(\beta) < \max(\alpha)$ if and only if $a_j = \max(\alpha) \neq a_k \neq \max(\alpha) - 1$ and $z(\alpha, \max) = 1$.*

Proof: Combine Lemmas 2.13, 2.14 and 2.15. □

Lemma 2.17. *Let $a_j = \max(\alpha)$, $a_k = \min(\alpha)$ and $a_j \geq a_k + 2$. Then:*

(i) *$t(\alpha) > t(\beta)$.*
(ii) *If $z(\alpha, \max) = 1$ then $\max(\beta) < \max(\alpha)$.*
(iii) *If $z(\alpha, \max) \geq 2$ then $\max(\beta) = \max(\alpha)$ and $z(\beta, \max) = z(\alpha, \max) - 1$.*

Proof: (i) is Lemma 2.10, (ii) is Lemma 2.13 (iii) and (iii) is Lemma 2.13 (ii). □

Lemma 2.18. *Let* $a_j = \max(\alpha)$, $a_k = \min(\alpha)$ *and* $a_j = a_k + 1$. *Then* $t(\alpha) = t(\beta)$, $\max(\alpha) = \max(\beta)$ *and* $z(\alpha, \max) = z(\beta, \max)$.

Proof: See Lemmas 2.11 and 2.15. □

Lemma 2.19. *Let* $a_j = \max(\alpha) = \min(\alpha) = a_k$. *Then* $t(\alpha) < t(\beta)$, $\max(\beta) = \max(\alpha) + 1$ *and* $z(\beta, \max) = 1 \leq z(\alpha, \max) = n$.

Proof: Using Lemma 2.12, this is obvious. □

Example 2.20.

(i) Let $a_j \geq 2$ and $a_k = 0$. Then $z(\beta) = z(\alpha) - 1$ and $t(\alpha) = t(\beta) + 2a_j - 3$, see Lemma 2.9. If $a_j = 2$ then $t(\alpha) = t(\beta) + 1$. If $a_j \geq 3$ then $t(\alpha) \geq t(\beta) + 3$.

(ii) Let $a_j \geq a_k + 2$ and $a_k \geq 1$ (so that $a_j \geq 3$). Then $z(\beta) = z(\alpha)$ and $t(\alpha) = t(\beta) + 2(a_j - a_k) - 2$. If $a_j = a_k + 2$ then $t(a) = t(\beta) + 2$. If $a_j \geq a_k + 3$ then $t(\alpha) \geq t(\beta) + 4$.

Observation 2.21. Assume that $s(\alpha) \geq n$, $a_j = \max(\alpha)$ and $a_k = \min(\alpha)$. Clearly, $a_j \geq 1$. If $a_j = 1$ then $a_1 = \cdots = a_n = 1$ and $t(\alpha) = 0$. Assume, therefore, that $a_j \geq 2$.

Let $a_j = a_k = a$. Then $t(\alpha) = na(a-1) \geq 2n \geq 4$ ($t(\alpha) = 4$ just for $n = 2$, $a = 2$).

Let $a_j > a_k = a$. If $a_j = a + 1$ then $a \geq 1$. For $u = z(\alpha, a+1)$ and $v = z(\alpha, a)$, we have $t(\alpha) = u(a+1)^2 + va^2 - u(a+1) - va = ua^2 + 2ua + u + va^2 - ua - u - va = (u+v)a^2 + (u-v)a$. However $u + v = n$, and therefore $t(\alpha) = na^2 + na - 2va \geq na^2 + na - 2(n-1)a = na(a-1) + 2a \geq 2$. In this case, $t(\alpha) = 2$ if and only if $a = 1$ and $z(\alpha, \max) = 1$.

Finally, let $a_j \geq a_k + 2$. We have $t(\alpha) = t(\beta) + 2(a_j - a_k - 1) + z(\beta) - z(\alpha)$ by Lemma 2.9. If $a_j \geq 2$ and $a_k \leq -2$ then $t(\alpha) \geq t(\beta) + 6$. If $a_j \geq 2$ and $a_k = -1$ then $t(\alpha) \geq t(\beta) + 5$. If $a_j \geq 2$ and $a_k = 0$ then $t(\alpha) \geq t(\beta) + 1$ (in this case $t(\alpha) = t(\beta) + 1$ just for $a_j = 2$). If $a_j \geq 2$ and $a_k \geq 1$ then $t(\alpha) \geq t(\beta) + 2$ (in this case, $t(\alpha) = t(\beta) + 2$ just for $a_j = a_k + 2$).

Observation 2.22. If $a_j = 1$ and $a_k = -1$ then $t(\alpha) = t(\beta) + 4$. If $a_j = 1$ and $a_k \leq -2$ then $t(\alpha) \geq t(\beta) + 5$. Notice also that $\max(\beta) \leq \max(\alpha)$. If $z(\alpha, \max) = 1$ then $\max(\beta) = \max(\alpha) - 1$. If $z(\alpha, \max) \geq 2$ then $\max(b) = \max(\alpha)$ and $z(\beta, \max) = z(\alpha, \max) - 1$.

3. Second bunch of technical results

In this section, let $n \geq 2$, $\alpha = (a_1, \ldots, a_n)$ be an ordered n-tuple of integers, $1 \leq j \leq n$, $b_i = a_i$ for $1 \leq i \leq j-1$, $b_i = a_{i+1}$ for $j \leq i \leq n$ and $\beta = (b_1, \ldots, b_{n-1})$.

The following two assertions are obvious.

Lemma 3.1.

(i) *If* $a_j = 0$ *then* $z(\beta) = z(\alpha) - 1$.
(ii) *If* $a_j \neq 0$ *then* $z(\beta) = z(\alpha)$.

Lemma 3.2. $s(\beta) = s(\alpha) - a_j$ *and* $r(\beta) = r(\alpha) - a_j^2$.

Lemma 3.3. *Let* $a_j = 0$. *Then* $t(\beta) = t(\alpha) + 1$.

Proof: We have $t(\beta) = r(\beta) - s(\beta) - z(\beta) = r(\alpha) - s(\alpha) + 1 = t(\alpha) + 1$. □

Lemma 3.4. *Let* $a_j \neq 0$. *Then* $t(\beta) = t(\alpha) - (a_j^2 - a_j) \leq t(\alpha)$. *The equality occurs if and only if* $a_j = 1$.

Proof: We have $t(\beta) = r(\beta) - s(\beta) - z(\beta) = r(\alpha) - a_j^2 - s(\alpha) + a_j - z(\alpha) = t(\alpha) - (a_j^2 - a_j) \leq t(\alpha)$. Now, $a_j^2 = a_j$ only for $a_j = 1$. □

Lemma 3.5. *Let* $a_j = 0$. *Then* $t(\beta) - z(\beta) = t(\alpha) - z(\alpha) + 2$.

Proof: We have $t(\beta) - z(\beta) = t(\alpha) + 1 - z(\beta) = t(\alpha) + 1 - z(\alpha) + 1$ by Lemmas 3.3 and 3.1 (i). □

Lemma 3.6. *Let* $a_j \neq 0$. *Then* $t(\beta) - z(\beta) \leq t(\alpha) - z(\alpha)$. *The equality occurs if and only if* $a_j = 1$.

Proof: Use Lemmas 3.4 and 3.1 (ii). □

Lemma 3.7.

(i) *If* $a_j > 0$ *then* $z(\beta, +) = z(\alpha, +) - 1$ *and* $z(\beta, -) = z(\alpha, -)$.
(ii) *If* $a_j = 0$ *then* $z(\beta, +) = z(\alpha, +)$ *and* $z(\beta, -) = z(\alpha, -)$.
(iii) *If* $a_j < 0$ *then* $z(\beta, +) = z(\alpha, +)$ *and* $z(\beta, -) = z(\alpha, -) - 1$.

Proof: This is obvious. □

Lemma 3.8. $0 \leq z(\alpha, +) - z(\beta, +) \leq 1$ *and* $0 \leq z(\alpha, -) - z(\beta, -) \leq 1$.

Proof: This follows from Lemma 3.7. □

4. Third bunch of technical results

In this section, let $n \geq 2$, $\alpha = (a_1, \dots, a_n)$ be an ordered n-tuple of integers such that $a_n = 0$ and put $\beta = (a_1 - 1, a_2, \dots, a_{n-1})$. The following two assertions are obvious.

Lemma 4.1.

(i) *If* $a_1 = 1$ *then* $z(\beta) = z(\alpha)$.
(ii) *If* $a_1 = 0$ *then* $z(\beta) = z(\alpha) - 2$.
(iii) *If* $a_1 \neq 0, 1$ *then* $z(\beta) = z(\alpha) - 1$.

Lemma 4.2.

(i) $s(\beta) = s(\alpha) - 1$.
(ii) $r(\beta) = r(\alpha) - 2a_1 + 1$.

Lemma 4.3.

(i) *If* $a_1 = 1$ *then* $t(\beta) = t(\alpha)$.
(ii) *If* $a_1 = 0$ *then* $t(\beta) = t(\alpha) + 4$.
(iii) *If* $a_1 \neq 0, 1$ *then* $t(\beta) = t(\alpha) - 2a_1 + 3$.
(iv) $t(\beta) < t(\alpha)$ *if and only if* $a_1 \geq 2$.

Proof: By Lemma 4.2, we have $t(\beta) = r(\beta) - s(\beta) - z(\beta) = r(\alpha) - s(\alpha) - 2a_1 + 2 - z(\beta)$. The rest follows from Lemma 4.1. □

Lemma 4.4.

(i) *If* $a_1 = 1$ *then* $t(\beta) - z(\beta) = t(\alpha) - z(\alpha)$.
(ii) *If* $a_1 = 0$ *then* $t(\beta) - z(\beta) = t(\alpha) - z(\alpha) + 6$.
(iii) *If* $a_1 \neq 0, 1$ *then* $t(\beta) - z(\beta) = t(\alpha) - z(\alpha) - 2a_1 + 4$.

Proof: This follows from Lemmas 4.3 and 4.1. □

Lemma 4.5. $t(\alpha) - z(\alpha) \geq t(\beta) - z(\beta)$ *if and only if* $a_1 \geq 1$. *The equality occurs if and only if* $a_1 = 1, 2$.

Proof: This follows from Lemma 4.4. □

5. Fourth bunch of technical results

Let $n \geq 2$ and α, β, j, k be as in the second section. Now, we denote by A (B, respectively) the set of ordered pairs (i_1, i_2) ((i_3, i_4), respectively) of indices such that $a_{i_1} > a_{i_2}$ ($b_{i_3} > b_{i_4}$, respectively) and we put $u(\alpha) = \sum_{(i_1,i_2)\in A}(a_{i_1} - a_{i_2})$ and $u(\beta) = \sum_{(i_3,i_4)\in B}(b_{i_3} - b_{i_4})$.

5.1. Let A_1 (B_1, respectively) designate the set of the pairs $(i_5, i_6) \in A$ ($(i_7, i_8) \in B$, respectively) such that $j \neq i_5 \neq k \neq i_6 \neq j$ ($j \neq i_7 \neq k \neq i_8 \neq j$, respectively). Put $u_1(\alpha) = \sum(a_{i_5} - a_{i_6})$ and $u_1(\beta) = \sum(b_{i_7} - b_{i_8})$. One sees readily that $u_1(\alpha) = u_1(\beta)$ (we have $A_1 = B_1$).

5.2. Let A_2 (B_2, respectively) designate the set of the pairs $(j, i_9) \in A$ ($(j, i_{10}) \in B$, respectively) such that $a_j \geq a_{i_9} + 2$ ($b_j \geq b_{i_{10}} + 1$, respectively) and $i_9 \neq k$ ($i_{10} \neq k$, respectively). Since $b_j = a_j - 1$ and $a_{i_9} = b_{i_9}$ ($a_{i_{10}} = b_{i_{10}}$, respectively), we have $A_2 = B_2$. Furthermore, $(a_j - a_{i_9}) - 1 = b_j - b_{i_9}$. Thus $u_2(\alpha) = u_2(\beta) + q_1$, where $u_2(\alpha) = \sum(a_j - a_{i_9})$, $u_2(\beta) = \sum(b_j - b_{i_{10}})$ and $q_1 = |A_2|$ ($= |B_2|$).

5.3. Let A_3 (B_3, respectively) designate the set of the pairs $(i_{11}, j) \in A$ ($(i_{12}, j) \in B$, respectively) such that $a_{i_{11}} \geq a_j + 1$ ($b_{i_{12}}) \geq b_j + 2$, respectively) and $i_{11} \neq k$ ($i_{12} \neq k$, respectively). Since $b_j = a_j - 1$ and $a_{i_{11}} = b_{i_{11}}$ ($a_{i_{12}} = b_{i_{12}}$, respectively), we have $A_3 = B_3$. Furthermore, $(a_{i_{11}} - a_j) + 1 = b_{i_{11}} - b_j$. Thus $u_3(\alpha) = u_3(\beta) - q_2$, where $u_3(\alpha) = \sum(a_{i_{11}} - a_j)$, $u_3(\beta) = \sum(b_{i_{11}} - b_j)$ and $q_2 = |A_3|$ $(= |B_3|)$.

5.4. Let A_4 (B_4, respectively) designate the set of the pairs $(k, i_{13}) \in A$ ($(k, i_{14}) \in B$, respectively) such that $a_k \geq a_{i_{13}} + 1$, ($b_k \geq b_{i_{14}} + 2$, respectively) and $i_{13} \neq j$ ($i_{14} \neq j$, respectively). Since $b_k = a_k + 1$ and $a_{i_{13}} = b_{i_{13}}$ ($a_{i_{14}} = b_{i_{14}}$, respectively), we have $A_4 = B_4$. Furthermore, $(a_k - a_{i_{13}}) + 1 = b_k - b_{i_{14}}$. Thus $u_4(\alpha) = u_4(\beta) - q_3$, where $u_4(\alpha) = \sum(a_k - a_{i_{13}})$, $u_4(\beta) = \sum(b_k - b_{i_{14}})$ and $q_3 = |A_4|$ $(= |B_4|)$.

5.5. Let A_5 (B_5, respectively) designate the set of the pairs (i_{15}, k) ((i_{16}, k), respectively) such that $a_{i_{15}} \geq a_k + 2$ ($b_{i_{16}} \geq b_k + 1$, respectively) $i_{15} \neq j$ ($i_{16} \neq j$, respectively). Since $b_k = a_k + 1$ and $a_{i_{15}} = b_{i_{15}}$ ($a_{i_{16}} = b_{i_{16}}$, respectively), we have $A_5 = B_5$. Furthermore, $(a_{i_{15}} - a_k) - 1 = b_{i_{15}} - b_k$. Thus $u_5(\alpha) = u_5(\beta) + q_4$, where $u_5(\alpha) = \sum(a_{i_{15}} - a_k)$, $u_5(\beta) = \sum(b_{i_{15}} - b_k)$ and $q_4 = |A_5|$ $(= |B_5|)$.

5.6. Put $A_6 = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$. Notice that this union is disjoint and that $|A_6| = q_0 + q_1 + q_2 + q_3 + q_4 = |B_6|$, where $q_0 = |A_1|$ $(= |B_1|)$. Notice also that A_6 $(= B_6)$ is just the set of pairs $(i_{17}, i_{18}) \in A \cap B$ such that $\{i_{17}, i_{18}\} \neq \{j, k\}$. Put $u_6(\alpha) = \sum(a_{i_{17}} - a_{i_{18}})$ and $u_6(\beta) = \sum(b_{i_{17}} - b_{i_{18}})$. We have $u_6(\alpha) = \sum_{i=1}^{5} u_i(\alpha)$, $u_6(\beta) = \sum_{i=1}^{5} u_i(\beta)$ and $u_6(\alpha) - u_6(\beta) = q_1 - q_2 - q_3 + q_4 = |A_2 \cup A_5| - |A_3 \cup A_4|$.

5.7. Put $A_7 = A \setminus A_6$ and $B_7 = B \setminus B_6$ (of course, we have $A_6 = B_6$).

5.8. Let $(i_{19}, i_{20}) \in A_7$. Then $a_{i_{19}} \geq a_{i_{20}}$ and just one of the following six cases takes place:

(a) $i_{19} = j,\ i_{20} \neq j, k,\ a_{i_{19}} = a_j - 1$;
(b) $i_{19} \neq j,\ i_{20} = k,\ a_{i_{19}} = a_k + 1$;
(c) $i_{19} = j,\ i_{20} = k,\ a_j = a_k + 1$;
(d) $i_{19} = j,\ i_{20} = k,\ a_j = a_k + 2$;
(e) $i_{19} = j,\ i_{20} = k,\ a_j \geq a_k + 3$;
(f) $i_{19} = k,\ i_{20} = j,\ a_k > a_j + 1$.

5.9. We put $A_8 = \{(j, i)\colon i \neq k,\ a_i = a_j - 1\}$, $q_5 = |A_8|$, $A_9 = \{(i, k)\colon i \neq j,\ a_i = a_k + 1\}$, $q_6 = |A_9|$, $A_{10} = \{(j, k)\colon a_j = a_k + 1\}$, $q_7 = |A_{10}|$ $(= 0, 1)$, $A_{11} = \{(j, k)\colon a_j = a_k + 2\}$, $q_8 = |A_{11}|$ $(= 0, 1)$, $A_{12} = \{(j, k)\colon a_j \geq a_k + 3\}$, $q_9 = |A_{12}|$ $(= 0, 1)$, $A_{13} = \{(k, j)\colon a_k \geq a_j + 1\}$, $q_{10} = |A_{13}|$ $(= 0, 1)$.

Now, $A_7 = A_8 \cup A_9 \cup A_{10} \cup A_{11} \cup A_{12} \cup A_{13}$ and this union is disjoint. Henceforth, $|A_7| = \sum_{i=5}^{10} q_i$.

Put $u_7 = \sum(a_{i_{19}} - a_{i_{20}})$. Since $A = A_6 \cup A_7$, we have $u(\alpha) = u_6(\alpha) + u_7(\alpha)$. Now, $u(\alpha) = q_5+q_6+q_7+2q_8+u_8(\alpha)+u_9(\alpha)$, where $u_8(\alpha) = a_j - a_k$ for $a_j \geq a_k+3$, $u_8(\alpha) = 0$ for $a_j \leq a_k + 2$, $u_9(\alpha) = a_k - a_j$ for $a_k \geq a + j + 1$ and $u_9(\alpha) = 0$ for $a_k \leq a_j$. Consequently, $u(\alpha) = u_6(\alpha) + q_5 + q_6 + q_7 + 2q_8 + u_8(\alpha) + u_9(\alpha)$.

5.10. Let $(i_{21}, i_{22}) \in B_7$. Then $b_{i_{21}} > b_{i_{22}}$ and just one of the following six cases takes place:

(a) $i_{21} = k,\ i_{22} \neq j, k,\ a_{i_{22}} = a_k$;
(b) $i_{21} \neq j, k,\ i_{22} = j,\ a_{i_{21}} = a_j$;
(c) $i_{21} = k,\ i_{22} = j,\ a_k = a_j - 1$;
(d) $i_{21} = k,\ i_{22} = j,\ a_k = a_j$;
(e) $i_{21} = k,\ i_{22} = j,\ a_k \geq a_j + 1$;
(f) $i_{21} = j,\ i_{22} = k,\ a_j \geq a_k + 3$.

5.11. We put $B_8 = \{(k, i)\colon i \neq j, k,\ a_i = a_k\}$, $q_{11} = |B_8|$, $B_9 = \{(i, j)\colon i \neq j, k,$ $a_i = a_j\}$, $q_{12} = |B_9|$, $B_{10} = \{(k, j)\colon a_j = a_k + 1\}$, $q_{13} = |B_{10}|$, $B_{11} = \{(k, j)\colon$ $a_j = a_k\}$, $q_{14} = |B_{11}|$, $B_{12} = \{(k, j)\colon a_k \geq a_j + 1\}$, $q_{15} = |B_{12}|$, $B_{13} =$ $\{(j, k)\colon a_j \geq a_k + 3\}$, $q_{16} = |B_{13}|$.

Now, $B_7 = B_8 \cup B_9 \cup B_{10} \cup B_{11} \cup B_{12} \cup B_{13}$ and this union is disjoint. Henceforth, $|B_7| = \sum_{i=11}^{16} q_i$.

Put $u_7(\beta) = \sum(b_{i_{21}} - b_{i_{22}})$. Since $B = B_6 \cup B_7$, we have $u(\beta) = u_6(\beta) + u_7(\beta)$. Now, $u_7(\beta) = q_{11} + q_{12} + q_{13} + 2q_{14} + u_8(\beta) + u_9(\beta)$, where $u_8(\beta) = a_k - a_j + 2$ for $a_k \geq a_j+1$, $u_8(\beta) = 0$ for $a_k \leq a_j$, $u_9(\beta) = a_j - a_k - 2$ for $a_j \geq a_k+3$ and $u_9(\beta) = 0$ for $a_j \leq a_k+2$. Consequently, $u(\beta) = u_6(\beta)+q_{11}+q_{12}+q_{13}+2q_{14}+u_8(\beta)+u_9(\beta)$.

5.12. We have $q_7 = q_{13}$, and so $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 + 2q_8 + u_8(\alpha) + u_9(\alpha) - q_2 - q_3 - q_{11} - q_{12} - 2q_{14} - u_8(\beta) - u_9(\beta)$.

5.13. If $a_j \geq a_k + 3$ then $q_7 = q_8 = q_{10} = u_9(\alpha) = q_{13} = q_{14} = q_{15} = u_8(\beta) = 0$, $q_9 = q_{16} = 1$, $u_8(\alpha) = a_j - a_k$ and $u_9(\beta) = a_j - a_k - 2$. If $a_j = a_k + 2$ then $q_7 = q_9 = q_{10} = u_8(\alpha) = u_9(\alpha) = q_{13} = q_{14} = q_{15} = q_{16} = u_8(\beta) = u_9(\beta) = 0$, $q_8 = 1$. In both these cases we get $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 - q_2 - q_3 - q_{11} - q_{12} + 2$.

5.14. If $a_j = a_k + 1$ then $q_7 = q_{13} = 1$, $q_8 = q_9 = q_{10} = u_8(\alpha) = u_9(\alpha) = q_{14} = q_{15} = q_{16} = u_8(\beta) = u_9(\beta) = 0$. In this case we get $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 - q_2 - q_3 - q_{11} - q_{12}$.

If $a_j = a_k$ then $q_7 = q_8 = q_9 = q_{10} = u_8(\alpha) = u_9(\alpha) = q_{13} = q_{14} = q_{15} = q_{16} = u_8(\beta) = u_9(\beta) = 0$, $q_{14} = 1$. In this case we get $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 - q_2 - q_3 - q_{11} - q_{12} - 2$.

5.15. In the sequel, let $a_j = \max(\alpha)$ and $a_k = \min(\alpha) = 0$. If $a_j = 0$ then $a_i = 0$ for every i and we have $u(\alpha) = 0 < 2n - 2 = u(\beta)$, $u(\alpha) - u(\beta) = 2 - 2n \leq -2$. If $a_j = 1$ then $u(\alpha) = s(\alpha)z(\alpha) = u(\beta)$, and so $u(\alpha) - u(\beta) = 0$.

5.16. Let $a_j = 2$. It follows from 5.13 that $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 - q_2 - q_3 - q_{11} - q_{12} + 2$. Now, $q_1 = z(\alpha) - 1$, $q_2 = 0$, $q_3 = 0$, $q_4 = z(\alpha, \max) - 1 = z(\alpha, 2) - 1$, $q_5 = z(\alpha, 1)$, $q_6 = z(\alpha, 1)$, $q_{11} = z(\alpha) - 1$, $q_{12} = z(\alpha, \max) - 1 = z(\alpha, 2) - 1$. Altogether, we arrive at $u(\alpha) - u(\beta) = z(\alpha) - 1 + z(\alpha, 2) - 1 + z(\alpha, 1) + z(\alpha, 1) - z(\alpha) + 1 - z(\alpha, 2) + 1 + 2 = 2z(\alpha, 1) + 2 \geq 2$. Of course, $u(\alpha) - u(\beta) = 2$ if and only if $z(\alpha, 1) = 0$. That is, $a_i \in \{0, 2\}$ for every $i \in \{1, \dots, n\}$.

5.17. And now, let $a_j \geq 3$. Again, $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 - q_2 - q_3 - q_{11} - q_{12} + 2$, $q_2 = 0$, $q_3 = 0$, so that $u(\alpha) - u(\beta) = q_1 + q_4 + q_5 + q_6 - q_{11} - q_{12} + 2$. Furthermore, q_1 is the number of indices i such that $i \neq k$ and $a_j = \max(\alpha) \geq a_i + 2$, q_4 is the number of indices i such that $i \neq j$ and $a_i \geq 2$, $q_5 = z(\alpha, a_j - 1)$, $q_6 = z(\alpha, 1)$, $q_{11} = z(\alpha) - 1$, $q_{12} = z(\alpha, \max) - 1 = z(\alpha, a_j) - 1$. Of course, $q_4 = n - q_6 - z(\alpha) - 1$, and so $u(\alpha) - u(\beta) = q_1 + q_5 - q_{12} + n$, where $q_1 \geq z(\alpha, 1) + z(\alpha) - 1$, $q_5 \geq 0$, $q_{12} \geq n - 2$. Thus $u(\alpha) - u(\beta) \geq z(\alpha, 1) + z(\alpha) + 1 \geq 2$. The equality $u(\alpha) - u(\beta) = 2$ is achieved if and only if $a_i = a_j = \max(\alpha)$ for every $i = 1, \dots, n$, $i \neq k$ ($(a_k = 0 = \min(\alpha))$. In this case, $s(\alpha) = (n-1)a_j \geq 3n - 3$.

Lemma 5.18. *If $a_j = \max(\alpha) \geq 2$ and $a_k = \min(\alpha) = 0$ then $u(\alpha) > u(\beta)$.*

PROOF: See 5.16 and 5.17. □

6. Inequalities

Throughout this section, let $n \geq 1$, $a_1, \dots, a_n$ be real numbers and let z denote the number of indices i such that $a_i = 0$.

Proposition 6.1. *Let $|a_j| \geq 2$ whenever $1 \leq j \leq n$ and $a_j \neq 0, \pm 1$. Then:*

(i) $\sum_{i=1}^n a_i^2 \geq 2z - 2n + 3\sum_{i=1}^n |a_i| \geq 2z - 2n + 3\sum_{i=1}^n a_i$.

(ii) $\sum_{i=1}^n a_i^2 = 2z - 2n + 3\sum_{i=1}^n |a_i|$ *if and only if* $a_i \in \{0, \pm 1, \pm 2\}$ *for every* $i = 1, \dots, n$.

(iii) $\sum_{i=1}^n a_i^2 = 2z - 2n + 3\sum_{i=1}^n a_i$ *if and only if* $a_i \in \{0, 1, 2\}$ *for every* $i = 1, \dots, n$.

PROOF: All the assertions follow easily from Lemma 1.16. □

Proposition 6.2. *Let $|a_j| \geq 2$ whenever $1 \leq j \leq n$ and $a_j \neq 0, \pm 1$, and let $\sum_{i=1}^n |a_i| \geq n$. Then:*

(i) $\sum_{i=1}^n a_i^2 \geq 2z + \sum_{i=1}^n |a_i| \geq 2z + \sum_{i=1}^n a_i$.

(ii) $\sum_{i=1}^n a_i^2 = 2z+\sum_{i=1}^n |a_i|$ *if and only if* $\sum_{i=1}^n |a_i| = n$ *and* $a_i \in \{0,\pm1,\pm2\}$ *for every* $i=1,\dots,n$.

(iii) $\sum_{i=1}^n a_i^2 = 2z + \sum_{i=1}^n a_i$ *if and only if* $\sum_{i=1}^n a_i = n$ *and* $a_i \in \{0,1,2\}$ *for every* $i=1,\dots,n$.

Proof: (i) This follows easily from Proposition 6.1 (i).

(ii) First, assume that $\sum_{i=1}^n a_i^2 = 2z + \sum_{i=1}^n |a_i|$. According to Proposition 6.1 (i), we have $2n + \sum_{i=1}^n |a_i| \geq 3\sum_{i=1}^n |a_i| \geq 3n$, and hence $\sum_{i=1}^n |a_i| = n$ and $\sum_{i=1}^n a_i^2 = 2z - 2n = 3\sum_{i=1}^n |a_i|$. Now, it follows from Proposition 6.1 (ii) that $a_i = 0,\pm1,\pm2$.

Conversely, if $\sum_{i=1}^n |a_i| = n$ and $a_i = 0,\pm1,\pm2$ then $\sum_{i=1}^n a_i^2 = 2z+\sum_{i=1}^n |a_i|$ by Proposition 6.1 (ii).

(iii) This follows from (i) and (ii). □

Proposition 6.3. *Let* $\sum_{i=1}^n |a_i| \geq n$. *Then:*

(i) $\sum_{i=1}^n a_i^2 \geq z + \sum_{i=1}^n |a_i| \geq \sum_{i=1}^n a_i$.

(ii) $\sum_{i=1}^n a_i^2 \geq z+n$.

(iii) $\sum_{i=1}^n a_i^2 = z + \sum_{i=1}^n |a_i|$ *(or* $= z+n$*) if and only if* $a_i \in \{1,-1\}$ *for every* n.

(iv) $\sum_{i=1}^n a_i^2 = z + \sum_{i=1}^n a_i$ *if and only if* $a_1 = \cdots = a_n = 1$.

Proof: Easy, see Remark 1.14. □

Remark 6.4.

(i) If $\sum_{i=1}^n a_i \geq n+z$ then $\sum_{i=1}^n a_i^2 \geq \sum_{i=1}^n a_i + 2z \geq n+3z$ by Lemma 1.15.

(ii) If $\sum_{i=1}^n |a_i| \geq n+z$ then $\sum_{i=1}^n a_i^2 \geq \sum_{i=1}^n |a_i| + 2z \geq n+3z$ (it follows from (i) and Lemma 1.2 (iv)).

Example 6.5. Let $n \geq 3$, $a_1 = \cdots = a_{n-1} = \frac{n}{n-1}$ and $a_n = 0$. We have $\sum_{i=1}^n a_i^2 = \frac{n^2}{n-1}$, $\sum_{i=1}^n a_i = n$, $z=1$, $2z-2n+3\sum_{i=1}^n a_i = n+2$ and $\sum_{i=1}^n a_i^2 - 2z + 2n - 3\sum_{i=1}^n a_i = \frac{2-n}{n-1} \leq -\frac{1}{2}$. Besides, $\sum_{i=1}^n a_i^2 - 2z - \sum_{i=1}^n a_i = \frac{2-n}{n-1}$ and $\sum_{i=1}^n a_i^2 - z - \sum_{i=1}^n a_i = \frac{1}{n-1}$.

Example 6.6. Let $n \geq 2$, $a_1 = \cdots = a_n = \frac{n-1}{n}$. Then $\sum_{i=1}^n a_i^2 = \frac{(n-1)^2}{n}$, $\sum_{i=1}^n a_i = n-1$, $z=0$, $\sum_{i=1}^n a_i^2 - 2z + 2n - 3\sum_{i=1}^n a_i = \frac{n+1}{n}$, $\sum_{i=1}^n a_i^2 - 2z - \sum_{i=1}^n a_i = \frac{1-n}{n} \leq -\frac{1}{2}$, $\sum_{i=1}^n a_i^2 - z - \sum_{i=1}^n a_i = \frac{1-n}{n}$.

Proposition 6.7. *Let* $a_1,\dots,a_n$ *be integers such that* $\sum_{i=1}^n |a_i| \leq n$. *Then:*

(i) $\sum_{i=1}^n a_i^2 \leq z + n^2 - n + 1$ *(*$\leq n^2$ *if* $a_j \neq 0$ *for at least one* j*).*

(ii) $\sum_{i=1}^n a_i^2 = z + n^2 - n + 1$ *if and only if there is* $k \in \{1,\dots,n\}$ *such that* $a_k \in \{n,-n\}$ *and* $a_i = 0$ *for* $i \neq k$ *(then* $z = n-1$*).*

Proof: Put $\alpha = (a_1, \dots, a_n)$. Without loss of generality we can assume that $n \geq 2$ and all numbers $a_1, \dots, a_n$ are nonnegative. If $z(\alpha) = n$ then $a_1 = \dots = a_n = 0 = \sum_{i=1}^n a_i^2$ and both (i) and (ii) are true. If $z(\alpha) = n-1$ then $a_k \neq 0$ just for one index k, $|a_k| \leq n$ and $\sum_{i=1}^n a_i^2 = a_k^2 \leq n^2 = n-1+n^2-n+1 = z+n^2-n+1$. Again, (i) and (ii) are true. Assume, therefore, that $z(\alpha) \leq n-2$. First, observe that $r(\alpha) < ns(\alpha) \leq n^2$ and we will proceed by induction on $n^2 - r(\alpha)$. Since $z(\alpha) \leq n-2$, we can find indices j and k such that $j \neq k$ and $1 \leq a_j \leq a_k$. Now, consider the n-tuple β treated in the second section of the paper. By Lemma 2.8, $r(\alpha) - r(\beta) = 2(a_j - a_k - 1) \leq -2$, $n^2 - r(\beta) \leq -2 + n^2 - r(\alpha) < n^2 - r(\alpha)$. Now, either due to the first part of the proof or due to the induction hypothesis, we get $r(\beta) \leq z(\beta) + n^2 - n + 1$. Consequently, $r(\alpha) \leq r(\beta) - 2 \leq z(\beta) + n^2 - n - 1$. If $a_j \geq 2$ then $z(\alpha) = z(\beta)$, and hence $r(\alpha) \leq z(\alpha) + n^2 - n - 1 < z(\alpha) + n^2 - n + 1$. If $a_j = 1$ then $z(\beta) = z(\alpha) + 1$ and $r(\alpha) \leq z(\alpha) + n^2 - n < z(\alpha) + n^2 - n + 1$. $\square$

Example 6.8. Let $n = 2$, $a_1 = \frac{19}{10}$, $a_2 = \frac{1}{10}$. Then $a_1 + a_2 = 2$, $z = 0$ and $a_1^2 + a_2^2 = \frac{181}{50} > \frac{150}{50} = z + n^2 - n + 1$.

Proposition 6.9. *Let $a_1, \dots, a_n$ be integers such that $\sum_{i=1}^n |a_i| = n$. Then:*

(i) $n^2 \geq n^2 - n + 1 + z \geq \sum_{i=1}^n a_i^2 \geq n + 2z \geq n$.

(ii) $\sum_{i=1}^n a_i^2 = n$ *if and only if* $a_i \in \{1, -1\}$ *for every* $i = 1, \dots, n$.

(iii) $\sum_{i=1}^n a_i^2 = n + 2z$ *if and only if* $a_i \in \{0, \pm 1, \pm 2\}$ *for every* $i = 1, \dots, n$.

(iv) $\sum_{i=1}^n a_i^2 = n^2$ *if and only if* $\sum_{i=1}^n a_i^2 = n^2 - n + 1 + z$ *and this is equivalent to the fact that there is* $k \in \{1, \dots, n\}$ *such that* $a_k \in \{n, -n\}$ *and* $a_i = 0$ *for* $i \neq k$.

Proof: Combine Proposition 6.2 (i), (ii) and Proposition 6.7 (i), (ii). $\square$

Remark 6.10. Let $a_1, \dots, a_n$ be integers and let $m = \sum_{i=1}^n |a_i| < n$. Clearly, $a_j = 0$ for at least one $j \in \{1, \dots, n\}$. Put $\alpha = (a_1, \dots, a_n)$ and define $\beta = (b_1, \dots, b_n)$ by $b_i = a_i$ for $i \neq j$ and $b_j = n - m$. Clearly, $s^+(\beta) = \sum_{i=1}^n |b_i| = n$ and $z(\beta) = z(\alpha) - 1$. Now, by Proposition 6.1 (i), we have $r(\alpha) + n^2 - 2nm + m^2 = r(\beta) \geq s^+(\beta) + 2z(\beta) = n + 2z(\alpha) - 2$. Consequently, $\sum_{i=1}^n a_i^2 \geq \left(2n - \sum_{i=1}^n |a_i|\right) \times \left(\sum_{i=1}^n |a_i|\right) + n - n^2 - 2 + 2z(\alpha)$.

For example, if $m = n - 1$ then we get $\sum_{i=1}^n a_i^2 \geq n - 3 + 2z(\alpha)$.

Remark 6.11. There are other ways of proving Proposition 6.1.

(i) Let $n \geq 2$, $\alpha = (a_1, \dots, a_n)$, where all the numbers a_i are nonnegative and let $s(\alpha) = n$. If $z(\alpha) = 0$ then $a_1 = \dots = a_n = 1$ and $t(\alpha) - z(\alpha) = 0$. If $z(\alpha) \geq 1$ then $\max(\alpha) \geq 2$ and $\min(\alpha) = 0$. Choose j and k such that $a_j = \max(\alpha)$ and $a_k = \min(\alpha) = 0$, and consider the n-tuple β from the second section. Clearly, $z(\beta) = z(\alpha) - 1$. Consequently, by Lemma 2.9, $t(\alpha) - t(\beta) = 2a_j - 3$ and $(t(\alpha) - z(\alpha)) - (t(\beta) - z(\beta)) = 2a_j - 4$. If

$a_j \geq 3$ then $t(\alpha) - z(\alpha) > t(\beta) - z(\beta)$. On the other hand, if $a_j = 2$ then $a_i \in \{0, 1, 2\}$ for every i and we have $t(\alpha) - z(\alpha) = r(\alpha) - s(\alpha) - 2z(\alpha) = 4z(\alpha, 2) + z(\alpha, 1) - 2z(\alpha, 2) - z(\alpha, 1) - 2z(\alpha) = 2z(\alpha, 2) - 2z(\alpha) = 0$, since $z(\alpha, 2) + z(\alpha, 1) + z(\alpha) = n = s(\alpha) = 2z(\alpha, 2) + z(\alpha, 1)$.

(ii) Taking into account (i), we can proceed by induction on $z(\alpha)$ to show Proposition 6.1 (for $s(\alpha) = n$). We can also proceed by induction on $\max(\alpha) + z(\alpha, \max(\alpha))$, see Lemma 2.17.

(iii) Assume that $a_1 \geq 1$, $a_n = 0$, and put $\beta = (a_1 - 1, a_2, \ldots, a_{n-1})$. Then $s(\beta) = s(\alpha) - 1 = n - 1$ and $t(\alpha) - z(\alpha) \geq t(\beta) - s(\beta)$, see Lemma 4.5. We see that we can proceed by induction on n.

Remark 6.12. Let $a_1, \ldots, a_n$ be integers such that $m = \sum_{i=1}^n |a_i| \geq n$. Now, put $\alpha = (a_1, \ldots, a_n)$ and consider the m-tuple $\beta = (\alpha, 0, \ldots, 0)$. We have $s^+(\beta) = m$, $z(\beta) = z(\alpha) + m - n$. Now, it follows from Proposition 6.9 that $3m - 2n + 2z(\alpha) \leq \sum_{i=1}^n a_i^2 \leq m^2 - n + 1 + z(\alpha)$.

For instance, if $m = n + 1$ then we obtain $n + 3 + 2z(\alpha) \leq \sum_{i=1}^n a_i^2 \leq n^2 + n + 2 + z(\alpha)$.

Remark 6.13. Let $a_1, \ldots, a_n$ be integers such that $m = \sum_{i=1}^n |a_i| < n$. Now, put $\alpha = (a_1, \ldots, a_n)$ and consider the m-tuple β used in Remark 6.10. We have $s^+(\beta) = n$, $z(\beta) = z(\alpha) - 1$. Now, it follows from Proposition 6.9 that $n + 2z(\beta) \leq \sum_{i=1}^n b_i^2 \leq n^2 - n + 1 + z(\beta)$. We have $\sum_{i=1}^n b_i^2 = \sum_{i=1}^n a_i^2 + (n - m)^2$. Thus $n(2m + 1) - n^2 - m^2 - 2 + 2z(\alpha) \leq \sum_{i=1}^n a_i^2 \leq n(2m - 1) - m^2 + z(\alpha)$.

For instance, if $m = n - 1$ then we get $n - 3 + 2z(\alpha) \leq \sum_{i=1}^n a_i^2 \leq n^2 - n + 1 + z(\alpha)$.

Observation 6.14.

(i) Of course, we can proceed also in the following way: $\sum_{i=1}^n a_i^2 - 3\sum_{i=1}^n a_i + 2n - 2z \geq \sum_{i=1}^n (|a_i| - 1)^2 - \sum_{i=1}^n |a_i| + n - 2z = \sum_{i=1, a_i \neq 0}^n (|a_i| - 1)^2 - \sum_{i=1, a_i \neq 0}^n |a_i| + n - z = \sum_{i=1, a_i \neq 0}((|a_i| - 1)^2 - (|a_i| - 1))$.

(ii) Due to (i), we have $\sum_{i=1}^n a_i^2 - 3\sum_{i=1}^n a_i + 2n - 2z \geq 0$, provided that $a_i \in (-\infty, -2\rangle \cup \langle -1, 1\rangle \cup \langle 2, \infty)$ for every i.

(iii) Another (slightly different) way: $\sum_{i=1}^n a_i^2 - 3\sum_{i=1}^n a_i + 2n - 2z = \sum_{i=1, a_i \neq 0}^n a_i^2 - 3\sum_{i=1, a_i \neq 0}^n a_i + 2(n - z) = \sum_{i=1, a_i \neq 0}^n (a_i^2 - 3a_i + 2)$.

(iv) Due to (iii), we have $\sum_{i=1}^n a_i^2 - 3\sum_{i=1}^n a_i + 2n - 2z \geq 0$, provided that $a_i \in (-\infty, 1\rangle \cup \langle 2, \infty)$ for every i.

Example 6.15. Let $n = 2$ and $a_1 = \frac{3}{2}$. Then $a_1^2 + a_2^2 - (a_1 + a_2) + 4 - 2z \geq 0$ if and only if $a_2 \neq 0$ (so that $z = 0$) and $a_2 \in (-\infty, (3 - \sqrt{2})/2\rangle \cup \langle (3 + \sqrt{2})/2, \infty)$ (so that either $a_2 < 1$ or $a_2 > 2$).

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