Normality, nuclear squares and Osborn identities

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Abstract. Let Q be a loop. If $S \leq Q$ is such that $\varphi(S) \subseteq S$ for each standard generator of $\operatorname{Inn} Q$, then S does not have to be a normal subloop. In an LC loop the left and middle nucleus coincide and form a normal subloop. The identities of Osborn loops are obtained by applying the idea of nuclear identification, and various connections of Osborn loops to Moufang and CC loops are discussed. Every Osborn loop possesses a normal nucleus, and this nucleus coincides with the left, the right and the middle nucleus. Loops that are both Buchsteiner and Osborn are characterized as loops in which each square is in the nucleus.

Keywords: loop; normal subloop; LC loop; Buchsteiner loop; Osborn loop; nuclear identification

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Bruck's paper [11] can be regarded as the beginning of the systematic theory of loops. A relatively long development notwithstanding, there still arise various lacunae in the foundations. The intent of this paper is to fill some of these. Standard references for loop theory are [10], [12], [29] and essentially all uncited claims can be found therein.

1. Normality and inner mappings

Let S be a subloop of a loop Q. It is well known, see [29], that S is normal if and only if $\varphi(S) = S$ for all $\varphi \in \operatorname{Inn} Q$, where $\operatorname{Inn} Q$ denotes the group of inner mappings. Since $\operatorname{Inn} Q$ is closed for inverses, the condition $\varphi(S) = S$ may be replaced by $\varphi(S) \subseteq S$. The *standard generators* of $\operatorname{Inn} Q$ are $L_{xy}^{-1} L_x L_y$, $R_{yx}^{-1} R_x R_y$ and $L_x^{-1} R_x$, where $L_x \colon y \mapsto xy$ is the *left translation* of the element $x \in Q$, and $R_x \colon y \mapsto yx$ is the *right translation*. Recall that $\operatorname{Inn} Q$ is defined as $\{\varphi \in \operatorname{Mlt} Q \colon \varphi(1) = 1\}$, where $\operatorname{Mlt} Q = \langle L_x, R_x \colon x \in Q\rangle$. The question addressed in this section is whether for S to be normal it suffices to assume that $\varphi(S) \subseteq S$ holds for all standard generators φ.

It does not seem to be really surprising that the answer is negative. Nevertheless, some effort seems to be needed to obtain an example that is easy to describe.

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The example presented below was obtained while investigating the structure of multiplicative equivalences, see [16].

An equivalence "$\sim$" upon the loop Q is said to be *multiplicative* if

$$x \sim y \ \text{ and } \ u \sim v \implies xu \sim yv \tag{1.1}$$

holds for all $x, y, u, v \in Q$. If "$\sim$" is multiplicative then $[x]_\sim \cdot [y]_\sim = [xy]_\sim$ is a well defined operation upon $Q/\sim$ and the set $S = [1]_\sim$ is closed under multiplication. However, S is not necessarily closed under (left or right) division.

Proposition 1.1. *Let "$\sim$" be a multiplicative equivalence upon a loop Q. Put $S = [1]_Q$ and assume that $xS = Sx = [x]_\sim$ for all $x \in Q$. Then $L_x^{-1}R_x(S) = S$ for all $x \in Q$, and if φ is equal to $L_{xy}^{-1}L_xL_y$ or $R_{yx}^{-1}R_xR_y$, then $\varphi(S) \subseteq S$ for all $x, y \in Q$. Furthermore, S is a subloop of Q.*

Proof: First note that $L_x^{-1}R_x(S) = S$ is the same as $xS = Sx$. Furthermore, $L_xL_y(S) = x[y]_\sim \subseteq [x]_\sim[y]_\sim \subseteq [xy]_\sim = L_{xy}(S)$ for all $x, y \in Q$. To see that S has to be a subloop, consider $s, t \in S$. Since $sS = S = tS$, there has to be $L_t^{-1}L_s(S) = S$, and thus $t\backslash s \in S$. $\square$

Corollary 1.2. *Let "$\sim$" be a multiplicative equivalence upon a loop Q that is not a loop congruence. Put $S = [1]_\sim$, and suppose that $xS = Sx = [x]_\sim$ for all $x \in Q$. Then there exist $x, y \in Q$ such that $L_{xy}^{-1}L_xL_y(S)$ or $R_{yx}^{-1}R_xR_y(S)$ is a proper subset of S.*

Proof: By Proposition 1.1 the latter two sets have to be subsets of S. If both of them are equal to S for all $x, y \in Q$, then $\varphi(S) = S$ for every standard generator φ of $\operatorname{Inn} Q$. In such a case $S \trianglelefteq Q$ and "$\sim$" is a loop congruence. $\square$

Consider now a construction of loops that generalizes an idea of G. E. Bates and F. Kiokemeister in [9]. The ingredients for the construction are a binary operation "$\cdot$" upon a set A, a loop Q and injective mappings $f_{a,b}\colon Q \to Q$ for $a, b \in A$. The underlying set is equal to $A \times Q$.

The following statement coincides with Theorem 4.1 of [16].

Proposition 1.3. *Let "$\cdot$" be a binary operation upon a set A and let 1 be the neutral element of this operation. Suppose that mappings $x \mapsto ax$ and $x \mapsto xa$ are surjective upon A for all $a \in A$. Set*

$$[c/b] = \{a \in A\colon ab = c\} \quad \text{and} \quad [a\backslash c] = \{b \in A\colon ab = c\} \tag{1.2}$$

for all $a, b, c \in A$. For all $a, b \in A$ let $f_{a,b}$ be an injective mapping $Q \to Q$, where Q is a loop. Suppose that $f_{a,b}$ is the identity mapping whenever $1 \in \{a, b\}$. The *element* $(1,1)$ *is then a neutral element of the binary operation that is defined upon* $A \times Q$ *by*

$$(a,x)\cdot(b,y) = (ab, f_{a,b}(xy)). \tag{1.3}$$

This operation yields a loop if and only if

(1) *for all* $c,b \in A$ *the sets* $\mathrm{Im}(f_{a,b})$, $a \in [c/b]$, *are pairwise distinct and partition* Q*; and*
(2) *for all* $a,c \in A$ *the sets* $\mathrm{Im}(f_{a,b})$, $b \in [a\backslash c]$, *are pairwise distinct and partition* Q.

In the next two lemmas assume that M is the loop upon $A \times Q$ that has been constructed by means of Proposition 1.3. Define an equivalence "$\sim$" upon M by

$$(a,x) \sim (b,y) \iff a = b. \tag{1.4}$$

Lemma 1.4. *If* $(a,x) \in M$, *then* $[(a,x)]_\sim = \{a\} \times Q = (a,x)(1 \times Q) = (1 \times Q)(a,x)$, *and* $1 \times Q = [(1,1)]_\sim$ *is a subloop of* M.

Proof: By (1.3), $(a,x)(1,y) = (a, f_{a,1}(xy)) = (a, xy)$. Each element of Q can be expressed as xy for some $y \in Q$. □

Lemma 1.5. *The equivalence "$\sim$" is a multiplicative equivalence of* M. *It is a congruence of* Q *if and only if* A *is a loop.*

Proof: The projection $M \to A$, $(a,x) \mapsto a$, is compatible with multiplication. Hence "$\sim$" is a multiplicative equivalence. This equivalence is a congruence of M if and only if A is a loop since the multiplication determines the divisions uniquely. □

In the following result, $[x]$ denotes the integer part of the real number x.

Theorem 1.6. *For a prime* p *define an operation* $\oplus = \oplus_p$ *upon* $\mathbb{Z}$ *by*

$$a \oplus b = \begin{cases} p\Big(\Big[\frac{a}{p^2}\Big] + \Big[\frac{b}{p^2}\Big]\Big) & \text{if } p \mid a+b \text{ and } p \nmid a; \\ a+b & \text{in every other case.}\end{cases} \tag{1.5}$$

Let an infinite loop Q *be partitioned to subsets* Q_i *of the same cardinality,* $i \in \mathbb{Z}_p$. *Thus* $Q = \bigcup Q_i$, *and* $Q_i \cap Q_j = \emptyset$ *if* $i,j \in \mathbb{Z}_p$ *and* $i \neq j$. *For each* $i \in \mathbb{Z}_p$ *choose a bijection* $\pi_i \colon Q \to Q_i$, *and set* $\pi_a = \pi_i$ *whenever* $a \equiv i \bmod p$.
Define an operation "$\cdot$" upon $\mathbb{Z} \times Q$ *by*

$$(a,x)(b,y) = \begin{cases} (a \oplus b, \pi_{(a+b-p)/p}(xy)) & \text{if } p \mid a+b \text{ and } p \nmid a; \\ (a \oplus b, xy) = (a+b, xy) & \text{in every other case.}\end{cases} \tag{1.6}$$

Then $M = (A \times Q, \cdot)$ is a loop with unit $(0,1)$ in which $(a,x) \sim (b,y) \implies a = b$ defines a multiplicative equivalence such that $S = [(0,1)]_\sim = \{0\} \times Q$ is subloop of M that is not a normal subloop of M. If φ is a standard generator of $\operatorname{Inn} M$, then $\varphi(S) \subseteq S$, and there exists a standard generator φ such that $\varphi(S) \neq S$. If Q is commutative, then M is commutative too.

Proof: Let us first verify that the construction of M conforms with Proposition 1.3. Consider $a, b \in \mathbb{Z}$. If $p \nmid a{+}b$ or if $p \mid a, b$ set $f_{a,b} = \mathrm{id}_Q$. In the other cases put $f_{a,b} = \pi_{(a+b-p)/p}$.

Fix $a, c \in \mathbb{Z}$ and consider, with respect to "$\oplus$", the set $[a\backslash c]$ as defined by (1.2). If $p \nmid c$ and $a \oplus b = c$, then $c = a{+}b$ and $[a\backslash c] = \{c{-}a\}$. The same is true if $p \mid a, c$. In these cases $f_{a,b} = \mathrm{id}_Q$. This means that (1.3) and (1.6) yield the same result and that condition (2) of Proposition 1.3 gets satisfied.

Suppose now that $p \mid c$, $c = a \oplus b$ and $p \nmid a$. Express a as $a_2p^2 + a_1p + a_0$ and b as $b_2p^2 + b_1p + b_0$, where $\{a_0, b_0, a_1, b_1\} \subseteq \{0, \dots, p{-}1\}$. Then $c = p(a_2{+}b_2)$ and $p = a_0 + b_0$. The set $[a\backslash c]$ is hence equal to $\{cp - a_2p^2 + xp + p - a_0 \colon 0 \le x < p\}$. That can be also expressed as $\{cp - a + p(a_1{+}x{+}1) \colon 0 \le x < p\}$.

The fact that $[a\backslash c]$ is always nonempty means that $u \mapsto a \oplus u$ is a surjective mapping $\mathbb{Z} \to \mathbb{Z}$ for any $a \in \mathbb{Z}$. Assume again that $p \mid c$, $p \nmid a$ and $b \in [a\backslash c]$. The definition of $(a,x)(b,y)$ from (1.5) agrees with that of (1.3), by the choice of $f_{a,b}$. What remains is to verify condition (2) of Proposition 1.3. If $b = cp - a + p(a_1{+}x{+}1)$, then $(a + b - p)/p = c + a_1 + x$. If x runs through $\mathbb{Z}_p$, then $c + a_1 + x$ runs through $\mathbb{Z}_p$ as well. Hence (2) holds.

Since mirror arguments are also true, $(M, \cdot)$ is a loop, by Proposition 1.3. Lemmas 1.4 and 1.5 imply that Corollary 1.2 can be used to prove the rest. ☐

2. Normal nuclei and LC loops

Left central loops (or *LC loops*) were introduced by F. Fenyves in [19]. There are several ways how they can be described, see [19], [30]. A unified treatment appears in Theorem 2.2. While Theorem 2.2 and Corollary 2.3 build upon the existing concepts [19], [30], [31], they contain several characterizations that seem to be new.

In a loop Q, we will denote the left and right inverses of the element x by $x^\lambda := 1/x$ and $x^\varrho := x\backslash 1$, respectively.

Let Q be a loop. Then $N_\lambda = \{a \in Q \colon\ a \cdot xy = ax \cdot y \text{ for all } x, y \in Q\}$ is known as the *left nucleus* of Q. Equations $x \cdot ay = xa \cdot y$ and $xy \cdot a = x \cdot ya$ yield the *middle* and the *right* nucleus, respectively. The main new result of this section is that in every LC loop the left nucleus is a normal subloop (Theorem 2.11).

The proof of this fact starts with Proposition 2.6 and does not depend upon the preceding statements.

In each LC loop $N_\lambda = N_\mu$. This is because every LC loop Q is a *left inverse property* (LIP) loop, i.e. it satisfies $x^\lambda \cdot xy = y$ for all $x, y \in Q$. An equivalent characterization of an LIP loop is that for each $x \in Q$ there exists $y \in Q$ such that $L_x^{-1} = L_y$, where $L_x\colon a \mapsto xa$ is the *left translation* of the element $x \in Q$. Similarly, Q is a RIP loop if the inverse of each right translation $R_x\colon a \mapsto ax$ is also a right translation. Loops that are both LIP and RIP are called *inverse property* (IP) loops.

A loop Q is said to be a *left alternative property* (LAP) loop if $x \cdot xy = xx \cdot y$ for all $x, y \in Q$. Clearly, Q satisfies the LAP if and only if L_x^2 is a left translation for all $x \in Q$. RAP loops are defined in a mirror way, and the intersection of LAP and RAP loops is the variety of *alternative property* (AP) loops.

Elements of nuclei can be characterized by means of autotopisms. Let Q be a loop. Denote by S_Q the symmetric group upon Q. A triple $(\alpha, \beta, \gamma) \in S_Q^3$ is an *autotopism* if $\alpha(x)\beta(y) = \gamma(xy)$ for all $x, y \in Q$. Autotopisms of Q form a group that is denoted by $\mathrm{Atp}(Q)$. For $x \in Q$ put $\lambda_x = (L_x, \mathrm{id}_Q, L_x)$, $\varrho_x = (\mathrm{id}_Q, R_x, R_x)$ and $\mu_x = (R_x^{-1}, L_x, \mathrm{id}_Q)$. The following facts are well known and easy to prove:

Lemma 2.1. *Let a be an element of a loop Q. Then $\lambda_a \in \mathrm{Atp}(Q) \iff a \in N_\lambda$, $\varrho_a \in \mathrm{Atp}(Q) \iff a \in N_\varrho$ and $\mu_a \in \mathrm{Atp}(Q) \iff a \in N_\mu$. Furthermore, consider $\sigma = (\alpha, \beta, \gamma) \in \mathrm{Atp}(Q)$:*

(i) *$\alpha = \mathrm{id}_Q \implies \sigma = \varrho_a$ for some $a \in N_\varrho$;*
(ii) *$\beta = \mathrm{id}_Q \implies \sigma = \lambda_a$ for some $a \in N_\lambda$; and*
(iii) *$\gamma = \mathrm{id}_Q \implies \sigma = \mu_a$ for some $a \in N_\mu$.*

Let Q be an LIP loop. Then $x^\lambda = x^\varrho$ for each $x \in Q$. Put $I(x) = x^{-1} = 1/x$. Then I permutes Q and $(\alpha, \beta, \gamma) \in \mathrm{Atp}(Q) \iff (I\alpha I, \gamma, \beta) \in \mathrm{Atp}(Q)$. Lemma 2.1 can be used to see that $a \in N_\lambda \iff (IL_aI, L_a, \mathrm{id}_Q) \iff a \in N_\mu$. Thus $N_\lambda = N_\mu$ in every LIP loop Q. Similarly $N_\varrho = N_\mu$ in an RIP loop. Each IP loop thus contains a *nucleus* $N_\lambda = N_\mu = N_\varrho$. An element belonging to the nucleus is said to be *nuclear*. Note that in a general setting the nucleus is defined as $N_\lambda \cap N_\mu \cap N_\varrho$.

Theorem 2.2. *Let Q be a loop. The following conditions are equivalent:*

(1) *$x(x \cdot yz) = (x \cdot xy)z$ for all $x, y, z \in Q$;*
(2) *$xx \cdot yz = (x \cdot xy)z$ for all $x, y, z \in Q$;*
(3) *$(xx \cdot y)z = x(x \cdot yz)$ for all $x, y, z \in Q$;*
(4) *$y(x \cdot xz) = (y \cdot xx)z$ for all $x, y, z, \in Q$;*
(5) *Q is an LAP loop such that $x^2 \in N_\lambda$ for each $x \in Q$;*
(6) *Q is an LAP loop such that $x^2 \in N_\mu$ for each $x \in Q$;*

(7) *Q is an LIP loop such that $x^2 \in N_\lambda$ for each $x \in Q$;*
(8) *$\lambda_x^2 = (L_x^2, \mathrm{id}_Q, L_x^2) \in \mathrm{Atp}(Q)$ for each $x \in Q$;*
(9) *$L_x^2 L_y$ is a left translation of Q for all $x, y \in Q$; and*
(10) *$L_y L_x^2$ is a left translation of Q for all $x, y \in Q$.*

Proof: First note that (8) is an equivalent expression of (1). The same is true for (9). Indeed, if $x(x \cdot yz) = wz$ for all $z \in Q$, then the substitution $z = 1$ implies that $w = x \cdot xy$. Similarly, (4) $\iff$ (10). Furthermore, note that setting $y = 1$ or $z = 1$ gives the LAP for all of the identities (1)–(4). Using LAP it is clear that (1) $\iff$ (2) and (1) $\iff$ (3). The equivalence (2) $\iff$ (5) is immediate as well. Now, setting $y = x^\varrho = x\backslash 1$ in (1) yields $x(x \cdot x^\varrho z) = xz$. Thus $x \cdot x^\varrho z = z$ for all $x, z \in Q$. Hence (1) $\Rightarrow$ (7). Since $N_\lambda = N_\mu$ in every LIP loop, the implication (7) $\Rightarrow$ (5) follows from $x \cdot xy = x(x^{-1}x^2y) = x^2y$. We have shown the equivalence of (1), (2), (3), (5), (7), (8) and (9). The next step is to prove (4) $\iff$ (6). An equivalent form of (4) is $(y/x^2)(x \cdot xz) = yz$. This can be expressed as $(R_{x^2}^{-1}, L_x^2, \mathrm{id}_Q) \in \mathrm{Atp}(Q)$ for all $x \in Q$. In an LAP loop $L_x^2 = L_{x^2}$. Thus (4) $\iff$ (6), by Lemma 2.1. Furthermore, (5) $\Rightarrow$ (6) since (5) $\iff$ (7), and $N_\lambda = N_\mu$ in every LIP loop. To finish it thus suffices to prove that every loop satisfying (6) is an LIP loop. Such a loop fulfils $x^\lambda x^2 = x$ since $x^\lambda x^2 x^\varrho = x^\lambda \cdot x^2 x^\varrho = x^\lambda(x \cdot xx^\varrho) = 1 = xx^\varrho$. By (4), $x^\lambda(x \cdot xz) = (x^\lambda x^2)z = xz$ for all $x, z \in Q$, i.e., Q satisfies the LIP. □

A loop is said to be an *LC loop* if it satisfies the conditions of Theorem 2.2. The mirror conditions yield the *RC loops*. A loop is said to be a *C loop*, see [19], [31], if it is both an LC loop and an RC loop. Theorem 2.2 easily yields the ensuing characterization of C loops.

Corollary 2.3. *Let Q be a loop. The following are equivalent:*

(1) *Q is a C loop;*
(2) *Q is an IP loop in which each square is nuclear;*
(3) *Q is an AP loop with $x^2 \in N_\mu$ for all $x \in Q$;*
(4) *$(yx \cdot x)z = y(x \cdot xz)$ for all $x, y, z \in Q$;*
(5) *$\mu_x^2 = (R_x^{-2}, L_x^2, \mathrm{id}_Q) \in \mathrm{Atp}(Q)$ for all $x \in Q$;*

Proof: Equivalences (1) $\iff$ (2) and (1) $\iff$ (3) follow from Theorem 2.2 in an immediate way. The implication (3) $\implies$ (4) is also clear. Setting $y = 1$ and $z = 1$ in (4) establishes the AP. Hence (3) $\implies$ (4). Substituting $(y/x)/x$ for y in (4) yields $yz = ((y/x)/x) \cdot x(xz)$. That is the same as (5). □

The identity $xy \cdot zx = x(yz \cdot x)$ defines Moufang loops and can be expressed by saying that $\lambda_x \varrho_x \in \mathrm{Atp}(Q)$ for all $x \in Q$. This observation served in [17] as an impetus to investigate all conditions of the form $\sigma_x^\varepsilon \tau_x^\eta \in \mathrm{Atp}(Q)$ for each

$x \in Q$, where $\{\varepsilon, \eta\} \subseteq \{-1, 1\}$, $\{\sigma, \tau\} \subseteq \{\lambda, \varrho, \mu\}$ and $\sigma \neq \tau$. It turns out that the varieties obtained in this way are the varieties of Moufang, left Bol, right Bol, left conjugacy closed (LCC), right conjugacy closed (RCC), Buchsteiner and extra loops. In other words, these are the varieties that can be obtained by *nuclear identification.*

The case when $\sigma = \tau$ and $\varepsilon = \eta$ was not investigated in [17]. This may be regarded as an omission. Assume $\varepsilon = \eta = 1$ and $\sigma = \tau$. The case $\sigma = \lambda$ describes the LC loops, by Theorem 2.2. The RC loops correspond to the case $\sigma = \varrho$, by a mirror argument. Corollary 2.3 implies that the C loops can be obtained from the case $\sigma = \mu$. The LC loops, RC loops and C loops thus result from a nuclear identification as well. It is easy to verify that the same varieties appear when $\sigma = \tau$ and $\varepsilon = \eta = -1$ is assumed.

Proposition 2.4. *If a loop Q satisfies two of the LCC, LC and LBol identities, then it satisfies all three. A left Bol loop Q is LC if and only if $x^2 \in N_\lambda$ for each $x \in Q$.*

PROOF: Every left Bol loop satisfies the LAP. Hence point (5) of Theorem 2.2 can be used to prove that a left Bol loop Q is an LC loop if and only if $x^2 \in N_\lambda$ for each $x \in Q$. The latter condition also characterizes those left Bol loops that are LCC, e.g. by formula (11) of [17]. A loop Q that is both LCC and LC is an LCC loop that satisfies the LAP and has all squares in N_λ. Such a loop is left Bol, e.g. by formula (13) of [17]. □

Left Bol LCC loops have been studied under the name *Burn loops*, see [27]

Proposition 2.5. *Let Q be a loop. The following are equivalent.*

(i) *Q is an extra loop;*
(ii) *Q is an LC loop that is also a right Bol loop, or an RCC loop, or a Buchsteiner loop; and*
(iii) *Q is a C loop that is also a left Bol loop, or an LCC loop.*

PROOF: Extra loops are the Moufang loops with squares in the nucleus, see [18]. Each extra loop is thus a C loop, by Corollary 2.3. Extra loops are conjugacy closed because a Moufang loop is conjugacy closed if and only if all squares are in the nucleus, as in [19] (cf. formula (13) of [17]). Thus (i) ⇒ (iii). The definition of extra loops is mirror symmetric. Thus also (i) ⇒ (iii'), where (iii') is the mirror version of (iii). Denote by (ii') the condition (ii) from which the Buchsteiner clause is removed. Trivially, (iii') ⇒ (ii'). An LAP loop that is RCC or Buchsteiner is an extra loop, by the mirror version of [17, Corollary 2.5]. Hence an LC loop has to be an extra loop if it is a Buchsteiner loop or an RCC loop. A right Bol loop with the LAP is Moufang, see [29]. Hence an LC loop is extra if it is right Bol.

Therefore (ii) $\Rightarrow$ (i). Both (i) $\iff$ (iii') and (i) $\iff$ (iii) follow. Thus if (iii) holds, then Q is conjugacy closed. Conjugacy closed loops with squares in the nucleus are Buchsteiner [17, Theorem 3.3]. Therefore (iii) $\Rightarrow$ (ii). $\square$

Let x be an element of loop Q. Then T_x is defined as $R_x^{-1}L_x$.

Proposition 2.6. *Let Q be a loop with a subloop S. Suppose that $S \leq N_\lambda \cap N_\mu$. If $T_x^{\pm 1}(s) \in S$ for each $x \in Q$ and $s \in S$, then $S \trianglelefteq Q$. Furthermore, $L_{xy}^{-1}L_xL_y(s) = T_{xy}^{-1}T_xT_y(s)$ for all $x, y \in Q$ and $s \in S$.*

Proof: Suppose that $s \in S$ and $x, y \in Q$. Then $sx \cdot y = s \cdot xy$ and so $R_{xy}^{-1}R_yR_x(s) = s$. Furthermore,

$$T_xT_y(s) \cdot xy = T_xT_y(s)x \cdot y = xT_y(s) \cdot y = x \cdot T_y(s)y = x \cdot ys.$$

Therefore $T_{xy}^{-1}T_xT_y(s) = (xy)\backslash(T_xT_y(s) \cdot xy) = xy\backslash(x \cdot ys) = L_{xy}^{-1}L_xL_y(s) \in S$. $\square$

Corollary 2.7. *Let Q be a loop such that $N_\mu = N_\lambda \leq Q$. Then N_μ is a normal subloop if and only if $T_x^{\pm 1}(a) \in N_\mu$ for all $x \in Q$ and $a \in N_\mu$.*

Lemma 2.8. *Let Q be a LIP loop such that $xax^{-1} \in N_\mu$ for all $x \in Q$ and $a \in N_\mu$. Then $T_x^{-1}(a) = x^{-1}ax$ and $T_x(a) = xax^{-1}$.*

Proof: We have $T_x^{-1}(y) = x\backslash(yx) = x^{-1} \cdot yx$ for all $x, y \in Q$ since Q is LIP. Put $y = xax^{-1}$. By our assumptions, $y \in N_\mu$, and so

$$\begin{aligned} T_x^{-1}(xax^{-1}) &= x^{-1}(xax^{-1} \cdot x) = (x^{-1} \cdot xax^{-1})x = (x^{-1}(x \cdot ax^{-1}))x \\ &= (ax^{-1})x = a(x^{-1}x) = a. \end{aligned}$$

Therefore $T_x(a) = xax^{-1}$. $\square$

Corollary 2.9. *Let Q be a LIP loop such that $xax^{-1} \in N_\mu$ for all $x \in Q$ and $a \in N_\mu$. Then $N_\mu \trianglelefteq Q$.*

Proof: This follows directly from Lemma 2.8 and Corollary 2.7 since $N_\lambda = N_\mu$ in every LIP loop Q. $\square$

If Q is an LIP loop, $x \in Q$ and $a \in N_\mu = N_\lambda$, then $(ax)^{-1} = x^{-1}a^{-1}$ as $(x^{-1}a^{-1})(ax) = x^{-1}x = 1$ and $(xa)^{-1} = a^{-1}x^{-1}$ as $(xa)(a^{-1}x^{-1}) = 1$.

Lemma 2.10. *Let Q be an LC loop. Then $xax^{-1} \in N_\mu$ for every $x \in Q$ and $a \in N_\mu$.*

Proof: An LC loop is an LIP loop. Hence $N_\lambda = N_\mu$. All squares of an LC loop belong to N_λ. Therefore $xax^{-1} \in N_\mu$ if and only if $y^2 \cdot xax^{-1} \in N_\mu$ for some $y \in Q$. Put $y = (xa)^{-1} = a^{-1}x^{-1}$. Then $y^2 \cdot xax^{-1} = y^2 \cdot y^{-1}x^{-1} = y \cdot (y \cdot y^{-1}x^{-1}) = a^{-1}x^{-1} \cdot x^{-1} = a^{-1}x^{-2} \in N_\mu$. $\square$

Theorem 2.11. *If Q is an LC loop, then $N_\lambda = N_\mu \trianglelefteq Q$.*

PROOF: This is a straightforward consequence of Lemma 2.10 and Corollary 2.9. □

By Theorem 2.11 the nucleus of a C loop is normal. This was first proved by J. D. Phillips and P. Vojtěchovský in [31].

3. Osborn loops

A major motivation for studying Osborn loops is that they are a broad structured variety of loops which include interesting classical varieties of loops, such as Moufang loops and conjugacy closed loops, as special cases. Before turning to that, we look at another approach. We informally mimic the scheme of nuclear identification studied in [17].

Let Q be a loop. For $x \in Q$, let $\alpha_x, \beta_x \in \operatorname{Sym} Q$ satisfy $\alpha_x(1) = \beta_x(1) = x$. If $(\alpha_x, \mathrm{id}_Q, \alpha_x)$ and $(\mathrm{id}_Q, \beta_x, \beta_x)$ are autotopisms, then it is easy to see that $\alpha_x = L_x$ and $\beta_x = R_x$, in which case the autotopisms are λ_x and ϱ_x, respectively. However, instead of assuming at the outset that α_x and β_x are translations, we leave them as permutations to be determined. Analogous with the approach in [17], we assume that for each $x \in Q$,

$$(\mathrm{id}_Q, \beta_x, \beta_x)(\alpha_x, \mathrm{id}_Q, \alpha_x) = (\alpha_x, \beta_x, \beta_x\alpha_x)$$

is an autotopism, i.e.,

$$\alpha_x(y) \cdot \beta_x(z) = \beta_x\alpha_x(yz)$$

for all $x, y, z \in Q$. (The arbitrariness of the choice of the order in which we multiplied the triples will be dealt with below.)

Setting $z = 1$, we get $\alpha_x = R_x^{-1}\beta_x\alpha_x$, and so $\beta_x = R_x$. Thus

$$\alpha_x(y) \cdot zx = \alpha_x(yz) \cdot x \tag{3.1}$$

for all $x, y, z \in Q$. Taking $y = 1$, we get $\alpha_x(z) \cdot x = x \cdot zx$. This gives us $\alpha_x = R_x^{-1}L_xR_x$, but in the interest of easily finding other expressions for α_x, we will rewrite (3.1) as

$$\alpha_x(y) \cdot zx = x(yz \cdot x). \tag{3.2}$$

Our autotopism is now

$$\psi_x := (\alpha_x, R_x, L_xR_x).$$

A loop Q is said to be an *Osborn loop* if for each $x \in Q$, there exists $\alpha_x \in \mathrm{Sym}(Q)$ such that (3.2) holds. As noted, α_x can be expressed in terms of translations so that (3.2) is an identity in the language of loops and thus Osborn loops form a variety. We next list a few different expressions for α_x.

Lemma 3.1. *Let Q be an Osborn loop and for each $x \in Q$, let $\alpha_x \in \mathrm{Sym}(Q)$ be such that* (3.2) *holds. Then*

$$\alpha_x = R_x^{-1} L_x R_x = L_x R_x R_{x^\lambda} = L_{x^\lambda}^{-1}.$$

Proof: We have already noted the first equality, which can be obtained from (3.2) by taking $z = 1$. Instead taking $z = x^\lambda$, we have $\alpha_x = L_x R_x R_{x^\lambda}$. Now set $y = x^\lambda$ and note that $\alpha_x(x^\lambda) = R_x^{-1} L_x R_x(x^\lambda) = 1$. Thus $zx = x(x^\lambda z \cdot x)$, that is, $R_x = L_x R_x L_{x^\lambda}$. Rearranging this, we have $L_{x^\lambda}^{-1} = R_x^{-1} L_x R_x$, which completes the proof. $\square$

It is worth recording separately a consequence of the last equality of the preceding lemma.

Corollary 3.2. *Let Q be an Osborn loop. For all $x \in Q$,*

$$R_x R_{x^\lambda} L_{x^\lambda} L_x = \mathrm{id}_Q\,. \tag{3.3}$$

Now for all x in an Osborn loop Q, we have $\psi_{x^\varrho}^{-1} = (L_x, R_{x^\varrho}^{-1}, R_{x^\varrho}^{-1} L_{x^\varrho}^{-1})$. From (3.3), we obtain $R_{x^\varrho}^{-1} L_{x^\varrho}^{-1} = R_x L_x$, and so

$$\psi_{x^\varrho}^{-1} = (L_x, R_{x^\varrho}^{-1}, R_x L_x) \tag{3.4}$$

is an autotopism.

Let Q be a loop and let $(Q^{\mathrm{op}}, *)$ denote the opposite loop defined by $x*y := yx$, with translations $L_x^{\mathrm{op}} := R_x$ and $R_x^{\mathrm{op}} := L_x$ and inverses $x^{\hat\lambda} = x^\varrho$ and $x^{\hat\varrho} = x^\lambda$. A triple (α, β, γ) is an autotopism of Q if and only if (β, α, γ) is an autotopism of Q^{op}. Thus $\psi_x \in \mathrm{Atp}(Q)$ if and only if $(L_x^{\mathrm{op}}, R_{x^{\hat\varrho}}^{\mathrm{op}\,-1}, R_x^{\mathrm{op}} L_x^{\mathrm{op}}) \in \mathrm{Atp}(Q^{\mathrm{op}})$, that is, if and only if $\psi_{x^{\hat\varrho}} \in \mathrm{Atp}(Q^{\mathrm{op}})$. Since we already have that $\psi_x \in \mathrm{Atp}(Q)$ if and only if $\psi_{x^\varrho} \in \mathrm{Atp}(Q)$, we conclude that a *loop Q is an Osborn loop if and only if its opposite loop Q^{op} is an Osborn loop.* In particular, any Osborn identity is equivalent to its mirror, which is the corresponding identity in Q^{op}.

Informally, if we had multiplied the triples $(\alpha_x, \mathrm{id}\,Q, \alpha_x)$ and $(\mathrm{id}\,Q, \beta_x, \beta_x)$ in the opposite order and assumed that the product is an autotopism, we would have obtained $\alpha_x = L_x$, the autotopism $(L_x, \beta_x, R_x L_x)$, and various expressions for the permutation β_x. This is precisely (3.4), and so we would have been led to the same variety of loops.

The various forms of the autotopisms ψ_x and ψ_{x^ϱ} lead to corresponding equivalent identities.

Theorem 3.3. *The following identities are equivalent in loops.*

(1) $(x \cdot yx)/x \cdot zx = x(yz \cdot x)$;
(2) $x(yx^\lambda \cdot x) \cdot zx = x(yz \cdot x)$;
(3) $x^\lambda \backslash y \cdot zx = x(yz \cdot x)$;
(4) $xy \cdot x\backslash(xz \cdot x) = (x \cdot yz)x$;
(5) $xy \cdot (x \cdot x^\varrho z)x = (x \cdot yz)x$;
(6) $xy \cdot z/x^\varrho = (x \cdot yz)x$;
(7) $x^\lambda \backslash (x^\lambda y \cdot z) = (y \cdot zx)/x$;
(8) $x\backslash(xy \cdot z) = (y \cdot zx^\varrho)/x^\varrho$.

PROOF: Any two components of an autotopism uniquely determine the third. Identities (1), (2) and (3) are equivalent because they all express that ψ_x is an autotopism with different forms of the first component α_x. Identities (4), (5) and (6) are the mirrors of (1), (2) and (3), respectively.

Next, starting with (3), replace y with $x^\lambda y$ to get $yz \cdot x = L_x R_x(x^\lambda y \cdot z) = R_x L_{x^\lambda}^{-1}(x^\lambda y \cdot z)$, using Lemma 3.1. Applying R_x^{-1} to both sides, we obtain (7). The steps are reversible, so (3) is equivalent to (7). Finally, (8) is the mirror version of (7). □

Any one of the identities in Theorem 3.3 may be taken as the definition of Osborn loops. When we use these in what follows, we shall often just refer to "an Osborn identity" rather than singling out the particular form.

From comparing the identities of Theorem 3.3 with the Moufang identity $xy \cdot zx = x(yz \cdot x)$ or its mirror image, we see that any Moufang loop is an Osborn loop. We now give a more thorough characterization.

A loop Q is *flexible* (FLX) if $xy \cdot x = x \cdot yx$ for all $x, y \in Q$.

Theorem 3.4. *Any of the following are necessary and sufficient for an Osborn loop Q to be a Moufang loop:* (i) *LIP*, (ii) *RIP*, (iii) *FLX*, (iv) *LAP*, (v) *RAP*.

PROOF: Moufang loops are *diassociative*, i.e., any 2-generated subloop is associative. Conditions (i)–(v) are all particular instances of diassociativity so their necessity is clear.

For sufficiency, (i), (ii) and (iii) are immediate from identities in Theorem 3.3. For (iv): If LAP holds, then for all $x \in Q$, $x^\lambda \backslash x = L_{x^\lambda}^{-1}(x) = R_x^{-1} L_x R_x(x) = (x \cdot xx)/x = (xx \cdot x)/x = xx$. Thus for all $z \in Q$, $x(x \cdot zx) = xx \cdot zx = x^\lambda \backslash x \cdot zx = x(xz \cdot x)$ using an Osborn identity in the last equality. Cancelling we have $x \cdot zx = xz \cdot x$, that is, FLX holds. The proof of the sufficiency of (v) is dual to this. □

Other instances of diassociativity are also sufficient for an Osborn loop to be a Moufang loop. For example, a loop Q satisfies the *antiautomorphic inverse* *property* if $x^\lambda = x^\varrho$ and $(xy)^\varrho = y^\varrho x^\varrho$ for all $x, y \in Q$. We omit the proof that an AAIP Osborn loop is Moufang as it is a little more involved than the proofs of the five cases of Theorem 3.4.

As we will see, an instance of diassociativity which is not sufficient for an Osborn loop Q to be Moufang is the *weak inverse property* (WIP): $x(yx)^\varrho = y^\varrho$ or equivalently, $(xy)^\lambda x = y^\lambda$ for all $x, y \in Q$.

The origin of Osborn loops lies in a paper of J. M. Osborn, see [28], who studied loops in which WIP holds in every loop isotope. He proved that such a loop must satisfy the identity $xy \cdot \theta_x(z)x = (x \cdot yz)x$ where for each x, θ_x is a permutation. A. S. Basarab in [1] dubbed a loop satisfying the identity an Osborn loop. In the same paper, Basarab also introduced generalized Moufang loops, which we discuss further below. These turn out to be precisely WIP Osborn loops, but not every Osborn loop has the WIP.

Independently and two years after Basarab's paper appeared, E. D. Huthnance also studied what are now called Osborn loops in his Ph.D. dissertation, see [22]. By an amusing coincidence, Huthnance reversed Basarab's terminology by using "Osborn loops" to refer to Basarab's generalized Moufang loops and "generalized Moufang loops" to refer to Basarab's Osborn loops. We follow Basarab since his paper appeared first and other papers have since been published following his convention [4], [5], [7], [21]. Many, but not all of the results in the remainder of this section can be found in Basarab's papers.

If $(\alpha, \beta, \gamma) \in \mathrm{Atp}(Q)$ is such that $\beta(1) = 1$, then it is well known and easy to show that $\alpha = \gamma = L_c\beta$ where $c = \alpha(1)$. In this case, β is called a *left pseudoautomorphism* with *companion c*. A left pseudoautomorphism is an automorphism if and only if the companion lies in the left nucleus.

Dually, if $(\alpha, \beta, \gamma) \in \mathrm{Atp}(Q)$ is such that $\alpha(1) = 1$, then $\beta = \gamma = R_c\alpha$ where $c = \beta(1)$. In this case, α is called a *right pseudoautomorphism* with *companion c*.

A loop Q is a *G-loop* if it is isomorphic to all of its loop isotopes. For instance, groups or more generally, conjugacy closed loops are G-loops. It is well known that a loop is a G-loop if and only if each element occurs as a companion of some left pseudoautomorphism and of some right pseudoautomorphism.

Lemma 3.5. *Let Q be a G-loop, i.e., for each $x \in Q$, there exists a left pseudoautomorphism φ_x and a right pseudoautomorphism ψ_x, each with companion x. Assume further that $\varphi_x\psi_x = \psi_x\varphi_x = \mathrm{id}_Q$ and $\varphi_x(x) = x = \psi_x(x)$. Then Q is an Osborn loop.*

Proof: Multiplying the autotopisms $(L_x\varphi_x, \varphi_x, L_x\varphi_x)$ and $(\psi_x, R_x\psi_x, R_x\psi_x)$, we get that $(L_x, \varphi_x L_x \psi_x, L_x\varphi_x R_x\psi_x)$ is an autotopism. For each $y \in Q$,

$$L_x\varphi_x R_x\psi_x(y) = L_x\varphi_x(\psi_x(y) \cdot x) = L_x\varphi_x\psi_x(y) \cdot \varphi_x(x) = xy \cdot x$$

since φ is a left pseudoautomorphism and using the assumptions of the lemma. Thus $L_x\varphi_x R_x\psi_x = R_xL_x$, and so for each $x \in Q$, $(L_x, \varphi_x L_x\psi_x, R_xL_x)$ is an autotopism. This autotopism has the form (L_x, β_x, R_xL_x) for a permutation β_x and so Q is an Osborn loop. □

Corollary 3.6. *Every conjugacy closed loop is an Osborn loop.*

PROOF: A loop Q is conjugacy closed if and only if for each $x \in Q$, $T_x = R_x^{-1}L_x$ is a right pseudoautomorphism with companion x (this is LCC) and T_x^{-1} is a left pseudoautomorphism with companion x (this is RCC). CC-loops are well known to be G-loops. Since $T_x(x) = x = T_x^{-1}(x)$, the lemma applies. □

It is, in fact, easy to show from working directly with the autotopisms defining LCC and RCC loops that an Osborn loop is CC if and only if it is LCC if and only if it is RCC.

A. S. Basarab in [8] defined a loop Q to be a *VD*-loop (probably named after Valentin Danilovich Belousov) if every T_x is a *left* pseudoautomorphism with companion x and every T_x^{-1} is a *right* pseudoautomorphism with companion x. Thus a VD-loop is defined by the identity $x(xy/x) \cdot (xz/x) = x((x \cdot yz)/x)$ and its mirror. Every VD-loop is a G-loop, as follows from the characterization of G-loops stated above. Moufang loops with nuclear fourth powers and CC-loops with nuclear squares are VD-loops.

Corollary 3.7 ([8])**.** *Every VD-loop is an Osborn loop.*

As discussed above, a *generalized Moufang loop* is a WIP Osborn loop. Such loops are characterized by the identity $x(yz \cdot x) = (y^\lambda x^\lambda)^\varrho \cdot zx$ or its equivalent mirror. Indeed, suppose this identity holds. Then for each x, $(\varrho R_{x^\lambda}\lambda, R_x, L_xR_x)$ is an autotopism. This has the form (α_x, R_x, L_xR_x) for a permutation α_x and so $\varrho R_{x^\lambda}\lambda = L_x^{-1}$ by Lemma 3.1. This is precisely WIP. The steps are clearly reversible, establishing the desired characterization.

J. M. Osborn showed that for what we now call a generalized Moufang loop Q, the factor loop $Q/N(Q)$ is a Moufang loop. A. S. Basarab in [1] proved that every isotope of a generalized Moufang loop is a generalized Moufang loop. Every WIP CC-loop is a generalized Moufang loop. In fact, combining results of A. S. Basarab in [2] and E. G. Goodaire and D. A. Robinson in [20], WIP CC-loops are precisely generalized Moufang loops with every square in the nucleus. A short proof of the latter fact appears in Section 4, cf. Theorem 4.3. Generalized Moufang loops satisfy a suitably generalized version of Moufang's theorem, see [3].

One can certainly obtain generalized Moufang loops which are neither CC-loops nor Moufang loops by, say, taking the direct product of a WIP CC-loop which is not an extra loop and a Moufang loop which is not an extra loop. The smallest example given by this construction is obtained from the nonassociative CC-loop of order 6 and the nonassociative Moufang loop of order 12. However it is not known what is the order of the smallest generalized Moufang loop which is neither a Moufang loop nor a CC-loop, nor has there ever been any effort at a classification of generalized Moufang loops of small orders.

Let Q be a loop. Recall that the *multiplication group* of Q is the permutation group $\mathrm{Mlt}(Q) = \langle L_x, R_x \colon x \in Q\rangle$. The *left* and *right multiplication groups* of Q are, respectively, $\mathrm{Mlt}_\lambda(Q) = \langle L_x \colon x \in Q\rangle$ and $\mathrm{Mlt}_\varrho(Q) = \langle R_x \colon x \in Q\rangle$.

Theorem 3.8. *Let Q be an Osborn loop. Then $\mathrm{Mlt}_\lambda(Q)$ and $\mathrm{Mlt}_\varrho(Q)$ are normal subgroups of $\mathrm{Mlt}(Q)$.*

PROOF: It is sufficient to show that for all $x, y \in Q$, $R_x^{-1} L_y R_x, R_x L_y R_x^{-1} \in \mathrm{Mlt}_\lambda(Q)$ and $L_x^{-1} R_y L_x, L_x R_y L_x^{-1} \in \mathrm{Mlt}_\varrho(Q)$. The first follow immediately from writing Theorem 3.3 (7), (8) in terms of translations:

$$R_x^{-1} L_y R_x = L_{x^\lambda}^{-1} L_{x^\lambda y}, \tag{3.5}$$

$$L_x^{-1} R_y L_x = R_{x^\varrho}^{-1} R_{y x^\varrho}. \tag{3.6}$$

Next rearrange (3.5) to get $R_x L_{x^\lambda y} R_x^{-1} = R_x L_{x^\lambda} R_x^{-1} L_y$. From Lemma 3.1, we have $R_x L_{x^\lambda} R_x^{-1} = L_x^{-1}$. Replacing y with $x^\lambda \backslash y$, we get $R_x L_y R_x^{-1} = L_x^{-1} L_{x^\lambda \backslash y}$. Similarly, $L_x R_y L_x^{-1} = R_x^{-1} R_{y/x^\varrho}$, completing the proof. □

The *left* and *right inner mapping groups* $\mathrm{Inn}_\lambda(Q)$ and $\mathrm{Inn}_\varrho(Q)$ of a loop Q are the stabilizers of 1 in $\mathrm{Mlt}_\lambda(Q)$ and $\mathrm{Mlt}_\varrho(Q)$, respectively. In terms of generators, it turns out that $\mathrm{Inn}_\lambda(Q) = \langle L_{xy}^{-1} L_x L_y \colon x, y \in Q\rangle$ and $\mathrm{Inn}_\varrho(Q) = \langle R_{yx}^{-1} R_x R_y \colon x, y \in Q\rangle$.

Theorem 3.9. *Let Q be an Osborn loop. Then for all $x, y \in Q$,*

$$[L_y, R_x] = (L_{x^\lambda y}^{-1} L_{x^\lambda} L_y)^{-1} = R_{xy^\varrho}^{-1} R_{y^\varrho} R_x.$$

Therefore $\mathrm{Inn}_\lambda(Q) = \langle [L_x, R_y] \colon x, y \in Q\rangle = \mathrm{Inn}_\varrho(Q)$.

PROOF: The first equality follows from multiplying (3.5) on the left by L_y^{-1}. The second equality follows from exchanging x and y in (3.6), multiplying on the left by R_x^{-1} and then taking inverses of both sides. The remaining assertion follows from the characterizations of the left and right inner mapping groups in terms of generators. □

Lemma 3.10. *Let Q be a loop. If $\mathrm{Mlt}_\lambda(Q) \trianglelefteq \mathrm{Mlt}(Q)$, then the right nucleus of Q is a normal subloop. If $\mathrm{Mlt}_\varrho(Q) \trianglelefteq \mathrm{Mlt}(Q)$, then the left nucleus of Q is a normal subloop.*

PROOF: See [15, Lemma 1.5]. □

Theorem 3.11. *Let Q be an Osborn loop. Then the three nuclei of Q coincide and the nucleus is a normal subloop.*

PROOF: The left nucleus of a loop Q is the fixed point set of $\mathrm{Inn}_\lambda(Q)$, the right nucleus is the fixed point set of $\mathrm{Inn}_\varrho(Q)$, and the middle nucleus is the fixed point set of $\langle [L_x, R_y] \colon x, y \in Q \rangle$. Thus the equality of the nuclei follows from Theorem 3.9. The normality follows from Theorem 3.8 and Lemma 3.10. □

Call an Osborn loop *proper* if it is neither conjugacy closed nor Moufang. By exhaustive computer search, there are no proper Osborn loops up through order 13. The smallest known ones have order 16, and again by exhaustive computer search, they are the smallest which have nontrivial nucleus. There are two of them, up to isomorphism. Each is a G-loop and each contains the dihedral group D_8 as a subloop. Each has center of order 2, coinciding with the nucleus, and the factor by the center is a nonassociative WIP CC-loop of order 8. Each is nilpotent of class 3; the second center is a copy of $\mathbb{Z}_2$ and the factor by the second center is $\mathbb{Z}_4$. The two loops can be distinguished equationally; one satisfies $L_x^4 = R_x^4 = \mathrm{id}$ but the other does not.

Lemma 3.12. *Let Q be an Osborn loop. Then for each $x, y \in Q$,*

(1) *$L_{xy}^{-1} L_x L_y$ is a right pseudoautomorphism with companion $y/x^\varrho \cdot (xy)^\varrho$;*
(2) *$R_{yx}^{-1} R_x R_y$ is a left pseudoautomorphism with companion $(yx)^\lambda \cdot x^\lambda \backslash y$.*

PROOF: We compute the autotopism

$$\psi_{(xy)^\varrho} \psi_{x^\varrho}^{-1} \psi_{y^\varrho}^{-1} = (L_{xy}^{-1} L_x L_y, R_{(xy)^\varrho} R_{x^\varrho}^{-1} R_{y^\varrho}^{-1}, \omega_{x,y}),$$

where we do not need the particular expression for $\omega_{x,y}$. The first component $L_{xy}^{-1} L_x L_y$ fixes 1 and hence is a right pseudoautomorphism. The companion is $R_{(xy)^\varrho} R_{x^\varrho}^{-1} R_{y^\varrho}^{-1}(1) = y/x^\varrho \cdot (xy)^\varrho$, as claimed. □

Corollary 3.13. *Let Q be an Osborn loop. For each $x \in Q$, $L_{x^\lambda} L_x = L_x L_{x^\varrho}$ and $R_x R_{x^\lambda} = R_{x^\varrho} R_x$ are automorphisms.*

PROOF: By Lemma 3.12, $L_{x^\lambda} L_x$ is a left pseudoautomorphism with companion $x/(x^\lambda)^\varrho \cdot (x^\lambda x)^\varrho = 1$. Now $L_{x^\lambda} L_x L_{x^\varrho}(y) = L_{x^\lambda} L_x(x^\varrho) \cdot L_{x^\lambda} L_x(y) = x^\lambda \cdot L_{x^\lambda} L_x(y) = L_{x^\lambda} L_{x^\lambda} L_x(y)$. Canceling L_{x^λ}, we obtain $L_{x^\lambda} L_x = L_x L_{x^\varrho}$ as claimed. The remaining assertions follow dually. □

A loop Q satisfies the *crossed inverse property* (CIP) if $xy \cdot x^\varrho = y$ or equivalently $x^\lambda \cdot yx = y$ for all $x, y \in Q$.

Lemma 3.14. *A CIP Osborn loop is a commutative Moufang loop.*

PROOF: For such a loop Q, from $y = xy \cdot x^\varrho$, we have $x^\varrho y = x^\varrho(xy \cdot x^\varrho) = (x^\varrho)^\lambda \backslash x \cdot yx^\varrho = 1 \cdot yx^\varrho = yx^\varrho$, and so Q is commutative. Commutative loops are flexible, so Q is Moufang by Theorem 3.4. □

A loop is *left automorphic* or a *left A-loop* if $\mathrm{Inn}_\lambda(Q) \leq \mathrm{Aut}(Q)$. Left automorphic loops form a variety because the defining condition can be expressed equationally using the generators: $L_{xy}^{-1}L_xL_y(zu) = L_{xy}^{-1}L_xL_y(z) \cdot L_{xy}^{-1}L_xL_y(u)$. *Right* automorphic loops are defined dually.

Theorem 3.15. *Let Q be a left or right automorphic Osborn loop. Then $Q/N(Q)$ is a commutative Moufang loop.*

PROOF: We prove the left case. We use Lemma 3.12 and the assumption that each $L_{x,y}$ is an automorphism to conclude that the companion $y/x^\varrho \cdot (xy)^\varrho$ lies in $N(Q)$. Thus in $Q/N(Q)$, the identity $y/x^\varrho \cdot (xy)^\varrho = 1$ holds. This is equivalent to $xy = y/x^\varrho$ or $xy \cdot x^\varrho = y$ for all $x, y \in Q$. By Lemma 3.14, we have the desired result. □

As a corollary, we obtain yet another proof of Basarab's CC-loop theorem, see [6]. Basarab's proof was explicated in a simplified form in [25] and another proof was given in [14].

Corollary 3.16. *Let Q be a CC-loop. Then $Q/N(Q)$ is an abelian group.*

PROOF: CC-loops are left automorphic, so by Corollary 3.6 and Theorem 3.15, $Q/N(Q)$ is commutative. Commutative CC-loops are abelian groups, see [26]. □

A. S. Basarab in [1] proved that if Q is a loop such that every loop isotope is Osborn, then $Q/N(Q)$ has WIP and hence is a generalized Moufang loop (and hence $(Q/N(Q))/N(Q/N(Q))$ is a Moufang loop. However, he did not address the following, which is still the outstanding open problem in the theory of Osborn loops.

Problem 3.17 ([24])**.** If Q is an Osborn loop, is every loop isotopic to Q an Osborn loop?

Problem 3.18. Does there exist a simple, proper Osborn loop?

Note that an affirmative answer to Problem 3.18 would be a counterexample to Problem 3.17: if Q is a simple, proper Osborn loop, then by Theorem 3.11, $N(Q) = \{1\}$. If every loop isotopic to Q is an Osborn loop, then by the discussion above, $Q \cong Q/N(Q)$ is a generalized Moufang loop and hence a Moufang loop, contradicting the assumption that Q is proper.

4. Loops that are both Osborn and Buchsteiner

As discussed in the last section, there are many equivalent ways to define Osborn loops. We shall use the following characterization:

$$(4.1) \qquad Q \text{ is an Osborn loop} \iff \psi_x = (L_{x^\lambda}^{-1}, R_x, L_x R_x) \in \mathrm{Atp}(Q)$$

for all $x \in Q$.

Buchsteiner loops are characterized by the law $x\backslash(xy \cdot z) = (y \cdot zx)/x$. Setting $y = x$ yields $L_x^{-1}L_{x^2} = R_x^{-1}L_xR_x$, while $z = x$ gives $L_x^{-1}R_xL_x = R_x^{-1}R_{x^2}$. Thus

$$(4.2) \qquad L_{x^2} = L_xR_x^{-1}L_xR_x \quad \text{and} \quad R_{x^2} = R_xL_x^{-1}R_xL_x$$

if $x \in Q$, Q a Buchsteiner loop.

A straightforward calculation using (4.2) yields

$$(4.3) \qquad R_x^2L_{x^2}^{-1}L_x^2 = R_{x^2} \qquad \text{if } x \in Q,\ Q \text{ a Buchsteiner loop.}$$

Both Osborn and Buchsteiner loops are known to have the property that $N_\lambda = N_\varrho = N_\mu$. In the following we shall thus consider only the nucleus $N = N(Q)$.

$$(4.4) \qquad L_{x^2} = L_xL_{x^\lambda}^{-1} \qquad \text{if } x^2 \in N(Q),\ Q \text{ a loop.}$$

This is true because (a) $x^2 \cdot x^\lambda = x$ since $(x^2 \cdot x^\lambda)x = x^2(x^\lambda \cdot x) = x^2$, and so (b) $x^2 \cdot x^\lambda y = (x^2 \cdot x^\lambda)y = xy$.

$$(4.5) \qquad L_{x^2} = L_xR_x^{-1}L_xR_x \qquad \text{if } x^2 \in N(Q),\ Q \text{ an Osborn loop.}$$

This follows from Lemma 3.1 since $L_{x^\lambda}^{-1} = L_x^{-1}L_{x^2}$, by (4.4).

The following fact can be obtained directly from the definition of Buchsteiner loops and is well known:

$$(4.6) \qquad Q \text{ is a Buchsteiner loop} \iff \varphi_x = (L_x,\, R_x^{-1},\, L_xR_x^{-1}) \in \mathrm{Atp}(Q)$$

for all $x \in Q$.

Theorem 4.1 (Kinyon)**.** *Let Q be a loop. Consider the following three properties:*

(a) *Q is an Osborn loop;*
(b) *Q is a Buchsteiner loop;*
(c) *Q is a loop such that all squares are in the nucleus.*

Any two of these properties imply the third property.

Proof: Let φ_x and ψ_x be the triples from (4.1) and (4.6), $x \in Q$. Then

$$\gamma_x = \varphi_x \psi_x = (L_x L_{x^\lambda}^{-1},\, \mathrm{id}_Q,\, L_x R_x^{-1} L_x R_x).$$

If (a) and (b) are true, then $\gamma_x \in \mathrm{Atp}(Q)$, and so $L_x R_x^{-1} L_x R_x(1) = x^2 \in N(Q)$. For the rest we can assume that $x^2 \in N(Q)$ for all $x \in Q$. It will suffice to show that $\gamma_x = (L_{x^2}, \mathrm{id}_Q, L_{x^2})$ if Q is an Osborn loop or a Buchsteiner loop. First note that $x^2 \in N(Q)$ implies $L_x L_{x^\lambda}^{-1} = L_{x^2}$, by (4.4). If Q is Buchsteiner, then $L_{x^2} = L_x R_x^{-1} L_x R_x$ by (4.2). The same equality follows from (4.5) if Q is Osborn. □

Theorem 4.2 (G. T. Jaiyeola & J. O. Adeniran [23]). *Let Q be a loop. Consider the following three properties:*

(a) *Q is an Osborn loop;*
(b) *Q is a Buchsteiner loop;*
(c) *Q satisfies the law $(x \cdot xy)(x^\lambda \cdot xz) = x(x \cdot yz)$.*

Any two of these properties imply the third property.

Proof: First note that (c) is equivalent to the assumption that

$$\delta_x = (L_x^2, L_{x^\lambda} L_x, L_x^2) \in \mathrm{Atp}(Q) \qquad \text{for all } x \in Q. \tag{4.7}$$

For every $x \in Q$

$$\varphi_x^{-2} \delta_x = (\mathrm{id}_Q, R_x^2 L_{x^\lambda} L_x, R_x L_x^{-1} R_x L_x).$$

If Q is both Osborn and Buchsteiner, then $(\mathrm{id}_Q, R_{x^2}, R_{x^2}) \in \mathrm{Atp}(Q)$, by Theorem 4.1. In such a loop $\delta_x \in \mathrm{Atp}(Q)$ if $\varphi_x^{-2}\delta_x = (\mathrm{id}_Q, R_{x^2}, R_{x^2})$. Now, $R_{x^2} = R_x L_x^{-1} R_x L_x$ by (4.2), while (4.4) and (4.3) imply $R_x^2 L_{x^\lambda} L_x = R_x^2 L_{x^2}^{-1} L_x^2 = R_{x^2}$. Thus (a) and (b) imply (c).

If (a) and (c) hold, then $\varphi_x^{-2}\delta_x \in \mathrm{Atp}(Q)$, and that yields

$$x^2 = R_x L_x^{-1} R_x L_x(1) \in N(Q).$$

It remains to consider the case when both (b) and (c) are true. For that use that

$$\psi_{x^\varrho}^2 \delta_x = (\mathrm{id}_Q, R_{x^\varrho}^2 L_{x^\lambda} L_x, (L_{x^\varrho} R_{x^\varrho})^2 L_x^2).$$

We obtain that $R_{x^\varrho}^2 L_{x^\lambda} L_x(1) = (x^\varrho)^2 \in N(Q)$ for all $x \in Q$. □

Theorem 4.3. *Let Q be a loop. Consider the following three properties:*

(a) *Q is a generalized Moufang loop;*
(b) *Q is a WIP CC loop;*
(c) *Q is a loop such that all squares are in the nucleus.*

Any two of these properties imply the third property.

Proof: This is a direct consequence of Theorem 4.1 since a WIP loop is a CC loop if and only if it is a Buchsteiner loop, by [17, Theorem 5.5], and since a loop is a generalized Moufang loop if and only if it is a WIP Osborn loop. □

Note that Theorem 4.3 remains true if point (c) is replaced by the point (c) of Theorem 4.2. Note also that by [17, Theorem 5.5] a WIP LCC loop is CC, and a WIP RCC loop is also CC.

References

[1] Basarab A. S., *A class of WIP loops*, Mat. Issled. **2** (1967), vyp. 2, 3–24 (Russian).
[2] Basarab A. S., *A certain class of G-loops*, Mat. Issled. **3** (1968), vyp. 2 (8), 72–77 (Russian).
[3] Basarab A. S., *Moufang's theorem*, Bul. Akad. Štiince RSS Moldoven, 1968, (1968), no. 1, 16–24 (Russian).
[4] Basarab A. S., *Isotopy of WIP loops*, Mat. Issled. **5** (1970), vyp. 2 (16), 3–12 (Russian).
[5] Basarab A. S., *The Osborn loop*, Studies in the theory of quasigroups and loops, 193, Izdat. "Štiinca", Kishinev, 1973, pages 12–18 (Russian).
[6] Basarab A. S., *A class of LK-loops*, Mat. Issled. **120** Bin. i n-arnye Kvazigruppy (1991), 3–7, 118 (Russian).
[7] Basarab A. S., *Osborn's G-loops*, Quasigroups Related Systems **1** (1994), no. 1, 51–56.
[8] Basarab A. S., *Generalized Moufang G-loops*, Quasigroups Related Systems **3** (1996), 1–5.
[9] Bates G. E., Kiokemeister F., *A note on homomorphic mappings of quasigroups into multiplicative systems*, Bull. Amer. Math. Soc. **54** (1948), 1180–1185.
[10] Belousov V. D., *Foundations of the Theory of Quasigroups and Loops*, Nauka, Moscow, 1967 (Russian).
[11] Bruck R. H., *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.
[12] Bruck R. H., *A Survey of Binary Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, 20, Reihe: Gruppentheorie, Springer, Berlin, 1958.
[13] Csörgő P., Drápal A., Kinyon M. K., *Buchsteiner loops*, Internat. J. Algebra Comput. **19** (2009), no. 8, 1049–1088.
[14] Drápal A., *Conjugacy closed loops and their multiplication groups*, J. Algebra **272** (2004), no. 2, 838–850.
[15] Drápal A., *On multiplication groups of left conjugacy closed loops*, Comment. Math. Univ. Carolin. **45** (2004), no. 2, 223–236.
[16] Drápal A., *On multiplicative equivalences that are totally incompatible with division*, Algebra Universalis **80** (2019), no. 3, Paper No. 32, 9 pages.
[17] Drápal A., Jedlička P., *On loop identities that can be obtained by a nuclear identification*, European J. Combin. **31** (2010), no. 7, 1907–1923.
[18] Fenyves F., *Extra loops. I.*, Publ. Math. Debrecen **15** (1968), 235–238.
[19] Fenyves F., *Extra loops. II. On loops with identities of Bol–Moufang type*, Publ. Math. Debrecen **16** (1969), 187–192.
[20] Goodaire E. G., Robinson D. A., *Some special conjugacy closed loops*, Canad. Math. Bull. **33** (1990), no. 1, 73–78.
[21] Hrůza B., *Sur quelques propriétés des inverse-faibles*, Knižnice Odborn. Věd. Spisů Vysoké Učení Tech. v Brně **B-56** (1975), 101–107 (French).
[22] Huthnance E. D., Jr., *A Theory of Generalized Moufang Loops*, Thesis (Ph.D.)–Georgia Institute of Technology, Georgia, 1969.
[23] Jaiyéolá T. G., Adéníran J. O., *A new characterization of Osborn–Buchsteiner loops*, Quasigroups Related Systems **20** (2012), no. 2, 233–238.

[24] Kinyon M., *A survey of Osborn loops*, plenary talk at the First Milehigh Conference on Loops, Quasigroups, & Nonassociative Systems, University of Denver, Denver, CO, 2005, `https://www.cs.du.edu/~petr/milehigh/2005/kinyon_talk.pdf`.

[25] Kinyon M. K., Kunen K., Phillips J. D., *Diassociativity in conjugacy closed loops*, Comm. Algebra **32** (2004), no. 2, 767–786.

[26] Kunen K., *The structure of conjugacy closed loops*, Trans. Amer. Math. Soc. **352** (2000), no. 6, 2889–2911.

[27] Nagy P. T., Strambach K., *Loops as invariant sections in groups, and their geometry*, Canad. J. Math. **46** (1994), no. 5, 1027–1056.

[28] Osborn J. M., *Loops with the weak inverse property*, Pacific J. Math. **10** (1960), 295–304.

[29] Pflugfelder H. O., *Quasigroups and Loops: Introduction*, Sigma Series in Pure Mathematics, 7, Heldermann, Berlin, 1990.

[30] Phillips J. D., Vojtěchovský P., *The varieties of loops of Bol–Moufang type*, Algebra Universalis **54** (2005), no. 3, 259–271.

[31] Phillips J. D., Vojtěchovský P., *C-loops: an introduction*, Publ. Math. Debrecen **68** (2006), no. 1–2, 115–137.

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