The operation ABA in operator algebras

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Abstract. The binary operation aba, called Jordan triple product, and its variants (such as e.g. the sequential product $\sqrt{a}b\sqrt{a}$ or the inverted Jordan triple product $ab^{-1}a$) appear in several branches of operator theory and matrix analysis. In this paper we briefly survey some analytic and algebraic properties of these operations, and investigate their intimate connection to Thompson type isometries in different operator algebras.

Keywords: loop; gyrogroup; Jordan triple product; Thompson metric; JB-algebra

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1. Introduction

This paper is of survey character and it is organized as follows.

The first part is involved with the standard K-loop structure on the positive definite cone of a C^*-algebra. We investigate some analytic and algebraic properties of the sequential product.

In the second part we point out the geometric origin of the motivation of studying Jordan triple product isomorphisms. Moreover, as an application we describe the structure of isometries with respect to a special class of distance measures incorporating the Thompson part metric as well.

Finally, in the last section the corresponding structural result concerning the Thompson part metric is also presented in the setting of JB-algebras.

2. The standard K-loop structure on positive definite operators

The operation $\sqrt{a}b\sqrt{a}$ called sequential product was introduced first by Gudder and Nagy, originally on the so-called Hilbert space effect algebra meaning the

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operator interval $[0, I]$, where I stands for the identity operator on the underlying Hilbert space, with respect to the usual Löwner ordering. In their quantum mechanical interpretation the operator $\sqrt{a}b\sqrt{a}$ represents a sequential measurement in which a is performed first and b is second.

In what follows A denotes a unital C^*-algebra, if not stated otherwise, and A_+^{-1} stands for its positive invertible elements. Considering the operation $a \circ b = \sqrt{a}b\sqrt{a}$ on A_+^{-1} we arrive at a rich mathematical object as $(A_+^{-1}, \circ)$ provides a fundamental example of K-loop. To recall the concept of K-loop, we note that a set S equipped with a binary operation "$\star$" is called a quasigroup whenever the equations

$$a \star x = b, \tag{2.1}$$

$$y \star a = b \tag{2.2}$$

have unique solutions in S for every $a, b \in S$. A quasigroup with unit is called a loop. A loop satisfying the identity

$$a \star (b \star (a \star c)) = (a \star (b \star a)) \star c$$

is called a Bol loop. A Bol loop with the so-called automorphic inverse property

$$(a \star b)^{-1} = a^{-1} \star b^{-1}$$

is called a K-loop, or in another words a Bruck loop. Let us mention that the abstract K-loop structure determines exactly the same structure as the so-called gyrogroups, see [11]. The theory of such objects has been developed by A. A. Ungar, see [12]. Note that the Einstein gyrogroup provides a particular important example of gyrogroups, which is defined on the set of admissible velocities

$$\mathbb{B} = \{v \in \mathbb{R}^3 \colon \|v\| < 1\}$$

equipped with the velocity addition

$$v \oplus u = \frac{1}{1 + \langle v, u\rangle}\left(v + \frac{1}{c_v}u + \frac{c_v}{1 + c_v}\langle v, u\rangle v\right)$$

where $c_v = (1 - \|v\|^2)^{-1/2}$ is the Lorentz factor.

The most challenging part in verifying that $(A_+^{-1}, \circ)$ is indeed a K-loop is to establish (2.2), that is, one needs to furnish that the equation

$$ya^{-1}y = b$$

has a unique solution $y \in A_+^{-1}$ for every $a, b \in A_+^{-1}$. This is provided by the Anderson–Trap theorem which states that the Pusz–Woronowitz geometric mean

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}$$

is the unique solution of the equality in question. After examining the standard K-loop structure, the question arises naturally when $(A_+^{-1}, \circ)$ became a Moufang loop, that is, when it satisfies

$$(a \star a) \star b = a \star (a \star b), \tag{2.3}$$

$$(a \star b) \star b = a \star (b \star b), \tag{2.4}$$

$$a \star (b \star a) = (a \star b) \star a. \tag{2.5}$$

The identity (2.3) is always satisfied, but the first two hold exactly when the algebra A is commutative, see [2]. Another notable property concerning commutativity is that for $a, b \in A_+^{-1}$, we have $a \circ b = b \circ a$ if and only if $ab = ba$, that is, commutativity with respect to the K-loop product is exactly the same as commutativity with respect to the usual product of the underlying algebra. The proof of this statement is surprisingly short, it rests on Jacobson's lemma, which asserts that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, and further elementary manipulations.

3. The (inverted) Jordan triple product and generalized Mazur–Ulam type theorems

Before moving on to the formulation of generalized Mazur–Ulam type theorem, let us recall the classical version and share some ideas of the proof.

Theorem 3.1 (Mazur–Ulam, 1932)**.** *Let X, Y be real linear normed spaces. Every surjective isometry $T\colon X \to Y$ is affine.*

In the first proof of the Mazur–Ulam theorem, one major step is to show that an isometry $T\colon X \to Y$ (more precisely, isometric isomorphism) preserves the geometric midpoint, the arithmetic mean as well. From this it follows that T respects dyadic convex combinations and thus, by the continuity of the isometry T we infer that T is affine. In this way a geometric preserver problem in fact can be reduced to an algebraic one. This idea has been used by L. Molnár and his coauthors and led to a number of generalized Mazur–Ulam type theorems. Here we present the most general version. To this end, we need the concept of point-reflection geometries. Let $\mathcal{X}$ be a set equipped with a binary operation "$\diamond$" which satisfies the following conditions:

(p1) $a \diamond a = a$ for every $a \in \mathcal{X}$;
(p2) $a \diamond (a \diamond b) = b$ for every $a, b \in \mathcal{X}$;

(p3) the equation $x \diamond a = b$ has a unique solution $x \in \mathcal{X}$ for any given $a, b \in \mathcal{X}$.

Then the pair $(\mathcal{X}, \diamond)$ is said to be a point-reflection geometry. The announced Mazur–Ulam type result of Molnár concerning generalized distance measures[1] reads as follows.

Theorem 3.2 (L. Molnár, 2015). *Assume that $(\mathcal{X}, \diamond)$ and $(\mathcal{Y}, \star)$ form point-reflection geometries. Let d and ϱ be two generalized distance measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Select $a, b \in \mathcal{X}$ and set*

$$L_{a,b}\colon = \{x \in \mathcal{X} \colon d(a,x) = d(x, b \diamond a) = d(a,b)\}.$$

Furthermore, we shall assume the following.

(b1) $d(b \diamond x, b \diamond x') = d(x', x)$ *for* $x, x' \in \mathcal{X}$;
(b2) $\sup\{d(x,b) \colon x \in L_{a,b}\} < \infty$;
(b3) *there is a constant* $K > 1$ *such that*

$$d(x, b \diamond x) \geq K \cdot d(x,b), \qquad x \in L_{a,b}.$$

Suppose that $\varphi \colon \mathcal{X} \to \mathcal{Y}$ *is a surjective map with*

$$\varrho(\varphi(x), \varphi(x')) = d(x,x'), \qquad x, x' \in \mathcal{X},$$

and also that

(b4) *the element* $c \in \mathcal{Y}$ *with* $c \star \varphi(a) = \varphi(b \diamond a)$ *satisfies* $\varrho(c \star y, c \star y') = \varrho(y', y)$ *for* $y, y' \in \mathcal{Y}$.

Then we necessarily have

$$\varphi(b \diamond a) = \varphi(b) \star \varphi(a).$$

As noted in [10], the above theorem is of Mazur–Ulam type in the sense that from this the usual Mazur–Ulam theorem can be concluded with little effort. Indeed, defining the operations $x_1 \diamond x_2 = 2x_1 - x_2$ and $y_1 \star y_2 = 2y_1 - y_2$ makes the real linear spaces X and Y point-reflection geometries. Then by the theorem, a real linear isometry T satisfies $T(2x_1 - x_2) = 2T(x_1) - T(x_2)$ which implies that T preserves the arithmetic mean, and the standard argument at the end of the original proof of the Mazur–Ulam theorem (see above) can be applied to show that T is affine. In the following we show how the above Mazur–Ulam type theorems are applicable in the study of certain isometries and distance measures on the positive definite cones of different operator algebras.

Consider the distance measure between positive operators

$$d_{N,f}(a,b) = N(f(a^{-1/2} b a^{-1/2}))$$

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[1]If $\mathcal{X}$ is a set, a function $d \colon \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a generalized distance measure if $d(x,y) = 0$ holds exactly when $x = y$.

where N is a complete, symmetric norm[2] and $f\colon]0,\infty[\to\mathbb{R}$ is a function satisfying

(c1) $f(x)=0$ exactly when $x=1$;

(c2) there is a real number $K>1$ such that $|f(x^2)|\geq K\cdot|f(x)|$ for $x\in]0,\infty[$.

Note that from the distance measure $d_{N,f}$ one can recover the usual Thompson metric, by taking $N=\|\cdot\|$ and $f=\log$.

Concerning the above type distance measure, it is shown in [10] that a so-called generalized isometry T, meaning

$$d_{N,f}(a,b)=d_{N,f}(T(a),T(b)) \qquad \text{for } \ a,b\in A_+^{-1},$$

preserves the inverted Jordan triple product, that is, we have

$$T(ab^{-1}a)=T(a)T(b)^{-1}T(a)$$

for every $a,b\in A_+^{-1}$. Then composing our map with a suitable congruence transformation (namely, the congruence by the element $T(e)^{-1/2}$), it can also be assumed that the transformation in question is a unital Jordan triple isomorphism as well, that is, $T(e)=e$ and

$$T(aba)=T(a)T(b)T(a) \qquad \text{for } \ a,b\in A_+^{-1}.$$

The question arises naturally how to proceed. In fact, we have two possibilities. The first is that we try to describe the Jordan triple isomorphisms directly. This has been done in [10] in the setting of von Neumann factors that are not of type I_2. More precisely, we have the following result.

Theorem 3.3 (L. Molnár, 2015)**.** *Assume that A,B are von Neumann algebras such that A is a factor not of type I_2. Let $T\colon A_+^{-1}\to B_+^{-1}$ be a continuous Jordan triple isomorphism. Then there exists either an algebra *-isomorphism, or there exists an algebra *-antiisomorphism $\theta\colon A\to B$, a number $\varepsilon\in\{-1,1\}$, and a continuous tracial*[3] *linear functional $\tau\colon A\to\mathbb{C}$ which is real valued on the set of self-adjoint elements and $\tau(e)\neq-\varepsilon$ such that*

$$T(a)=\exp(\tau(\log a))\theta(a^\varepsilon)$$

holds for every $a\in A_+^{-1}$. If A is of one of the types I_∞, II_∞, III, then the functional τ vanishes.

Note that the converse of the above theorem is also true, namely, the last displayed formula defines a continuous Jordan triple isomorphism. Further we

[2]A norm on the C^*-algebra A is called symmetric if $N(axb)\leq\|a\|N(x)\|b\|$ holds for every $a,b,x\in A$.

[3]A functional $\tau\colon A\to\mathbb{C}$ is called tracial whenever $\tau(ab)=\tau(ba)$ holds for all $a,b\in A$.

mention that in the proof a rather strong result on commutativity preserving maps between centrally closed prime algebras has been applied. This is the main reason why the arguments cannot be carried out to a more general setting.

From the above result it can be derived the following structure theorem concerning distance measures. For more details we refer the reader to [10].

Theorem 3.4 (L. Molnár, 2015). *Let A, B be von Neumann algebras with complete, symmetric norms N, M, respectively. Assume that $f, g\colon]0,\infty[\to \mathbb{R}$ are continuous functions both satisfying* (c1) *and f also fulfilling* (c2). *Suppose that A is a factor not of type I_2. Let $T\colon A_+^{-1} \to B_+^{-1}$ be a surjective map with*

$$d_{N,f}(a,b) = d_{M,g}(T(a),T(b)) \qquad \text{for all} \quad a,b \in A_+^{-1}.$$

Then there exists either an algebra *-isomorphism, or there exists an algebra *-antiisomorphism $\theta\colon A \to B$, a number $\varepsilon \in \{-1,1\}$, an element $d \in B_+^{-1}$, and a continuous tracial linear functional $\tau\colon A \to \mathbb{C}$ which is real valued on the set of self-adjoint elements and $\tau(e) \neq -\varepsilon$ such that

$$T(a) = \exp(\tau(\log a))d\theta(a^\varepsilon)d$$

holds for every $a \in A_+^{-1}$. If A is an infinite factor, then the functional τ is missing.

As for the second possibility, we can apply the so-called commutative diagram argument involving the original Mazur–Ulam theorem to obtain that

$$T(a) = \exp(h(\log a))$$

for every $a \in A_+^{-1}$ where h is a linear isometry on the set of self-adjoint elements. Once the structure of h is known, it can be applied to describe T. This was the approach of the proof of [7, Theorem 4.]. To formulate this result, let us consider the following properties concerning the continuous function $f\colon]0,\infty[\to \mathbb{R}$:

(c2') for some real number $c > 0$, we have $|f(x)| \geq c$ outside a neighborhood of 1;
(c3') function f is differentiable at $x = 1$ with nonvanishing derivative;
(c4') $|f(t)| \neq |f(t^{-1})|$ for some $t \in]0,\infty[$.

Theorem 3.5 (O. Hatori and L. Molnár, 2017). *Let A, B be C^*-algebras. Suppose that $T\colon A_+^{-1} \to B_+^{-1}$ is a surjection, and consider the following statements.*

(1) *There are continuous functions $f, g\colon]0,\infty[\to \mathbb{R}$ which satisfy* (c1) *and* (c2')–(c3'), *and we have*

$$\|f(a^{-1/2}ba^{-1/2})\| = \|g(T(a)^{-1/2}T(b)T(a)^{-1/2})\|$$

for every $a, b \in A_+^{-1}$.

(2) *There is a Jordan* **-isomorphism* $J\colon A\to B$*, an element* $d\in B_+^{-1}$*, a central projection* p *and a real number* $c>0$ *such that*

$$T(a)=d(pJ(a)^c+p^\perp J(a)^{-c})d$$

is satisfied for all $a\in A_+^{-1}$.

(3) *There is a Jordan* **-isomorphism* $J\colon A\to B$*, an element* $d\in B_+^{-1}$ *and a central projection* p *such that*

$$T(a)=d(pJ(a)+p^\perp J(a)^{-1})d$$

holds for all $a\in A_+^{-1}$.

(4) *There is a Jordan* **-isomorphism* $J\colon A\to B$ *and an element* $d\in B_+^{-1}$ *such that*

$$T(a)=dJ(a)d \qquad \text{for all } \ a\in A_+^{-1}.$$

Then we have (1) $\Longrightarrow$ (2)*. If* $f=g$*, we have* (1) $\Longrightarrow$ (3)*. Moreover, if* $f=g$ *and* (c4') *holds, then* (1) $\Longleftrightarrow$ (4)*.*

Here the result of Kadison on linear norm isometries between self-adjoint parts of C^*-algebras has been employed, which asserts that any such map is necessarily implemented by a Jordan *-isomorphism and a multiplication by a central symmetry.

Note that the above result significantly extends the former structural result [5, Theorem 5] on Thompson isometries on the spaces of positive invertible elements where only the function $f=g=\log$ appeared. In addition, the original formulation of the above mentioned Hatori–Molnár theorem contains further characterizations of Jordan *-isomorphisms incorporating the spectrum and the spectral radius as well. For more details see [7, (4.1) and (4.2) in Theorem 4].

We remark that the proof techniques surveyed in the current section with smaller modifications can be applied to obtain structural result on isometries of certain compact Lie groups. As for investigations in this direction, we mention the publications [1], [3], [4], [6].

4. Thompson isometries of JB algebras

In the previous section the structural results were formulated for C^*-algebras or for certain class of von Neumann algebras. Very recently B. Lemmens, M. Roelands and M. Wortel in [9] pointed out that as for the structural result concerning the Thompson isometries the C^*-algebra setting is slightly restrictive, since the result of O. Hatori and L. Molnár remains valid in the more abstract setting of JB algebras too. Recall that a Jordan algebra $(A,\circ)$ is a commutative, not necessarily associative algebra such that

$$a\circ(b\circ a^2)=(a\circ b)\circ a^2$$

holds for every element $a, b \in A$. A JB algebra is a normed, complete real Jordan algebra satisfying

$$\|a \circ b\| \leq \|a\| \cdot \|b\|,$$
$$\|a^2\| = \|a\|^2,$$
$$\|a^2\| \leq \|a^2 + b^2\|$$

for all $a, b \in A$. Important examples of JB algebras are given by the Euclidean Jordan algebras and by the self-adjoint part of a C^*-algebra whenever it is equipped with the Jordan product $a \circ b = (ab + ba)/2$.

Denote by $\mathrm{JB}[a, e]$ the JB algebra generated by a and the unit element e. Then the spectrum of a consists of those real numbers λ such that $a - \lambda e$ is not invertible in $\mathrm{JB}[a, e]$. An element with nonnegative spectrum is called positive. The cone of positive elements is denoted by A_+ and its interior, which consists of positive elements with strictly positive spectrum, is denoted by A_+°. In any JB algebra A_+° makes A an order unit space with order unit e, that is, we have

$$\|a\| = \inf\{t > 0 \colon -te \leq a \leq te\}.$$

So the Thompson metric d_T can be defined on A_+° as follows. For $a, b \in A_+^\circ$, we set

$$M(a/b) = \inf\{t > 0 \colon a \leq tb\}$$

and then

$$d_T(a, b) = \log \max\{M(a/b), M(b/a)\}.$$

In terms of the quadratic representation, one can derive a straightforward formula for d_T. To do so, define the triple product $\{\cdot, \cdot, \cdot\}$ as

$$\{a, b, c\} := (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c)$$

for every $a, b, c \in A$. Then the linear transformation $U_a \colon A \to A$ which is given by

$$U_a(b) := \{a, b, a\}$$

is called the quadratic representation of a, and we can write

$$d_T(a, b) = \|\log U_{b^{-1/2}}(a)\|$$

for every $a, b \in A_+^\circ$. In [9] the authors achieved the following result concerning Thompson isometries in the setting of JB algebras.

Theorem 4.1 (B. Lemmens, M. Roelands and M. Wortel, 2018)**.** *Let A, B be JB algebras. A map $T \colon A_+^\circ \to B_+^\circ$ is a bijective isometry with respect to the Thompson metric if and only if there is a Jordan isomorphism $J \colon A \to B$, an* *element $b \in B_+^\circ$, and a central projection p such that*

$$T(a) = U_b(pJ(a) + p^\perp J(a)^{-1})$$

for all $a \in A_+^\circ$.

The proof basically follows the arguments given in [5], but it is adjusted to the setting of JB algebras. For instance, the Jordan triple product is replaced by $U_a b$, the Pusz–Woronowitz geometric mean turns to $U_{a^{1/2}}(U_{a^{-1/2}}b)^{1/2}$ and so on. Moreover, the proof rests heavily on the forthcoming result [8, Theorem 1.4] on bijective linear isometries of JB algebras, which plays the role of Kadison's result on linear norm isometries of self-adjoint elements.

Theorem 4.2 (J. M. Isidro and Á. Rodríguez-Palacios, 1995)**.** *Let $h\colon A \to B$ be a bijective linear isometry. Then $h(a) = sJ(a)$ where s is a central symmetry and $J\colon A \to B$ is a Jordan isomorphism.*

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