Automorphic loops and metabelian groups

Mark Greer, Lee Raney

Abstract. Given a uniquely 2-divisible group G, we study a commutative loop $(G, \circ)$ which arises as a result of a construction in "Engelsche elemente noetherscher gruppen" (1957) by R. Baer. We investigate some general properties and applications of "$\circ$" and determine a necessary and sufficient condition on G in order for $(G, \circ)$ to be Moufang. In "A class of loops categorically isomorphic to Bruck loops of odd order" (2014) by M. Greer, it is conjectured that G is metabelian if and only if $(G, \circ)$ is an automorphic loop. We answer a portion of this conjecture in the affirmative: in particular, we show that if G is a split metabelian group of odd order, then $(G, \circ)$ is automorphic.

Keywords: metabelian groups; automorphic loops; Bruck loops; Moufang loops

Classification: 20N05

1. Introduction

A loop $(Q, \cdot)$ consists of a set Q with a binary operation $\cdot : Q \times Q \to Q$ such that (i) for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$, and (ii) there exists $1 \in Q$ such that $1x = x1 = x$ for all $x \in Q$. Standard references for loop theory are [3], [14].

Let G be a uniquely 2-divisible group, that is, a group in which the map $x \mapsto x^2$ is a bijection. On G we define a new binary operation as follows:

$$x \circ y = xy[y, x]^{1/2}. \tag{1.1}$$

Here $a^{1/2}$ denotes the unique $b \in G$ satisfying $b^2 = a$ and $[y, x] = y^{-1}x^{-1}yx$. Though it is not obvious, $(G, \circ)$ is a commutative loop with neutral element 1. Moreover, this loop is *power-associative*, which informally means that integer powers of elements can be defined unambiguously, and powers in G and powers in $(G, \circ)$ coincide. It turns out that $(G, \circ)$ lives in a variety of loops called Γ-loops (defined in Section 2), which include commutative RIF loops, see [10], and commutative automorphic loops, see [8] and [13].

If G is nilpotent of class at most 2, then $(G, \circ)$ is an abelian group. In this case, the passage from G to $(G, \circ)$ is called the "Baer trick", see [7]. This construction

DOI 10.14712/1213-7243.2020.043

seems to first appear in [1]. It was utilized by H. Bender in [2] to provide an alternative proof of the following result due to J. G. Thompson in [15].

Theorem 1.1. *Let p be an odd prime and let A be the semidirect product of a p-subgroup P with a normal p'-subgroup Q. Suppose that A acts on a p-group G such that*

$$C_G(P) \leq C_G(Q).$$

Then Q acts trivially on G.

Our goal is to study $(G, \circ)$ with different restrictions on G. We show that $(G, \circ)$ is a commutative Moufang loop *if and only if* G is uniquely 2-divisible 2-Engel (Theorem 2.9) and give an alternative proof to Baer that if $(G, \circ)$ is an abelian group then G has nilpotency class at most 2 (Corollary 3.11). Our main result is that if G is uniquely 2-divisible split-metabelian then $(G, \circ)$ is a commutative automorphic loop (Theorem 3.3). Finally we end with some general facts about $(G, \circ)$ when G is metabelian and open problems.

2. Preliminaries

To avoid excessive parentheses, we use the following convention:

- ◦ multiplication "$\cdot$" will be less binding than divisions "$\backslash$", "$/$";
- ◦ divisions are less binding than juxtaposition.

For example $xy/z \cdot y\backslash xy$ reads as $((xy)/z)(y\backslash(xy))$. To avoid confusion when both "$\cdot$" and "$\circ$" are in a calculation, we denote divisions by "$\backslash_.$" and "$\backslash_\circ$", respectively.

In a loop Q, the left and right translations by $x \in Q$ are defined by $yL_x = xy$ and $yR_x = yx$, respectively. We thus have "$\backslash$", "$/$" as $x\backslash y = yL_x^{-1}$ and $y/x = yR_x^{-1}$. We define the *left section* of Q, $L_Q = \{L_x \colon x \in Q\}$, *left multiplication group* of Q, $\mathrm{Mlt}_\lambda(Q) = \langle L_x \colon x \in Q\rangle$ and *multiplication group* of Q, $\mathrm{Mlt}(Q) = \langle R_x, L_x \colon x \in Q\rangle$. We define the *inner mapping group* of Q, $\mathrm{Inn}(Q) = \mathrm{Mlt}(Q)_1 = \{\theta \in \mathrm{Mlt}(Q) \colon 1\theta = 1\}$. It is well known that $\mathrm{Inn}(Q)$ has the standard generators $L_{x,y}, R_{x,y}$, and T_x, see [3], where

$$L_{x,y} = L_x L_y L_{yx}^{-1}, \qquad R_{x,y} = R_x R_y R_{xy}^{-1}, \qquad T_x = R_x L_x^{-1}.$$

A loop Q is an *automorphic loop* if every inner mapping of Q is an automorphism of Q, $\mathrm{Inn}(Q) \leq \mathrm{Aut}(Q)$. A loop is Moufang if it satisfies $xy \cdot zx = x(yz \cdot x)$ and is a Bruck loop if it satisfies both $x(y \cdot xz) = (x \cdot yx)z$ and $(xy)^{-1} = x^{-1}y^{-1}$ where x^{-1} is the unique two-sided inverse of x.

Definition 2.1. A loop $(Q, \cdot)$ is a Γ-loop if the following hold:

(Γ_1) Loop Q is commutative.

(Γ_2) Loop Q has the automorphic inverse property (AIP): $\forall\, x, y \in Q,\ (xy)^{-1} = x^{-1}y^{-1}$.
(Γ_3) $\forall\, x \in Q,\ L_x L_{x^{-1}} = L_{x^{-1}} L_x$.
(Γ_4) $\forall\, x, y \in Q,\ P_x P_y P_x = P_{yP_x}$ where $P_x = R_x L_{x^{-1}}^{-1} = L_x L_{x^{-1}}^{-1}$.

We recall some definitions and notation, which is standard in most group theory books. We define $[x_0, x_1, \ldots, x_n] = [[[x_0, x_1], \ldots], x_n]$. Hence, $[x, y, z] = [[x, y], z]$. The following identities are well-known:

Lemma 2.2. *Let $x, y, z \in G$ for a group G:*

- $[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z]$;
- $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$;
- $[x, y^{-1}] = [y, x]^{y^{-1}}$, *and* $[x^{-1}, y] = [y, x]^{x^{-1}}$;
- $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = [x, y, z^x][z, x, y^z][y, z, x^y] = 1$.

Recall that the *lower central series* of a group is $G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots$, with $\gamma_i(G)$ defined inductively by

$$\gamma_1(G) = G, \qquad \gamma_{i+1}(G) = [\gamma_i(G), G]$$

and the *upper central series* of a group G is $1 = \zeta^0(G) \leq \zeta^1(G) \leq \ldots$, with $\zeta^i(G)$ defined inductively by

$$\zeta^0(G) = 1, \qquad \frac{\zeta^{i+1}(G)}{\zeta^i(G)} = Z\Big(\frac{G}{\zeta^i(G)}\Big)$$

where if $\pi_i \colon G \to \zeta^i(G)$ is the natural projection map, then $\zeta^{i+1}(G)$ is the inverse image of the center.

Finally, a group G is *nilpotent* if its upper central series has finite length, it means that its lower central series has finite length. Therefore, we have G is *nilpotency of class n* if and only if $[x_0, x_1 \ldots, x_n] = 1$ for all $x_i \in G$. A group G is 2-*Engel* if $[x, y, y] = 1$, alternatively $xx^y = x^y x$ for all $x, y \in G$. Lastly recall the *derived subgroup* of G, $G' = \langle [x, y] \colon x, y \in G \rangle$. A group is *metabelian* if $G'' = 1$ (or $[x, y][u, v] = [u, v][x, y]$ for all $x, y, u, v \in G$).

Theorem 2.3 ([1])**.** *Let G be a uniquely 2-divisible group. For all $x, y \in G$, define $x \circ y = xy[y, x]^{1/2}$. Then $(G, \circ)$ is an abelian group if and only if G is nilpotency class 2. Moreover, powers in G coincide with powers in $(G, \circ)$.*

Note that in the proof of the above theorem the restriction to class 2 only appears in the proof of associativity. An immediate question is what properties does $(G, \circ)$ have without the restriction that G be nilpotent of class 2?

Theorem 2.4 ([6])**.** *Let G be a uniquely 2-divisible group. Then $(G, \circ)$ is a Γ-loop. Moreover, powers coincide in G and $(G, \circ)$.*

The main goal of [6] was to establish a connection to Bruck loops and Γ-loops of odd order.

Theorem 2.5 ([6]). *There is a one-to-one correspondence between left Bruck loops of odd order n and Γ-loops of odd order n. That is:*

(i) *If $(Q, \cdot)$ is a left Bruck loop of odd order n with $1 \in Q$ identity element, then $(Q, \circ)$ is a Γ-loop of order n where $x \circ y = (1)L_x L_y [L_y, L_x]^{1/2}$.*
(ii) *If $(Q, \cdot)$ is a Γ-loop of odd order n, then $(Q, \oplus)$ is a left Bruck loop of order n where $x \oplus y = (x^{-1}\backslash(y^2x))^{1/2}$.*
(iii) *The mappings in* (i) *and* (ii) *are mutual inverses.*

In general, not much can be said about $(G, \circ)$ without any restrictions on G. However, we do have the following.

Lemma 2.6. *Let G be a uniquely 2-divisible group. Then $Z(G) \leq Z(G, \circ)$.*

Proof: Let $g \in Z(G)$. Then we have

$$\begin{aligned} g \circ (x \circ y) &= gxy[y,x]^{1/2}[xy[y,x]^{1/2}, g]^{1/2} = gxy[y,x]^{1/2} \\ &= gxy[y,gx]^{1/2} = (g \circ x) \circ y, \\ x \circ (g \circ y) &= xgy[gy,x]^{1/2} = xgy[y,x]^{1/2} = xgy[y,xg]^{1/2} = (x \circ g) \circ y, \\ x \circ (y \circ g) &= xyg[yg,x]^{1/2} = xyg[y,x]^{1/2} = xy[y,x]^{1/2}g \\ &= xy[y,x]^{1/2}g[g, xy[y,x]^{1/2}]^{1/2} = (x \circ y) \circ g. \end{aligned}$$

Thus $g \in Z(G, \circ)$. □

It turns out that $(G, \circ)$ has a lot of structure if G is 2-Engel.

Lemma 2.7. *Let G be uniquely 2-divisible. Then $xy[y,x]^{1/2} = (xy^2x)^{1/2}$ if and only if G is 2-Engel.*

Proof: Before beginning the proof, we first note that if G is uniquely 2-divisible and $a, b \in G$ commute, then a commutes with $b^{1/2}$. Indeed, since $(a^{-1}b^{1/2}a)^2 = a^{-1}ba$, it follows that $(a^{-1}ba)^{1/2} = a^{-1}b^{1/2}a$. Thus, since a and b commute, we have that $b^{1/2} = a^{-1}b^{1/2}a$, as desired.

Suppose G is 2-Engel. Hence, both x and y commute with $[y,x]$. Then by the note above,

$$(xy[y,x]^{1/2})^2 = xy[y,x]^{1/2}xy[y,x]^{1/2} = (xy)^2[y,x] = xy^2x.$$

Taking square roots of both sides gives the desired results.

For the reverse direction, set $u = [y,x]^{1/2}$. By hypothesis, $xyuxyu = xyyx$ and canceling gives $uxyu = yx$. Multiplying both sides on the right by u gives

$yxu = uxyu^2 = uxyy^{-1}x^{-1}yx = uyx$. Since yx commutes with u (Theorem 2.4) it commutes with any power of u. Thus $yx[y,x] = [y,x]yx$. Replacing x with $y^{-1}x$ to get $x[y,y^{-1}x] = [y,y^{-1}x]x$. But $[y,y^{-1}x] = y^{-1}x^{-1}yyy^{-1}x = [y,x]$. Therefore $x[y,x] = [y,x]x$, that is, $[y,x,x] = 1$. Thus, G is 2-Engel. □

Defining multiplication with $x \oplus y = (xy^2x)^{1/2}$ has been well studied by R. H. Bruck, G. Glaubermann, and others.

Theorem 2.8 ([5]). *Let G be uniquely 2-divisible group. For all $x, y \in G$, define $x \oplus y = (xy^2x)^{1/2}$. Then $(G, \oplus)$ is a Bruck loop. Moreover, powers in G coincide with powers in $(G, \circ)$.*

Finally, it is well known that commutative Bruck loops are Moufang, see [3].

Theorem 2.9. *Let G be uniquely 2-divisible. Then G is 2-Engel if and only if $(G, \circ)$ is a commutative Moufang loop.*

Proof: If G is 2-Engel then $(G, \circ) = (G, \oplus)$, and hence a commutative Bruck loop, so Moufang.

Alternatively, set $u = [x,y]^{1/2}$. Using the inverse property,

$$y = x^{-1} \circ (x \circ y) = x^{-1}xyu^{-1}[xyu^{-1}, x^{-1}]^{1/2}.$$

Cancel and multiply on the left by u to get $u = [xyu^{-1}, x^{-1}]^{1/2}$. Squaring both sides gives $u^2 = [xyu^1, x^{-1}] = [yu^{-1}, x^{-1}] = uy^{-1}xyu^{-1}x^{-1}$. Hence $u = y^{-1}xyu^{-1}xy$ after canceling. Multiplying on the left by x^{-1} to get $x^{-1}u = [x,y]u^{-1}x^{-1} = u^2u^{-1}x^{-1} = ux^{-1}$. Since x^{-1} commutes with u it commutes with $u^2 = [x,y]$. Similarly, since $[x,y]$ commutes with x^{-1}, it commutes with x. Hence, G is 2-Engel. □

3. Split metabelian groups

Let G be the semidirect product of a normal abelian subgroup H of odd order acted on (as a group of automorphisms) by an abelian group F of odd order. Products in H and in F are written multiplicatively. We use exponential notation for the action of $\mathrm{Aut}(H)$ on H: given $\theta \in \mathrm{Aut}(H)$, $h \in H$, define $h^\theta = \theta(h)$.

Further, given $m, n \in \mathbb{Z}$ with m and n relatively prime to $|H|$, we make special use of the notation $h^{(m/n)\theta} = (h^{m/n})^\theta = (h^\theta)^{m/n}$. Note that since H is abelian, this convention is consistent with an additional notation: given commuting automorphisms $\theta, \psi \in \mathrm{Aut}(H)$, $h^{\theta+\psi} = h^\theta h^\psi$. Then $G = H \rtimes F = HF$, where

$$h_1f_1h_2f_2 = h_1f_1 \cdot h_2f_2 = h_1h_2^{f_1}f_1f_2$$

for all $h_1, h_2 \in H$, $f_1, f_2 \in F$. Note that G is metabelian (we refer to such groups as *split metabelian*). To proceed, we need a proposition.

Proposition 3.1. *Let H be an abelian group of odd order. Suppose α and β are commuting automorphisms of H with odd order in* $\mathrm{Aut}(H)$. *Then the map $h \mapsto h^{\alpha+\beta}$ is an automorphism of H.*

Proof: Define $\phi\colon H \to H$ by $\phi(h) = h^{\alpha+\beta}$. Clearly, ϕ is a homomorphism. We will show that ϕ is injective. Suppose $h_0 \in H$ such that $\phi(h_0) = 1$. It follows that $h_0^{\alpha} = h_0^{-\beta}$, and thus

$$h_0^{\alpha^2} = (h_0^{\alpha})^{\alpha} = (h_0^{-\beta})^{\alpha} = (h_0^{\alpha})^{-\beta} = (h_0^{-\beta})^{-\beta} = h_0^{\beta^2}.$$

Now, since α, β are commuting, odd-ordered automorphisms of H, there exists some positive, odd integer k such that $\alpha^k = \mathrm{id}_H = \beta^k$. In particular,

$$\begin{aligned} h_0^{\alpha^k} &= h_0^{\beta^k}; \\ (h_0^{\alpha^2})^{\alpha^{k-2}} &= (h_0^{\beta^2})^{\beta^{k-2}}; \\ (h_0^{\beta^2})^{\alpha^{k-2}} &= (h_0^{\beta^2})^{\beta^{k-2}}; \\ (h_0^{\alpha^{k-2}})^{\beta^2} &= (h_0^{\beta^{k-2}})^{\beta^2}. \end{aligned}$$

Since $\beta^2 \in \mathrm{Aut}(H)$, it follows that $h_0^{\alpha^{k-2}} = h_0^{\beta^{k-2}}$. Continuing in this manner, we have that $h_0^{\alpha} = h_0^{\beta}$, and hence $h_0^{\beta} = h_0^{-\beta}$. Since $|H|$ is odd, this implies that $h_0 = 1$. Therefore, ϕ is an injective homomorphism $H \to H$ and is thus an automorphism of H. $\square$

Since F is abelian, Proposition 3.1 implies that if θ is a $\mathbb{Q}$-linear combination of elements of F (where the numerators and denominators of the coefficients are relatively prime to $|H|$), the map $H \to H$, $h \mapsto h^{\theta}$ is an automorphism of H which commutes with any other such linear combination ψ. In particular, note that the aforementioned automorphism has an inverse in $\mathrm{Aut}(H)$. We denote this inverse by $h \mapsto h^{\theta^{-1}}$, and this map also commutes with ψ. We will use this fact throughout the following calculations without specific reference.

Lemma 3.2. *Let $u = hf$, $x = h_1 f_1$, $y = h_2 f_2 \in G$. Then*

- $u^{-1} = h^{-f^{-1}} f^{-1}$;
- $u^{1/2} = h^{(1+f^{1/2})^{-1}} f^{1/2}$;
- $[x, y] = h_1^{f_1^{-1}(-1+f_2^{-1})} h_2^{f_2^{-1}(-f_1^{-1}+1)} \in H$;
- $x \circ y = h_1^{(1+f_2)/2} h_2^{(1+f_1)/2} f_1 f_2$;
- $x \backslash y = x \backslash_{\circ} y = \left(h_1^{-1-f_1^{-1}f_2} h_2^{2}\right)^{(1+f_1)^{-1}} f_1^{-1} f_2$;
- $uL_{x,y} = \left(h^{(1+f_1)(1+f_2)} h_2^{1+ff_1-f-f_1}\right)^{(1+f_1f_2)^{-1}/2} f$.

Proof: First, we compute

$$u\cdot h^{-f^{-1}}f^{-1}=hf\cdot h^{-f^{-1}}f^{-1}=hh^{-f^{-1}f}ff^{-1}=hh^{-1}ff^{-1}=1,$$

and first item is proved.

Next, we compute

$$\begin{aligned}\big(h^{(1+f^{1/2})^{-1}}f^{1/2}\big)^2&=h^{(1+f^{1/2})^{-1}}f^{1/2}\cdot h^{(1+f^{1/2})^{-1}}f^{1/2}\\&=h^{(1+f^{1/2})^{-1}}h^{(1+f^{1/2})^{-1}f^{1/2}}f^{1/2}f^{1/2}.\end{aligned}$$

Setting $k=h^{(1+f^{1/2})^{-1}}\in H$ gives

$$\big(h^{(1+f^{1/2})^{-1}}f^{1/2}\big)^2=kk^{f^{1/2}}f=k^{1+f^{1/2}}f=hf=u,$$

and thus $u^{1/2}=h^{(1+f^{1/2})^{-1}}f^{1/2}$.

Now, we have

$$\begin{aligned}[x,y]&=x^{-1}y^{-1}xy\\&=\big(h_1^{-f_1^{-1}}f_1^{-1}\cdot h_2^{-f_2^{-1}}f_2^{-1}\big)\big(h_1f_1\cdot h_2\cdot f_2\big)\\&=\big(h_1^{-f_1^{-1}}h_2^{-f_2^{-1}f_1^{-1}}f_1^{-1}f_2^{-1}\big)\big(h_1h_2^{f_1}f_1f_2\big)\\&=h_1^{-f_1^{-1}}h_2^{-f_2^{-1}f_1^{-1}}\big(h_1h_2^{f_1}\big)^{f_1^{-1}f_2^{-1}}f_1^{-1}f_2^{-1}f_1f_2\\&=h_1^{-f_1^{-1}+(f_1f_2)^{-1}}h_2^{-(f_1f_2)^{-1}+f_2^{-1}}\cdot 1\\&=h_1^{f_1^{-1}(-1+f_2^{-1})}h_2^{f_2^{-1}(-f_1^{-1}+1)}.\end{aligned}$$

Next, we get

$$\begin{aligned}x\circ y&=h_1f_1\circ h_2f_2\\&=(h_1f_1)(h_2f_2)\cdot[h_2f_2,h_1f_1]^{1/2}\\&=\big(h_1h_2^{f_1}f_1f_2\big)\big(h_2^{f_2^{-1}(-1+f_1^{-1})}h_1^{f_1^{-1}(-f_2^{-1}+1)}\big)^{1/2}\\&=h_1h_2^{f_1}\big(h_2^{f_2^{-1}(-1+f_1^{-1})}h_1^{f_1^{-1}(-f_2^{-1}+1)}\big)^{f_1f_2/2}f_1f_2\\&=h_1^{1+(f_2(-f_2^{-1}+1))/2}h_2^{f_1+(f_1(-1+f_1^{-1})/2}f_1f_2\\&=h_1^{(1+f_2)/2}h_2^{(1+f_1)/2}f_1f_2.\end{aligned}$$

To compute $x\backslash y$, observe that

$$\begin{aligned}x\circ\big(h_1^{-1-f_1^{-1}f_2}h_2^2\big)^{(1+f_1)^{-1}}f_1^{-1}f_2&=h_1f_1\circ\big(h_1^{-1-f_1^{-1}f_2}h_2^2\big)^{(1+f_1)^{-1}}f_1^{-1}f_2\\&=h_1^{(1+f_1^{-1}f_2)/2}\big(h_1^{-1-f_1^{-1}f_2}h_2^2\big)^{(1+f_1)^{-1}((1+f_1)/2)}f_1f_1^{-1}f_2\end{aligned}$$

$$= h_1^{(1+f_1^{-1}f_2)/2+(-1-f_1^{-1}f_2)/2} h_2^{2/2} f_2$$
$$= h_2 f_2 = y,$$

and thus $x\backslash y = \left(h_1^{-1-f_1^{-1}f_2} h_2^2\right)^{(1+f_1)^{-1}} f_1^{-1} f_2$.

Finally, we have

$$\begin{aligned}
uL_{x,y} &= \frac{(u \circ x) \circ y}{x \circ y} \\
&= \frac{\left(h^{(1+f_1)/2} h_1^{(1+f)/2} f f_1\right) \circ h_2 f_2}{h_1^{(1+f_2)/2} h_2^{(1+f_1)/2} f_1 f_2} \\
&= \frac{\left(h^{(1+f_1)/2} h_1^{(1+f)/2}\right)^{(1+f_2)/2} h_2^{(1+ff_1)/2} f f_1 f_2}{h_1^{(1+f_2)/2} h_2^{(1+f_1)/2} f_1 f_2} \\
&= \frac{h^{(1+f_1)(1+f_2)/4} h_1^{(1+f)(1+f_2)/4} h_2^{(1+ff_1)/2} f f_1 f_2}{h_1^{(1+f_2)/2} h_2^{(1+f_1)/2} f_1 f_2} \\
&= \Big(\left(h_1^{(1+f_2)/2} h_2^{(1+f_1)/2}\right)^{-1-(f_1 f_2)^{-1}(f f_1 f_2)} \\
&\quad \cdot \left(h^{(1+f_1)(1+f_2)/4} h_1^{(1+f)(1+f_2)/4} h_2^{(1+ff_1)/2}\right)^2 \Big)^{(1+f_1 f_2)^{-1}} (f_1 f_2)^{-1} (f f_1 f_2) \\
&= \Big(\left(h_1^{(1+f_2)/2} h_2^{(1+f_1)/2}\right)^{-1-f} \\
&\quad \cdot \left(h^{(1+f_1)(1+f_2)/2} h_1^{(1+f)(1+f_2)/2} h_2^{1+ff_1}\right) \Big)^{(1+f_1 f_2)^{-1}} f \\
&= \Big(h^{(1+f_1)(1+f_2)/2} h_1^{(1+f_2)/2(-1-f)+(1+f)(1+f_2)/2} \\
&\quad \cdot h_2^{((1+f_1)/2)(-1-f)+(1+ff_1)} \Big)^{(1+f_1 f_2)^{-1}} f \\
&= \left(h^{(1+f_1)(1+f_2)/2} h_1^0 h_2^{(1+ff_1-f-f_1)/2} \right)^{(1+f_1 f_2)^{-1}} f \\
&= \left(h^{(1+f_1)(1+f_2)} h_2^{1+ff_1-f-f_1} \right)^{(1+f_1 f_2)^{-1}/2} f.
\end{aligned}$$

□

Theorem 3.3. *Let G be a split metabelian group of odd order. Then $(G, \circ)$ is an automorphic loop.*

Proof: Since $(G, \circ)$ is commutative for any $x, y \in G$, $L_{x,y} = R_{x,y}$ and $T_x = \mathrm{id}_G$. Thus, to prove that $(G, \circ)$ is automorphic, it suffices to show that $L_{x,y}$ is a loop homomorphism. We must show that $uL_{x,y} \circ vL_{x,y} = (u \circ v)L_{x,y}$ for all u, v, $x, y \in G$. Thus, let $u = hf$, $v = kg$, $x = h_1 f_1$, $y = h_2 f_2 \in G$. We first compute, by Lemma 3.2

$$\begin{aligned}
uL_{x,y} \circ vL_{x,y} &= \Big(\big(h^{(1+f_1)(1+f_2)} h_2^{1+ff_1-f-f_1}\big)^{(1+f_1f_2)^{-1}/2} f\Big)\\
&\quad \circ \Big(\big(k^{(1+f_1)(1+f_2)} h_2^{1+gf_1-g-f_1}\big)^{(1+f_1f_2)^{-1}/2} g\Big)\\
&= \Big(\big(h^{(1+f_1)(1+f_2)} h_2^{1+ff_1-f-f_1}\big)^{(1+f_1f_2)^{-1}/2}\Big)^{(1+g)/2}\\
&\quad \cdot \Big(\big(k^{(1+f_1)(1+f_2)} h_2^{1+gf_1-g-f_1}\big)^{(1+f_1f_2)^{-1}/2}\Big)^{(1+f)/2} fg\\
&= \big(h^{(1+f_1)(1+f_2)(1+g)/2} k^{(1+f_1)(1+f_2)(1+f)/2}\\
&\quad \cdot h_2^{(1+ff_1-f-f_1)(1+g)/2+(1+gf_1-g-f_1)(1+f)/2}\big)^{(1+f_1f_2)^{-1}/2} fg\\
&= \big(h^{(1+f_1)(1+f_2)(1+g)/2} k^{(1+f_1)(1+f_2)(1+f)/2}\\
&\quad \cdot h_2^{1-fg+fgf_1-f_1}\big)^{(1+f_1f_2)^{-1}/2} fg.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(u \circ v)L_{x,y} &= \big(h^{(1+g)/2} k^{(1+f)/2} fg\big) L_{x,y}\\
&= \Big(\big((h^{(1+g)/2} k^{(1+f)/2}\big)^{(1+f_1)(1+f_2)} h_2^{1+fgf_1-fg-f_1}\Big)^{(1+f_1f_2)^{-1}/2} fg\\
&= \Big(h^{(1+g)(1+f_1)(1+f_2)/2} k^{(1+f)(1+f_1)(1+f_2)/2}\\
&\quad \cdot h_2^{1+fgf_1-fg-f_1}\Big)^{(1+f_1f_2)^{-1}/2} fg\\
&= uL_{x,y} \circ vL_{x,y}.
\end{aligned}$$

□

As an immediate corollary, we see that if G is any group such that all groups of order $|G|$ are split metabelian, then $(G, \circ)$ is an automorphic loop. In particular, disregarding the cases where G is abelian, we obtain the following.

Corollary 3.4. *If $|G|$ is any one of the following (for distinct odd primes p and q), then $(G, \circ)$ is automorphic.*

- *pq (where p divides $q-1$),*
- *p^2q,*
- *p^2q^2.*

Corollary 3.5. *Let p and q be distinct odd primes with p dividing $q-1$. Then there is exactly one nonassociative, commutative, automorphic loop of order pq.*

PROOF: Let G be a group of order pq. Then $(G, \circ)$ is automorphic (Theorem 3.3). Suppose Q is a Γ-loop of order pq. Then $(Q, \oplus)$ is a Bruck loop. The only two options are (i) $(Q, \oplus)$ is abelian or (ii) $(Q, \oplus)$ is the unique nonassociative Bruck loop of order pq, see [9]. For (i), $Q = (Q, \oplus)$ and hence an abelian group (so automorphic). For (ii), $(G, \oplus_\circ) = (Q, \oplus)$ must be the same nonassociative Bruck loop, and hence, $Q = (G, \circ)$. □

The only known examples where $(G, \circ)$ is not automorphic occur when G is not metabelian.

Conjecture 3.6. *Let G be a uniquely 2-divisible group. Then $(G, \circ)$ is automorphic if and only if G is metabelian.*

For a general metabelian group G, we have the following results.

Lemma 3.7. *Let G be a uniquely 2-divisible, metabelian group. Then for all $x, y, z \in G$*

- $[[x, y]^{1/2}, z] = [[x, y], z]^{1/2}$;
- $[x, y, z][z, x, y][y, z, x] = 1$.

Theorem 3.8. *Let G be uniquely 2-divisible and metabelian. Then $\zeta^2(G) \trianglelefteq Z(G, \circ)$.*

Proof: If $g \in \zeta^2(G)$, then it is clear that $gT_x = x$. We show $gL_{x,y} = g$. First, it is clear that $[g, x, y] = 1 \Leftrightarrow [x, g, y] = 1 \Leftrightarrow [x, y, g] = 1$. Thus, we have $[g, x]y = y[g, x]$ and $[x, y]g = g[x, y]$. Now,

$$\begin{aligned}
y \circ (x \circ g) &= yxg[g, x]^{1/2}[xg, y]^{1/2}[[x, g]^{1/2}, y]^{1/2} \\
&= yxg[g, x]^{1/2}[xg, y]^{1/2} \\
&= yxg[g, x]^{1/2}[x, y]^{1/2}[g, y]^{1/2} \\
&= yxg[x, y]^{1/2}[g, y]^{1/2}[g, x]^{1/2} \\
&= yx[x, y]^{1/2}g[g, yx]^{1/2} \\
&= yx[x, y]^{1/2}g[g, yx]^{1/2}[g, [x, y]^{1/2}]^{1/2} \\
&= (y \circ x) \circ g.
\end{aligned}$$

Hence, $gL_{x,y} = g$. □

Theorem 3.9. *Let G be uniquely 2-divisible and of nilpotency class 3. Then $Z(G, \circ) = \zeta^2(G)$.*

Proof: By the previous theorem, we have $\zeta^2(G) \leq Z(G, \circ)$. From Lemma 3.7, we have $[y, x, z][z, y, x] = [y, [x, z]]$ by interchanging x and y. Thus,

$$[[y, x]^{1/2}, z]^{1/2}[[z, y]^{1/2}, x]^{1/2} = [y, [x, z]^{1/2}]^{1/2}. \tag{$*$}$$

Let $g \in Z(G,\circ)$. We show $[g,x,y]=1$ for all $x,y \in G$ and therefore, $g \in \zeta^2(G)$. Since $g \in Z(G,\circ)$, we have $g\circ(x\circ y) = x\circ(y\circ g)$. Hence, we have

$$\begin{aligned}
&gxy[y,x]^{1/2}[xy,g]^{1/2}[[y,x]^{1/2},g]^{1/2} = xyg[g,y]^{1/2}[yg,x]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & xyg[g,xy][y,x]^{1/2}[xy,g]^{1/2}[[y,x]^{1/2},g]^{1/2} = xyg[g,y]^{1/2}[yg,x]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & [g,xy][y,x]^{1/2}[xy,g]^{1/2}[[y,x]^{1/2},g]^{1/2} = [g,y]^{1/2}[yg,x]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & [g,xy]^{1/2}[y,x]^{1/2}[[y,x]^{1/2},g]^{1/2} = [g,y]^{1/2}[yg,x]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & [g,y]^{1/2}[g,x]^{1/2}[g,x,y]^{1/2}[y,x]^{1/2}[[y,x]^{1/2},g]^{1/2} = [g,y]^{1/2}[y,x]^{1/2}[y,x,g]^{1/2}\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times [g,x]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & [g,x,y]^{1/2}[[y,x]^{1/2},g]^{1/2} = [y,x,g]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & [g,x,y]^{1/2} = [[y,x]^{1/2},g]^{1/2}[[g,y]^{1/2},x]^{1/2}\\
\Leftrightarrow\ & [g,x,y]^{1/2} = [y,[x,g]^{1/2}]^{1/2} \qquad\qquad (*)\\
\Leftrightarrow\ & [[g,x]^{1/2},y]^{1/2}[[g,x]^{1/2},y]^{1/2}[[x,g]^{1/2},y]^{1/2} = 1\\
\Leftrightarrow\ & [[g,x]^{1/2},y]^{1/2} = 1\\
\Leftrightarrow\ & [g,x,y] = 1.
\end{aligned}$$

□

Corollary 3.10. *Let G be uniquely 2-divisible and of nilpotency class 3. Then $(G,\circ)$ is a commutative loop of nilpotency class 2.*

Proof: We have as sets, $G/\zeta^2(G) = (G,\circ)/Z(G,\circ)$ by Theorem 3.9. Now, since $G/\zeta^2(G)$ is an abelian group, the two sets have the same operation and thus, $(G,\circ)/Z(G,\circ)$ is an abelian group. □

Finally, we give an alternative proof of Baer's result that if $(G,\circ)$ is an abelian group, then G is of nilpotency class at most 2.

Corollary 3.11. *Let G be uniquely 2-divisible. If $(G,\circ)$ is an abelian group, then G is of class at most 2.*

Proof: Since $(G,\circ)$ is an abelian group, $(G,\circ)$ is a commutative Moufang loop. Thus, G is 2-Engel, which implies G is of class at most 3. Thus, by Theorem 3.9, $G = \zeta^2(G)$, and hence G has class at most 2. □

Acknowledgment. Some investigations in this paper were assisted by the automated deduction tool, Prover9, and the finite model builder, Mace4, both developed by McCune in [11]. Similarly, all presented examples were found and verified using the GAP system [4] together with the LOOPS package [12]. We also thank the anonymous referee for several comments that improved the manuscript.

References

[1] Baer R., *Engelsche Elemente Noetherscher Gruppen*, Math. Ann. **133** (1957), 256–270 (German).

[2] Bender H., *Über den größten p'-Normalteiler in p-auflösbaren Gruppen*, Arch. Math. (Basel) **18** (1967), 15–16 (German).

[3] Bruck R. H., *A Survey of Binary Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 20, Reihe: Gruppentheorie, Springer, Berlin, 1958.

[4] *GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra*, `https://www.gap-system.org`, 2008.

[5] Glauberman G., *On loops of odd order*, J. Algebra **1** (1964), 374–396.

[6] Greer M., *A class of loops categorically isomorphic to Bruck loops of odd order*, Comm. Algebra **42** (2014), no. 8, 3682–3697.

[7] Isaacs I. M., *Finite Group Theory*, Graduate Studies in Mathematics, 92, American Mathematical Society, Providence, 2008.

[8] Jedlička P., Kinyon M., Vojtěchovský P., *The structure of commutative automorphic loops*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 365–384.

[9] Kinyon M. K., Nagy G. P., Vojtěchovský P., *Bol loops and Bruck loops of order pq*, J. Algebra **473** (2017), 481–512.

[10] Kinyon M. K., Vojtěchovský P., *Primary decompositions in varieties of commutative diassociative loops*, Comm. Algebra **37** (2009), no. 4, 1428–1444.

[11] McCune W. W., *Prover9, Mace4*, `https://www.cs.unm.edu/~mccune/prover9/`, 2009.

[12] Nagy G. P., Vojtěchovský P., *Computing with small quasigroups and loops*, Quasigroups Related Systems **15** (2007), no. 1, 77–94.

[13] Stuhl I., Vojtěchovský P., *Enumeration of involutory latin quandles, Bruck loops and commutative automorphic loops of odd prime power order*, Nonassociative Mathematics and Its Applications, Contemp. Math., 721, Amer. Math. Soc., Providence, 2019, pages 261–276.

[14] Pflugfelder H. O., *Quasigroups and Loops: Introduction*, Sigma Series in Pure Mathematics, 7, Heldermann Verlag, Berlin, 1990.

[15] Thompson J. G., *Fixed points of p-groups acting on p-groups*, Math. Z. **86** (1964), 12–13.

M. Greer, L. Raney:

Department of Mathematics, One Harrison Plaza,

University of North Alabama, Florence, Alabama, AL 35632, USA

E-mail: mgreer@una.edu

E-mail: lraney@una.edu

(*Received* December 18, 2019*, revised* June 23, 2020)