On finite commutative IP-loops with elementary abelian inner mapping groups of order p^5

Markku Niemenmaa

Abstract. We show that finite commutative inverse property loops with elementary abelian inner mapping groups of order p^5 are centrally nilpotent of class at most two.

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1. Introduction

If Q is a loop then the mappings $L_a(x) = ax$ and $R_a(x) = xa$ are called the left and right translation. These two mappings are permutations on Q for every $a \in Q$ and the permutation group $M(Q) = \langle L_a, R_a \colon a \in Q \rangle$ is called the multiplication group of Q. The stabilizer of the neutral element of Q is the inner mapping group of Q and we denote it by $I(Q)$. If Q is a group then $I(Q) = \mathrm{Inn}(Q)$, the group of inner automorphisms of Q.

The centre $Z(Q)$ of a loop Q contains all elements a with the property that $ax = xa$, $(ax)y = a(xy)$, $(xa)y = x(ay)$ and $(xy)a = x(ya)$ for every $x, y \in Q$. The centre $Z(Q)$ is an abelian group and if we write $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, then we have a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q and $Z_n = Q$, then Q is said to be centrally nilpotent of class n. R. H. Bruck in [1] showed that if Q is centrally nilpotent of class at most two, then $I(Q)$ is an abelian group. P. Csörgő in [3] showed that the converse of Bruck's result is not true by constructing a centrally nilpotent loop Q whose nilpotency class is three and whose inner mapping group $I(Q)$ is an elementary abelian group of order 2^6. More examples and constructions of loops with nilpotency class three and elementary abelian inner mapping groups of order 2^6 were given by A. Drápal and P. Vojtěchovský in [4]. Earlier results by P. Csörgő, T. Kepka and M. Niemenmaa, see [2] and [10], cover the cases where $I(Q)$ is elementary abelian of order p^2 and p^3 and it turned out that Q is then centrally nilpotent of class at most two.

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A loop Q is an inverse property loop (in short, IP-loop) if Q has a unique left and right inverse x^{-1} and $x^{-1}(xy) = y = (yx)x^{-1}$ for every $x, y \in Q$. M. Niemenmaa in [7] proved that if Q is a finite commutative IP-loop and $I(Q)$ is elementary abelian of order p^4, then Q is centrally nilpotent of class at most two. The purpose of this paper is to show that in the case of finite commutative inverse property loops, the nilpotency class is also at most two provided that the inner mapping group is elementary abelian of order p^5.

We consider only finite loops and groups in this paper. The proofs of our main theorems rely on the use of connected transversals in finite groups and this notion and some basic results about these transversals are explained in the following section. For basic facts about loop theory and its connections to group theory the reader is advised to consult [1] and [9].

2. Connected transversals

We shall start with a brief discussion about connected transversals in a group and try to give some insight into the relationship between loops and groups given by this notion.

Let G be a group and $H \leq G$. If A and B are two left transversals to H in G and $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and for every $b \in B$, then we say that the two transversals are H-connected in G. If $A = B$, then we say that A is a selfconnected transversal to H in G. In the following lemmas and theorems we consider some basic properties of H-connected transversals A and B. We denote by H_G the core of H in G (it is the largest normal subgroup of G contained in H).

Lemma 2.1. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

For the proof see [9, Lemma 2.5].

Lemma 2.2. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

For the proof see [9, Proposition 2.7].

Theorem 2.3. *Let H be a nilpotent subgroup of G. If $G = \langle A, B\rangle$ and $H_G = 1$, then H is subnormal in G and $Z(G) > 1$.*

For the proof see [6, Theorem 2.8].

Theorem 2.4. *If H is cyclic and $G = \langle A, B\rangle$, then $G' \leq H$.*

For the proof see [9, Theorem 3.5].

Theorem 2.5. *Let p be a prime number. If $H \cong C_p \times C_p$ and $G = \langle A, B\rangle$, then $G' \leq N_G(H)$.*

For the proof see [10, Lemma 4.2].

Theorem 2.6. *Let p be a prime number. If $H \cong C_p \times C_p \times C_p$ and $G = \langle A, B\rangle$, then $G' \leq N_G(H)$.*

For the proof see [2, Theorem 3.7].

Theorem 2.7. *Let H be an elementary abelian subgroup of order p^4 of G and let A be selfconnected transversal to H in G. If $G = \langle A\rangle$ and $A = A^{-1}$, then $G' \leq N_G(H)$.*

For the proof see [7, Theorem 3.1].

Lemma 2.8. *Let $G = \langle A, B\rangle$. If H is nilpotent and $H_G = 1$, then the core of $HZ(G)$ in G properly contains $Z(G)$.*

For the proof see [8, Lemma 2.6].

Lemma 2.9. *Let H be a nontrivial subgroup of G, $H_G = 1$ and $G = \langle A, B\rangle$. Then $H \cap H^a > 1$ for every $a \in A \cup B$.*

For the proof see [5, Lemma 2.8].

We shall conclude this section by establishing the relation between connected transversals and loop theory. If $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are the sets of left and right translations, then A and B are $I(Q)$-connected transversals in $M(Q)$. Since $M(Q)$ is transitive on Q, it follows that the core of $I(Q)$ in $M(Q)$ is trivial. T. Kepka and M. Niemenmaa proved the following theorem in 1990 [9, Theorem 4.1].

Theorem 2.10. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H of G satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

If Q is a commutative loop, then $A = B$. Furthermore, if Q is a commutative inverse property loop, then $(L_a)^{-1} = L_{a^{-1}}$ and thus $A = A^{-1}$.

3. Main theorems

In this section we consider the situation that $A = B$, $A = A^{-1}$ and H is an elementary abelian group of order p^5. We first introduce the following two lemmas.

Lemma 3.1. *If $H_G = 1$, then $1 \in A$ and $Z(G) \subseteq A$.*

For the proof see [9, page 113] and [7, Lemma 2.3]

Lemma 3.2. *If $ab = ch$, where $a, b, c \in A$ and $h \in H$, then $h \in H \cap H^a \cap H^b$.*

Proof: Now $h = c^{-1}ab$ and $h^{a^{-1}} = (c^{-1}ab)^{a^{-1}} = ac^{-1}a^{-1}cc^{-1}abb^{-1}aba^{-1} \in H$. We also get $h^{b^{-1}} = (c^{-1}ab)^{b^{-1}} = bc^{-1}b^{-1}cc^{-1}ba = bc^{-1}b^{-1}cc^{-1}abh_1 \in H$ (here $h_1 \in H$). Thus $h \in H \cap H^a \cap H^b$. □

Theorem 3.3. *Let H be an elementary abelian subgroup of a finite group G and let H be of order p^5. If A is a selfconnected transversal to H in G, $G = \langle A \rangle$ and $A = A^{-1}$, then $G' \leq N_G(H)$.*

Proof: We shall prove the theorem by induction on the order of G. From Theorems 2.4, 2.5, 2.6 and 2.7 it follows immediately that $H_G = 1$. By Lemma 2.2, $N_G(H) = H \times Z(G)$ and $Z(G) > 1$ by Theorem 2.3. By Lemma 2.8 the core of $HZ(G)$ in G is equal to $KZ(G)$, where $1 < K \leq H$.

If $|K| \geq p^4$, then we conclude by Theorem 2.4 that $G' \leq HZ(G) = N_G(H)$. Thus we may assume that $|K| = p$ or $|K| = p^2$ or $|K| = p^3$. By applying Theorems 2.4–2.7 and Lemma 2.2 on $G/KZ(G)$ and $HZ(G)/KZ(G)$ it follows that $G' \leq N_G(HZ(G)) = HM$. Here $M/KZ(G) = Z(G/KZ(G))$, M is normal in G and $M \cap HZ(G) = KZ(G)$. We shall now divide the proof into three parts depending on the order of K:

1) Let $|K| = p$. Now we can proceed exactly in the same way as in part 1) of the proof of Theorem 3.1 in [7].

2) Now assume that $K \cong C_p \times C_p$. Let $a, b \in A$ and $ab = ch$, where $c \in A$ and $h \in H$. If $d \in A$, then $h^d = (c^{-1}ab)^d = h_1c^{-1}ah_2bh_3 = h_1hb^{-1}h_2bh_3 \in HH^bH$. As $HZ(G)$ is normal in HM and $H^b \leq HM$, we have $h^d \in HZ(G)H^b \leq G$ for every $d \in A$. Thus $h \in \bigcap[HZ(G)H^b]^g$, where g ranges over the elements of G. This intersection is a normal subgroup of G and we denote it by $N(b)$ (thus $N(b)$ is the core of $HZ(G)H^b$ in G). From Lemma 2.1 it follows that $HZ(G)H^b = HN(b)$.

If we write $ab = kf$, where $k \in H$ and $f \in A$, then likewise $k \in N(a)$, where $N(a)$ is naturally the core of $HZ(G)H^a$ in G. Clearly, $N(a) \geq KZ(G)$ for every $a \in A$, $ab \in AN(b)$ and also $ab \in N(a)A$.

If $|N(a) \cap H| \geq p^4$, then $HN(a)/N(a)$ is cyclic and by Theorem 2.4, $G' \leq HN(a) = HZ(G)H^a$. We now consider the conjugates $HZ(G)/KZ(G)$ and $H^aZ(G)/KZ(G)$ and write $HZ(G) \cap H^aZ(G) = LZ(G)$, where $L \leq H$. From Lemma 2.9 it follows that $LZ(G)$ is larger than $KZ(G)$. Now $LZ(G) = Z(HZ(G)H^a)$ and as $HZ(G)H^a$ is normal in G, it follows that the core of $HZ(G)$ is larger than $KZ(G)$, a contradiction. Thus we may assume that $|N(a) \cap H| \leq p^3$ for every $a \in A$.

Then consider the case that $ab = ch$, $N(a) \cap H \neq N(b) \cap H$ and $|N(a) \cap H| = p^3 = |N(b) \cap H|$. By Theorem 2.4, it follows that $G' \leq HN(a)N(b) = HZ(G)H^aH^b$. By Lemma 3.2, $h \in Z(HZ(G)H^aH^b) \leq N_G(H) = H \times Z(G)$. As $Z(HZ(G)H^aH^b)$ is normal in G, we conclude that $h \in K$. Thus $ab \in AK$.

If $ab = ch$ and $N(a) \cap H = K$ or $N(b) \cap H = K$, then $ab \in KA$ or $ab \in AK$. By Lemma 3.1, $AZ(G) \subseteq A$ and as $KZ(G)$ is normal in G, we conclude that $AK = KA$ is a subgroup of G. Thus we see that $A^2 \subseteq AK < G$, contradicting $\langle A \rangle = G$.

3) Now assume that $K \cong C_p \times C_p \times C_p$. In part two of the proof we showed that $|N(a) \cap H| \leq p^3$ for every $a \in A$. As $N(a) \cap H \geq K$, we must have $N(a) \cap H = K$ for every $a \in A$. But then $A^2 \subseteq AK < G$, a contradiction. $\square$

Let Q be a loop and $M(Q)' \leq N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$. This is equivalent of Q being centrally nilpotent of class at most two, see [1], also Section 6 in [11]. By combining Theorem 2.10 with Theorem 3.3 we thus get

Theorem 3.4. *Let Q be a finite commutative IP-loop and let $I(Q)$ be an elementary abelian group of order p^5. Then Q is centrally nilpotent of class at most two.*

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M. Niemenmaa:
Department of Mathematical Sciences, University of Oulu,
Pentti Kaiteran katu 1, Linnanmaa, PL 3000, 90014 Oulu, Finland

E-mail: markku.niemenmaa@oulu.fi

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