Comment.Math.Univ.Carolin. 61,4 (2020) 567–579

Classification of quasigroups according to directions of translations I

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Abstract. It is proved that every translation in a quasigroup has two independent parameters. One of them is a bijection of the carrier set. The second parameter is called a direction here. Properties of directions in a quasigroup are considered in the first part of the work. In particular, totally symmetric, semisymmetric, commutative, left and right symmetric and also asymmetric quasigroups are characterized within these concepts.

The sets of translations of the same direction are under consideration in the second part of the work. Coincidence of these sets defines nine varieties, among them are varieties of LIP, RIP, MIP and CIP quasigroups. Quasigroups in other five varieties also have some invertibility properties.

Keywords: quasigroup; parastrophe; parastrophic symmetry; parastrophic orbit; translation; direction

Classification: 20N05

Introduction

We shall refer to this paper as Part 1, and to its continuation as Part 2. Both papers investigate several fundamental concepts of quasigroup theory from the point of view of parastrophic symmetry, see [7], [9]. Part 1 is concerned with translations. In each quasigroup Q there are defined six types of translations: the left, right and middle translations and their inverses. Two translations may coincide as permutations of Q, and yet be different when considered upon the web of the quasigroup. We shall call each of the translation types a direction and will associate it with one of the elements ι, l, r, s, sl and rs, i.e., the elements of a symmetric group S_3 which are used when describing a parastrophy action. The parastrophy action defined upon translations transforms the middle translation M_a as follows:

$$
\begin{aligned}
{}^{\iota}M_a &= M_a, & {}^{s}M_a &= M_a^{-1}, & {}^{ls}M_a &= L_a,\\
{}^{l}M_a &= L_a^{-1}, & {}^{r}M_a &= R_a, & {}^{rs}M_a &= R_a^{-1}.
\end{aligned}
$$

DOI 10.14712/1213-7243.2021.002

A direction of a translation is defined accordingly—e.g., the direction of a left translation is ls. The main result of Part 1 is the classification of quasigroups Q for which there is a nontrivial subgroup of S_3 such that for each $a \in Q$ there coincide the translations of a with directions from that subgroup. As can be expected, this yields totally symmetric, semisymmetric, commutative, and left and right symmetric quasigroups.

Part 2 of this article is devoted to the sets ${}^{\sigma}\mathcal{M}$ of translations having the same direction $\sigma \in S_3$. It is easy to show that a quasigroup has: a left, right, middle inverse property if and only if the respective equality is true: ${}^{l}\mathcal{M} = {}^{ls}\mathcal{M}$, ${}^{r}\mathcal{M} = {}^{rs}\mathcal{M}$, ${}^{s}\mathcal{M} = \mathcal{M}$. It has been proved that all equalities ${}^{\sigma}\mathcal{M} = {}^{\kappa}\mathcal{M}$ with $\sigma \neq \kappa$ define nine different varieties. There are well-known varieties of LIP, RIP, MIP, CIP quasigroups and some unknown varieties of quasigroups among them. Each of them has some inverse property.

The obtained relations (10)–(15) imply that any translation could be chosen as basic reference. We choose the middle translation because in this case the left translation and its inverse have the directions ls and l, and the right translation and its inverse have the directions r and rs respectively. Moreover, if σ is the direction of a translation, then σs is the direction of its inverse. The middle translation was studied in [1]–[5]. In particular, in [2] the connections of translations with Latin squares were considered.

Preliminaries

In most cases of the article we consider the set of all invertible (quasigroup) operations defined on a set.

Invertible operations. Let Q be a common carrier set for operations being under consideration. A binary operation f is called *invertible* if it is an invertible element in both monoids: the *left symmetric monoid* $\big(\Omega; \underset{l}{\oplus}, e_l\big)$ and the *right symmetric monoid* $\big(\Omega; \underset{r}{\oplus}, e_r\big)$ of binary operations, where Ω is the set of all binary operations and

$$(1)\qquad \begin{aligned} \big(g \underset{l}{\oplus} h\big)(x,y) &:= g(h(x,y),y), \qquad & e_l(x,y) &:= x,\\ \big(g \underset{r}{\oplus} h\big)(x,y) &:= g(x,h(x,y)), \qquad & e_r(x,y) &:= y. \end{aligned}$$

The set of all binary invertible operations is denoted by Δ. In other words, an operation f is invertible, if there exist its left ${}^{l}f$ and right ${}^{r}f$ inverses, i.e., the equalities

$$(2)\qquad f \underset{l}{\oplus} {}^{l}f = e_l, \qquad {}^{l}f \underset{l}{\oplus} f = e_l, \qquad f \underset{r}{\oplus} {}^{r}f = e_r, \qquad {}^{r}f \underset{r}{\oplus} f = e_r$$

are true. Therefore, values of any two of variables in the equality $f(x_1, x_2) = x_3$ uniquely define the third one. Consequently, for every σ in S_3 the relationship

$$ {}^{\sigma}\!f(x_{1\sigma}, x_{2\sigma}) = x_{3\sigma} \quad :\Leftrightarrow \; f(x_1, x_2) = x_3 \tag{3} $$

defines an operation ${}^{\sigma}\!f$ called a *parastrophe* of f. It is easy to prove that for all $\sigma, \kappa \in S_3$ and for all invertible operations f

$$ {}^{\iota}\!f = f, \qquad {}^{\sigma}({}^{\kappa}\!f) = {}^{\sigma\kappa}\!f. \tag{4} $$

The first two equalities from (2) are equivalent to

$$ {}^{l}\!f(x_3, x_2) = x_1 \quad :\Leftrightarrow \; f(x_1, x_2) = x_3. $$

The relationship (3) with $\sigma = (13)$ is

$$ {}^{(13)}\!f(x_3, x_2) = x_1 \quad :\Leftrightarrow \; f(x_1, x_2) = x_3. $$

Thus, ${}^{l}\!f = {}^{(13)}\!f$. Analogously, ${}^{r}\!f = {}^{(23)}\!f$. Therefore, it is convenient to follow the notation $l := (13)$, $r := (23)$, $s := lrl = (12)$, $S_3 := \{\iota, l, r, s, sl, sr\}$.

By (4), $(\sigma, f) \mapsto {}^{\sigma}\!f$ is an action of the group S_3 on the set Δ of all binary invertible operations defined on the carrier set Q. This action will be called the *parastrophy action*.

Notation. As a rule we compose mappings from the right to the left. Thus $\alpha\beta(x)$ means

$$ \alpha\beta(x) = (\alpha \circ \beta)(x) = \alpha(\beta(x)). $$

We keep this rule even when mappings appear as exponents upon the right. Thus $M^{\alpha\beta}$ means $M^{\alpha\circ\beta}$. However, indices are treated differently. As dictated by the parastrophy action, $x_{i\sigma\kappa}$ means $x_{(i\sigma)\kappa}$. This is the only situation where we compose from the left to the right.

Quasigroups. An algebra $\big(Q; \cdot; \overset{l}{\cdot}; \overset{r}{\cdot}\big)$ is called a *quasigroup*, if $(\cdot)$ is an invertible operation and $\big(\overset{l}{\cdot}\big)$ and $\big(\overset{r}{\cdot}\big)$ are its left and right inverses. Using (1) the conditions of invertibility (2) can be written as the identities

$$ (x \overset{l}{\cdot} y) \cdot y = x, \quad (x \cdot y) \overset{l}{\cdot} y = x, \quad x \cdot (x \overset{r}{\cdot} y) = y, \quad x \overset{r}{\cdot} (x \cdot y) = y. \tag{5} $$

The algebra $\big(Q; \cdot; \overset{l}{\cdot}; \overset{r}{\cdot}\big)$ is called a *loop*, if in addition it has a neutral element e: $ex = xe = x$. The operation $(\cdot)$ is called *main*.

It is convenient to consider a quasigroup as an algebra with all parastrophes in its signature: $\big(Q; \cdot, \overset{l}{\cdot}, \overset{r}{\cdot}, \overset{s}{\cdot}, \overset{sl}{\cdot}, \overset{sr}{\cdot}\big)$, its brief notation here is $(Q; \cdot)$, where $(\cdot)$ is its main operation. Replacing the main operation $(\cdot)$ with its σ-parastrophe $\big(\overset{\sigma}{\cdot}\big)$, we obtain the algebra $\big(Q;\overset{\sigma}{\cdot},\overset{l\sigma}{\cdot},\overset{r\sigma}{\cdot},\overset{s\sigma}{\cdot},\overset{sl\sigma}{\cdot},\overset{sr\sigma}{\cdot}\big)$ which is called σ-parastrophe of the given quasigroup. Since its main operation is $\big(\overset{\sigma}{\cdot}\big)$, the σ-parastrophe is denoted by $\big(Q;\overset{\sigma}{\cdot}\big)$.

Bijections L_a, R_a, M_a of a quasigroup $(Q;\cdot)$ are called *left*, *right* and *middle translations*, if

$$L_a(x) := a\cdot x, \qquad R_a(x) := x\cdot a, \qquad M_a(x) := x \overset{r}{\cdot} a. \tag{6}$$

Therefore,

$$L_a^{-1}(x) = a \overset{r}{\cdot} x, \qquad R_a^{-1}(x) = x \overset{l}{\cdot} a, \qquad M_a^{-1}(x) = a \overset{l}{\cdot} x. \tag{7}$$

In [3], [4] the middle translation is denoted by T_a.

Parastrophic symmetry.

Definition 1. An action of the symmetric group $S_3 = \{\iota, l, r, s, sl, sr\}$, where $l := (13)$, $s := (12)$, $r := (23)$, on a set K will be called a *parastrophy action*, i.e., the equalities ${}^{\iota}k = k$ and ${}^{\sigma}({}^{\tau}k) = {}^{\sigma\tau}k$ hold for all $\sigma,\tau\in S_3$, where ${}^{\sigma}k$ denotes the image of the pair (σ,k). For an element k its stabilizer group $\mathrm{Ps}(k)$ is called a *parastrophic symmetry group* and its orbit $\mathrm{Po}(k)$ is a *parastrophy orbit*. Both the parastrophic symmetry group and its parastrophy orbit are defined by the following equalities

$$\mathrm{Ps}(k) := \{\sigma\colon {}^{\sigma}k = k\}, \qquad \mathrm{Po}(k) := \{k_1\colon (\exists\,\sigma\in S_3)\ k_1 = {}^{\sigma}k\}. \tag{8}$$

It is easy to verify the truth of the following theorem.

Theorem 1. *Let k be an element of a set K. Every parastrophy action on K satisfies the following:*

1) *the parastrophy orbits form a partition of K;*
2) *the parastrophic symmetry group $\mathrm{Ps}(k)$ is a subgroup of S_3;*
3) $\mathrm{Ps}({}^{\sigma}k) = \sigma\mathrm{Ps}(k)\sigma^{-1}$, $\sigma\in S_3$;
4) $|\mathrm{Ps}(k)|\cdot|\mathrm{Po}(k)| = 6$;
5) *the size of $|\mathrm{Po}(k)|$ is 1, 2, 3 or 6. Let us call k:*
   (a) *asymmetric, if $|\mathrm{Ps}(k)| = 1$. Then its orbit has six different elements*

   $$\mathrm{Po}(k) = \{k, {}^{l}k, {}^{r}k, {}^{s}k, {}^{sl}k, {}^{sr}k\}$$

   and each of them has $\{\iota\}$ as its parastrophic symmetry group;
   (b) *singly symmetric, if $|\mathrm{Ps}(k)| = 2$. Then its orbit has three different elements*

   $$\mathrm{Po}(k) = \{k, {}^{l}k, {}^{r}k\}$$

and their parastrophic symmetry groups are $\{\iota, s\}$, $\{\iota, l\}$, $\{\iota, r\}$ *which are conjugated;*

(c) *semisymmetric, if* $|\mathrm{Ps}(k)| = 3$*. Then its orbit has two different elements*

$$\mathrm{Po}(k) = \{k, {}^{s}k\}, \qquad k = {}^{sl}k = {}^{sr}k, \qquad {}^{s}k = {}^{l}k = {}^{r}k,$$

and each of them has A_3 *as its parastrophic symmetry group;*

(d) *totally symmetric, if* $|\mathrm{Ps}(k)| = 6$*. Then its orbit has one element whose parastrophic symmetry group is* S_3*.*

The parastrophy orbit of IP quasigroups was considered in [10].

Let F be a functional variable of a quasigroup, i.e., the variable takes on a value in the set of invertible operations of a carrier set. A σ-parastrophe of the variable is denoted by ${}^{\sigma}F$ and it takes on the value ${}^{\sigma}f$, if F takes on the value f. Notations $(\cdot)$, $(\circ)$ for functional variables are also possible.

In this article, we consider only predicates $P(F)$ of the second order logic such that $P(f)$ is a proposition of the first order logic, where F is a functional variable and f is a function defined on a set.

Definition 2. Let $P(F)$ be a predicate in a class of quasigroups $\mathfrak{A}$, where F takes on a value in the main operations of quasigroups from $\mathfrak{A}$. A predicate ${}^{\sigma}P(F)$ is said to be a *σ-parastrophe* of $P(F)$, if it can be obtained from $P(F)$ by replacing F with ${}^{\sigma^{-1}}F$.

"*$P(F)$ is a predicate in a class of quasigroups* $\mathfrak{A}$" implies that $P(F)$ contains neither functional nor individual constants and the functional variables F take their values among the main operations in $\mathfrak{A}$.

We say that "*$P(\cdot)$ is true in a quasigroup $(Q; f)$*", if $P(f)$ is a true proposition in $(Q; f)$ and "*$P(\cdot)$ is true in a class $\mathfrak{A}$ of quasigroups*", if it is true in each quasigroup of $\mathfrak{A}$.

Proposition 2. *${}^{\kappa}({}^{\sigma}P)$ and ${}^{\kappa\sigma}P$ are the identical predicate.*

Proof: Let $P(\cdot)$ be a predicate in a class of quasigroups $\mathfrak{A}$. According to the definition, ${}^{\sigma}P(\cdot)$ is obtained from $P(\cdot)$ by replacing $(\cdot)$ with $(\overset{\sigma^{-1}}{\cdot})$. To obtain ${}^{\kappa}({}^{\sigma}P)(\cdot)$, we have to replace $(\cdot)$ with $(\overset{\kappa^{-1}}{\cdot})$ in ${}^{\sigma}P(\cdot)$. Thus, ${}^{\kappa}({}^{\sigma}P)(\cdot)$ is obtained from $P(\cdot)$ by replacing $(\cdot)$ with $(\overset{\sigma^{-1}\kappa^{-1}}{\cdot}) = (\overset{(\kappa\sigma)^{-1}}{\cdot})$. That is why ${}^{\kappa}({}^{\sigma}P)$ and ${}^{\kappa\sigma}P$ are the identical predicate. $\square$

Let $(Q; \circ)$ be a quasigroup and P be a proposition defined in $(Q; \circ)$. The *parastrophy orbit* $\mathrm{Po}(P)$ and *parastrophic symmetry set* $\mathrm{Ps}^{\circ}(P)$ of the proposition P

are defined by

$$\mathrm{Po}(P) := \{{}^{\sigma}P\colon\ \sigma \in S_3\}, \qquad \mathrm{Ps}^{\circ}(P) := \{\sigma\colon\ {}^{\sigma}P \ \text{ is true in } \ (Q;\circ)\}.$$

We will write $\mathrm{Ps}^{\sigma}(P)$ instead of $\mathrm{Ps}^{\overset{\sigma}{\circ}}(P)$ if it does not lead to a misunderstanding.

The following statement is true.

Lemma 3. *Let $(Q;\circ)$ be a quasigroup, $P(\cdot)$ a proposition and $(\cdot)$ be a functional variable:*

1) *if $P(\cdot)$ is true in a quasigroup $(Q;\circ)$, i.e., $P(\circ)$ is true, then for all $\sigma \in S_3$ ${}^{\sigma}P(\cdot)$ is true in its σ-parastrophe $\big(Q;\overset{\sigma}{\circ}\big)$;*
2) *"${}^{\sigma}P(\cdot)$ is true in a ν-parastrophe of $(Q;\circ)$" is equivalent to $\sigma \in \nu\mathrm{Ps}^{\iota}(P)$;*
3) $\mathrm{Ps}^{\nu}(P) = \nu\mathrm{Ps}^{\iota}(P)$*;*
4) *if $\mathrm{Ps}^{\iota}(P)$ is a group, then S_3 acts on $\mathrm{Po}(P)$ with respect to the binary relation "to be true in a parastrophe of the given quasigroup".*

Proof: Let $(*) := \big(\overset{\sigma}{\circ}\big)$. Then the item 1) follows from the equality

$$P(\circ) = P\big({}^{\sigma^{-1}}\big(\overset{\sigma}{\circ}\big)\big) = P\big(\overset{\sigma^{-1}}{*}\big) = {}^{\sigma}P(*).$$

"$\sigma \in \mathrm{Ps}^{\overset{\nu}{\circ}}(P)$" is equivalent to "${}^{\sigma}P(\cdot)$ is true in a ν-parastrophe of $(Q;\circ)$". It means that "${}^{\nu^{-1}\sigma}P(\cdot)$ is true in the quasigroup $(Q;\circ)$", that is, "$\nu^{-1}\sigma$ belongs to $\mathrm{Ps}^{\iota}(P)$", i.e., $\sigma \in \nu\mathrm{Ps}^{\iota}(P)$. Therefore, the item 2) is proved. The item 3) immediately follows from the item 2).

To prove the item 4), suppose that $\mathrm{Ps}^{\iota}(P)$ is a group. Taking into account Proposition 2, it is enough to prove that the relation "to be true in the same parastrophe" is an equivalence. Since $\sigma \in \sigma\mathrm{Ps}^{\iota}(P)$ for all $\sigma \in S_3$, then by item 2), ${}^{\sigma}P(\cdot)$ is true in the σ-parastrophe of the quasigroup $(Q;\circ)$. That is why the relation is reflexive. Its symmetric property is evident. To prove transitivity, suppose that ${}^{\sigma}P(\cdot)$, ${}^{\kappa}P(\cdot)$ are true in ν-parastrophe for some $\nu \in S_3$ and ${}^{\kappa}P(\cdot)$, ${}^{\omega}P(\cdot)$ are true in π-parastrophe for some $\pi \in S_3$. According to item 2), the following relationships are true

$$\nu^{-1}\sigma \in \mathrm{Ps}^{\iota}(P), \qquad \nu^{-1}\kappa \in \mathrm{Ps}^{\iota}(P), \qquad \pi^{-1}\kappa \in \mathrm{Ps}^{\iota}(P), \qquad \pi^{-1}\omega \in \mathrm{Ps}^{\iota}(P).$$

Since $\mathrm{Ps}^{\iota}(P)$ is a subgroup of S_3, then

$$\theta := \nu^{-1}\pi = (\nu^{-1}\kappa)(\pi^{-1}\kappa)^{-1} \in \mathrm{Ps}^{\iota}(P).$$

Thence, $\pi = \nu\theta$ for some $\theta \in \mathrm{Ps}^{\iota}(P)$. Therefore, $\pi^{-1}\omega \in \mathrm{Ps}^{\iota}(P)$ is equivalent to $\theta^{-1}\nu^{-1}\omega \in \mathrm{Ps}^{\iota}(P)$ which means

$$\nu^{-1}\omega \in \theta\mathrm{Ps}^{\iota}(P) = \mathrm{Ps}^{\iota}(P).$$

This relationship with $\nu^{-1}\sigma \in \mathrm{Ps}^{\iota}(P)$ implies that ${}^{\sigma}P(\cdot)$ and ${}^{\omega}P(\cdot)$ are true in ν-parastrophe of the quasigroup $(Q;\circ)$. Thus, the given relation is transitive and so it is an equivalence. $\square$

Let ${}^{\sigma}\mathfrak{A}$ denote the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$. A set of all pairwise parastrophic classes is called a *parastrophy orbit of the class* $\mathfrak{A}$, see [9]:

$$\mathrm{Po}(\mathfrak{A}) := \{{}^{\sigma}\mathfrak{A} \colon \sigma \in S_3\} = \{\mathfrak{A}, {}^{l}\mathfrak{A}, {}^{r}\mathfrak{A}, {}^{s}\mathfrak{A}, {}^{sl}\mathfrak{A}, {}^{sr}\mathfrak{A}\}. \tag{9}$$

Since $(\sigma, \mathfrak{A}) \mapsto {}^{\sigma}\mathfrak{A}$ is a parastrophy action on $\mathrm{Po}(\mathfrak{A})$, then

$$|\mathrm{Ps}(\mathfrak{A})| \cdot |\mathrm{Po}(\mathfrak{A})| = 6.$$

The parastrophy orbit of varieties is uniquely defined by one of its varieties, i.e. if $\mathfrak{A}$ is a variety, then all elements of $\mathrm{Po}(\mathfrak{A})$ are varieties, and each of them determines $\mathrm{Po}(\mathfrak{A})$ completely.

Therefore, if an identity defines a variety, then it defines all varieties from its orbit.

Theorem 4 ([9]). *Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$.*

PROOF: The proof follows from Lemma 3. $\square$

Corollary 5 ([9]). *Let P be true in a class of quasigroups $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all $\sigma \in \mathrm{Ps}(\mathfrak{A})$.*

For example, let P be a proposition in the variety of all distributive quasigroups, then ${}^{\sigma}P$ is true in this variety for all $\sigma \in S_3$ because the variety of distributive quasigroups is totally symmetric, i.e., each parastrophe of a distributive quasigroup is also distributive, see [8].

Parastrophic translations

In this subsection, we consider some relations among translations defined by the same element in a quasigroup.

Every element a of a quasigroup $(Q;\cdot)$ defines six bijections: left, right, middle translations and their inverses

$$\mathscr{M}_a := \{M_a, M_a^{-1}, L_a, L_a^{-1}, R_a, R_a^{-1}\}. \tag{10}$$

It is well known that each element defines the same set of bijections in each parastrophe of a quasigroup, see, for example [5], [3], [4]. But what is the dependence between translations in a quasigroup and translations in σ-parastrophe of the quasigroup? In a totally symmetric quasigroup all translations defined by the same element coincide. In spite of this, the corresponding web can be constructed. That is why two translations might be considered as different even when they coincide as mappings. To achieve this we shall introduce an additional parameter (called direction) that will be assigned to each translation.

The definition of the *σ-parastrophe* of left, right and middle translations is obtained from the definitions of the corresponding translation by replacing the main operation with its σ^{-1}-parastrophe:

$$ {}^{\sigma}L_a(x) := a \overset{\sigma^{-1}}{\cdot} x, \qquad {}^{\sigma}R_a(x) := x \overset{\sigma^{-1}}{\cdot} a, \qquad {}^{\sigma}M_a(x) := x \overset{r\sigma^{-1}}{\cdot} a, \tag{11} $$

where $\sigma \in S_3$. Replacing the main operation with its κ^{-1}-parastrophe in these equalities, we obtain

$$ {}^{\kappa}({}^{\sigma}L_a) = {}^{\kappa\sigma}L_a, \qquad {}^{\kappa}({}^{\sigma}R_a) = {}^{\kappa\sigma}R_a, \qquad {}^{\kappa}({}^{\sigma}M_a) = {}^{\kappa\sigma}M_a. \tag{12} $$

Relations among parastrophes of different translations and their inverses are given in the following equalities:

$$ {}^{\sigma}L_a = {}^{\sigma sr}M_a, \qquad {}^{\sigma}R_a = {}^{\sigma r}M_a, \qquad {}^{\sigma s}M_a = ({}^{\sigma}M_a)^{-1}. \tag{13} $$

Indeed,

$$ {}^{\sigma}L_a(x) = a \overset{\sigma^{-1}}{\cdot} x = x \overset{s\sigma^{-1}}{\cdot} a = a \overset{r(\sigma sr)^{-1}}{\cdot} x = {}^{\sigma sr}M_a(x); $$

$$ {}^{\sigma}R_a(x) = x \overset{\sigma^{-1}}{\cdot} a = x \overset{r(\sigma r)^{-1}}{\cdot} a = {}^{\sigma r}M_a(x); $$

$$ {}^{\sigma s}M_a(x) = y \;\Leftrightarrow\; x \overset{(\sigma s)^{-1}}{\cdot} y = a \;\Leftrightarrow\; x \overset{s\sigma^{-1}}{\cdot} y = a \;\Leftrightarrow\; y \overset{\sigma^{-1}}{\cdot} x = a $$

$$ \Leftrightarrow\; {}^{\sigma}M_a(y) = x \;\Leftrightarrow\; ({}^{\sigma}M_a)^{-1}(x) = y. $$

Proposition 6. *In all parastrophes of a quasigroup $(Q;\cdot)$ an element a defines the same set of translations $\mathscr{M}_a$, see* (10)*. Moreover,*

$$ \mathscr{M}_a = \{{}^{\sigma}L_a \colon \sigma \in S_3\} = \{{}^{\sigma}R_a \colon \sigma \in S_3\} = \{{}^{\sigma}M_a \colon \sigma \in S_3\}. \tag{14} $$

Namely, the following equalities are true

$$ \begin{aligned} &{}^{\iota}M_a = M_a, && {}^{ls}M_a = L_a, && {}^{r}M_a = R_a, \\ &{}^{s}M_a = M_a^{-1}, && {}^{l}M_a = L_a^{-1}, && {}^{rs}M_a = R_a^{-1}. \end{aligned} \tag{15} $$

$$ \mathscr{M}_a = \{{}^{\iota}M_a, {}^{s}M_a, {}^{l}M_a, {}^{r}M_a, {}^{ls}M_a, {}^{rs}M_a\}. $$

Proof: The proof immediately follows from (13). □

The transformation ${}^{\kappa}M_a$ will be called *κ-translation* or *κ-parastrophe of the translation* M_a, the permutation κ will be called the *direction* of ${}^{\kappa}M_a$.

Note that the translations of the directions σ and σs are mutually inverse for all σ; ι and s are directions of the middle translation and its inverse; l and ls are directions of the left translation and its inverse; r and rs are directions of the right translation and its inverse.

Let us consider some additional explanation of the notion *direction*. Each quasigroup $(Q;\circ)$ has a property: "*In the equality*

$$x_1 \circ x_2 = x_3,$$

the value of each of the variables x_1, x_2, x_3 uniquely defines two bijections". Therefore, each element a defines six bijections of the carrier set:

$$(16) \qquad {}^{\sigma}M_a\colon x_{1\sigma^{-1}} \mapsto x_{2\sigma^{-1}}, \qquad x_{3\sigma^{-1}} = a, \ \ \sigma \in S_3.$$

In detail:

$$\begin{array}{llllll}
\iota & {}^{\iota}M_a: & x_{1\iota} \mapsto x_{2\iota}, & x_{3\iota} = a, & \text{i.e.,} & M_a: \ x_1 \mapsto x_2, \ \ x_3 = a,\\
s & {}^{s}M_a: & x_{1s} \mapsto x_{2s}, & x_{3s} = a, & \text{i.e.,} & M_a^{-1}: \ x_2 \mapsto x_1, \ \ x_3 = a,\\
ls & {}^{ls}M_a: & x_{1sl} \mapsto x_{2sl}, & x_{3sl} = a, & \text{i.e.,} & L_a: \ x_2 \mapsto x_3, \ \ x_1 = a,\\
l & {}^{l}M_a: & x_{1l} \mapsto x_{2l}, & x_{3l} = a, & \text{i.e.,} & L_a^{-1}: \ x_3 \mapsto x_2, \ \ x_1 = a,\\
r & {}^{r}M_a: & x_{1r} \mapsto x_{2r}, & x_{3r} = a, & \text{i.e.,} & R_a: \ x_1 \mapsto x_3, \ \ x_2 = a,\\
rs & {}^{rs}M_a: & x_{1sr} \mapsto x_{2sr}, & x_{3sr} = a, & \text{i.e.,} & R_a^{-1}: \ x_3 \mapsto x_1, \ \ x_2 = a.
\end{array}$$

In a TS quasigroup these transformations coincide as bijections of the carrier set but they have pairwise different directions. Consequently, each of these transformations (translations and their inverses) has two independent parameters: direction and permutation of the carrier set. Analogically, a vector also has two parameters: a direction and a length.

Notation. A translation of the direction κ defined by an element a of a quasigroup with the main operation "$\circ$" is denoted by ${}^{\kappa}M_a^{\circ}$. We omit $(\circ)$ if it does not lead to a misunderstanding. Namely, we write ${}^{\kappa}M_a$ instead of ${}^{\kappa}M_a^{\circ}$ and ${}^{\kappa}M_a^{\sigma}$ instead of ${}^{\kappa}M_a^{\overset{\sigma}{\circ}}$.

Lemma 7. *Let $(Q;\circ)$ be a quasigroup, then for all $a \in Q$ the following assertions are true:*

1) *if a translation or its inverse defined by a has the direction κ in σ-parastrophe, then it has the direction $\nu\kappa$ in the $\nu\sigma$-parastrophe, i.e.,*

$${}^{\kappa}M_a^{\sigma} = {}^{\nu\kappa}M_a^{\nu\sigma};$$

2) *the set of all translations defined by the same element is the same in each parastrophe of the given quasigroup.*

PROOF: For each $x \in Q$

$$^{\nu\kappa}M_a^{\nu\sigma}(x) \overset{(11)}{=} x \overset{r(\nu\kappa)^{-1}(\nu\sigma)}{\circ} a = x \overset{r\kappa^{-1}\sigma}{\circ} a \overset{(11)}{=} {}^{\kappa}M_a^{\sigma}(x).$$

Item 2) follows from item 1) and Proposition 6. □

Let $(Q;\circ)$ be a quasigroup. Consider the proposition $P^{\circ}(M_a)$:

$$\text{“}M_a = {}^{\kappa}M_a^{\iota} = \cdots = {}^{\omega}M_a^{\iota}\text{”},$$

(17) $$\mathrm{Ps}^{\circ}(M_a) := \{\kappa\colon {}^{\kappa}M_a^{\iota} = M_a\}.$$

Let $\mathrm{Ps}(M_a)$ be a group. According to Lemma 3, the group S_3 acts on the parastrophy orbit of the proposition $P^{\circ}(M_a)$. Thus, the number of different translations defined by a is equal to 1, 2, 3 or 6, i.e., the translations could be totally symmetric, semisymmetric, singly symmetric or asymmetric. Consider each of these cases.

Note that the set $\mathrm{Ps}(M_a)$ is not always a group.

Example 1. Consider the quasigroup $(\mathbb{Z}_5;\circ)$, where $\mathbb{Z}_5$ is the ring of modulo 5 and

$$x \circ y = x + 2y + 1.$$

All its translations defined by 2 are

$$^{\iota}M_2(x) = M_2(x) = 2x+3, \qquad {}^{ls}M_2(x) = L_2(x) = 2x+3,$$

$$^{r}M_2(x) = R_2(x) = x, \qquad {}^{s}M_2(x) = M_2^{-1}(x) = 3x+1,$$

$$^{l}M_2(x) = L_2^{-1}(x) = 3x+1, \qquad {}^{rs}M_2(x) = R_2^{-1}(x) = x.$$

Thus, $\mathrm{Ps}(M_2) = \{\iota, ls\} = \{\iota, sr\}$. It is not a group, because $(sr)^2 = sl \notin \mathrm{Ps}(M_2)$.

Totally symmetric translations. A translation defined by a is said to be *totally symmetric*, if all translations defined by a coincide. It means that $\mathrm{Ps}(M_a) = S_3$. Lemma 7 implies that all parastrophes of M_a coincide in all parastrophes of the quasigroup.

Proposition 8. *The translation M_a is totally symmetric if and only if for all x*

(18) $$ax = xa, \qquad x \cdot xa = a, \qquad a \cdot ax = x.$$

PROOF: Using (6) and (7), the equalities can be written as

$$L_a = R_a, \qquad M_a = R_a, \qquad L_a = L_a^{-1}.$$

It means that translations of all directions defined by a coincide as transformations of the carrier set Q. $\square$

Corollary 9. *All translations of a quasigroup are totally symmetric if and only if the quasigroup is totally symmetric.*

For example, each group of exponent two is totally symmetric.

Semisymmetric translations. A translation M_a is considered to be: *semi-symmetric*, if $\mathrm{Ps}(M_a) \supseteq A_3 := \{\iota, sl, sr\}$; *strongly semi-symmetric*, if $\mathrm{Ps}(M_a) = A_3$, i.e., each parastrophe of the quasigroup has at most two (exactly two, respectively) different translations defined by the element a. Therefore, each semisymmetric translation is either totally symmetric or strongly semisymmetric. Namely, according to Lemma 7, the ι-, sl-, sr-parastrophes of M_a coincide in ι-, sl-, sr-parastrophes of the quasigroup $(Q;\cdot)$ and s-, l-, r-parastrophes of the translation M_a coincide in s-, l-, r-parastrophes of the quasigroup.

Proposition 10. *The translation M_a is semi-symmetric if and only if for all x*

$$ax \cdot a = x, \qquad x \cdot ax = a. \tag{19}$$

Proof: Using (6) and (7), the equalities (19) are equivalent to

$$L_a = R_a^{-1}, \qquad L_a = M_a. \tag{20}$$

Applying (15), we have $M_a = {}^{sl}M_a = {}^{sr}M_a$ which means $A_3 \subseteq \mathrm{Ps}(M_a)$ and thus the translation M_a is semi-symmetric. $\square$

If all translations of a quasigroup are semisymmetric, then equations (19) are true for all x and a in Q, i.e., equations (19) are equivalent identities which define the variety of semisymmetric quasigroups. Identities implying semi-symmetricity is given in [6, Corollaries 11, 12].

Corollary 11. *A quasigroup is semisymmetric if and only if all of its middle translations are semi-symmetric.*

Singly symmetric translations. A translation M_a is considered to be *singly symmetric*, if its parastrophic symmetry set is a two-element group $\{\iota, \sigma\}$, where $\sigma^2 = \iota$, that is, the translation has only one symmetry ${}^{\iota}M_a = {}^{\sigma}M_a$.

The group S_3 has three two-element subgroups: $\{\iota, \sigma\}$, $\sigma = l, r, s$. The coset $S_3/\{\iota, \sigma\}$ has three blocks:

$$l\{\iota, \sigma\} = \{l, l\sigma\}, \qquad r\{\iota, \sigma\} = \{r, r\sigma\}, \qquad s\{\iota, \sigma\} = \{s, s\sigma\}.$$

Consequently, the element a defines three different translations. According to Lemma 7:

- the l- and $l\sigma$-parastrophes of the translation M_a coincide in l- and $l\sigma$-parastrophes $\big(Q;\overset{l}{\circ}\big)$, $\big(Q;\overset{l\sigma}{\circ}\big)$: ${}^{l}M_a = {}^{l\sigma}M_a$;
- the r- and $r\sigma$-parastrophes of the translation M_a coincide in r- and $r\sigma$-parastrophes $\big(Q;\overset{r}{\cdot}\big)$, $\big(Q;\overset{r\sigma}{\cdot}\big)$: ${}^{r}M_a^{r} = {}^{r\sigma}M_a^{r\sigma}$;
- the s- and $s\sigma$-parastrophes of the translation M_a coincide in s- and $s\sigma$-parastrophes $\big(Q;\overset{s}{\cdot}\big)$, $\big(Q;\overset{s\sigma}{\cdot}\big)$: ${}^{s}M_a^{s} = {}^{s\sigma}M_a^{s\sigma}$.

Proposition 12. *Let $(Q;\cdot)$ be a quasigroup, then the following assertions are true:*

1) $\mathrm{Ps}^{\cdot}(M_a) = \{\iota, s\}$ *if and only if* $x\big(a\overset{l}{\cdot}x\big) = a$ *for all* $x \in Q$;
2) $\mathrm{Ps}^{\cdot}(M_a) = \{\iota, l\}$ *if and only if* $ax\cdot x = a$ *for all* $x \in Q$;
3) $\mathrm{Ps}^{\cdot}(M_a) = \{\iota, r\}$ *if and only if* $x\cdot xa = a$ *for all* $x \in Q$.

Proof: The equality $\mathrm{Ps}(M_a) = \{\iota, s\}$ means $M_a = {}^{s}M_a$, i.e., $x\overset{r}{\cdot}a = a\overset{l}{\cdot}x$ that is equivalent to $x\,(a\overset{l}{\cdot}x\,) = a$.

$\mathrm{Ps}(M_a) = \{\iota, l\}$ means $M_a = {}^{l}M_a$. According to (6), (7), (15): $M_a(y) = L_a^{-1}(y)$ for all $y \in Q$, i.e., $y\overset{r}{\cdot}a = a\overset{r}{\cdot}y$. This equality is equivalent to $y\cdot(a\overset{r}{\cdot}y) = a$. Denote $x := a\overset{r}{\cdot}y$, then $y = ax$. Thus, $ax\cdot x = a$ holds for all x.

The equality $\mathrm{Ps}(M_a) = \{\iota, r\}$ means $M_a = {}^{r}M_a$, i.e., $x\overset{r}{\cdot}a = x\cdot a$, that is, $x\cdot xa = a$. □

Proposition 12 immediately implies the following corollary.

Corollary 13. *Each translation of a quasigroup has the group $\{\iota, s\}$ ($\{\iota, l\}$ and $\{\iota, r\}$) as a parastrophic symmetry set if and only if the quasigroup is commutative (left symmetric and right symmetric, respectively).*

Asymmetric translations. A translation M_a will be *asymmetric*, if its parastrophic symmetry set is trivial, i.e., $\mathrm{Ps}(M_a) = \{\iota\}$. In other words, the element a defines six different translations:

$$\mathrm{Po}(M_a) = \{M_a, {}^{l}M_a, {}^{r}M_a, {}^{s}M_a, {}^{sl}M_a, {}^{sr}M_a\}.$$

If all translations are asymmetric, the quasigroup is called asymmetric.

Acknowledgment. The authors are grateful to the reviewer of English Vira Obshanska for corrections and help.

References

[1] Belousov V. D., *The group associated with a quasigroup*, Mat. Issled. **4** (1969), no. 3, 21–39 (Russian).

[2] Dénes J., Keedwell A. D., *Latin Squares and Their Applications*, Academic Press, New York, 1974.

[3] Duplák J., *On quasi-identities of transitive quasigroups*, Math. Slovaca **34** (1984), no. 3, 281–294.

[4] Duplák J., *Quasigroups determined by balanced identities of length* $\leqslant 6$, Czechoslovak Math. J. **36(111)** (1986), no. 4, 599–616.

[5] Duplák J., *A parastrophic equivalence in quasigroups*, Quasigroups Related Systems **7** (2000), 7–14.

[6] Krainichuk H., Tarkovska O., *Semi-symmetric isotopic closure of some group varieties and the corresponding identities*, Bul. Acad. Ştiinţe Repub. Mold. Mat. Number **3(85)** (2017), 3–22.

[7] Smith J. D. H., *An introduction to quasigroups and their representations*, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, 2007.

[8] Sokhatsky F. M., *On pseudoisomorphy and distributivity of quasigroups*, Bul. Acad. Ştiinţe Repub. Mold. Mat. **2(81)** (2016), 125–142.

[9] Sokhatsky F. M., *Parastrophic symmetry in quasigroup theory*, Visnik DonNU. Ser. A Natural Sciences **1–2** (2016), 70–83.

[10] Sokhatsky F. M., Lutsenko A. V., *The bunch of varieties of inverse property quasigroups*, Visnik DonNU. Ser. A Natural Sciences **1–2** (2018), 56–69.

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(*Received* October 30, 2019, *revised* July 17, 2020)