On some imaginary triquadratic number fields k with $\mathrm{Cl}_2(k) \simeq (2,4)$ or $(2,2,2)$

Abdelmalek Azizi,
Mohamed Mahmoud Chems-Eddin, Abdelkader Zekhnini

Abstract. Let d be a square free integer and $L_d := \mathbb{Q}(\zeta_8, \sqrt{d})$. In the present work we determine all the fields L_d such that the 2-class group, $\mathrm{Cl}_2(L_d)$, of L_d is of type $(2,4)$ or $(2,2,2)$.

Keywords: 2-group rank; 2-class group; imaginary triquadratic number fields

Classification: 11R11, 11R16, 11R18, 11R27, 11R29

1. Introduction

Let k be a number field and $\mathrm{Cl}_2(k)$ its 2-class group, that is, the 2-Sylow subgroup of its ideal class group $\mathrm{Cl}(k)$. The problem of determining the structure of $\mathrm{Cl}_2(k)$ is one of the most interesting problems of algebraic number theory, accordingly many mathematicians treated this problem for some number fields of degree 2, 4. For example, in [18] using binary quadratic forms theory, P. Kaplan determined the 2-class group of some quadratic number fields. The authors of [11], [12] used genus theory and class field theory to characterize those imaginary quadratic number fields, k, with 2-class group of type $(2, 2^m)$ or $(2,2,2)$ and the 2-rank of the class group of its Hilbert 2-class field equal to 2. In [5], using units and the 2-part of the class number of subextensions of k, the authors determined the 2-class group of some real biquadratic number fields $k = \mathbb{Q}(\sqrt{m}, \sqrt{d})$ with d be an odd square free integer. Using similar techniques, the paper [7] characterizes all the fields $k = \mathbb{Q}(i, \sqrt{d})$ such that $\mathrm{Cl}_2(k)$ is of type $(2,4)$ or $(2,2,2)$ (here $(a_1, \ldots, a_r)$ denotes the direct sum of cyclic groups of order a_i for $i = 1, \ldots, r$). Whenever k is an imaginary multiquadratic number field, this problem is strongly related to the units of k and the class number of the 2-part of the class numbers of its subextensions as we will see later. This paper is, actually, a continuation and extension of our earlier work [4], in which we determined the rank of the 2-class group of all fields of the form $L_d := \mathbb{Q}(\zeta_8, \sqrt{d})$ with d being a positive square free integer and moreover we determined all fields L_d for which the 2-class

DOI 10.14712/1213-7243.2021.008

group, $\mathrm{Cl}_2(L_d)$, is of type $(2,2)$. In this work, we are interested in determining all positive square free integers d satisfying $\mathrm{Cl}_2(L_d)$ is of type $(2,4)$ or $(2,2,2)$.

Notations

The next notations will be used for the rest of this article:

- d: a positive odd square free integer;
- L_d: $\mathbb{Q}(\zeta_8, \sqrt{d})$;
- $\mathrm{Cl}_2(L)$: the 2-class group of some number field L;
- $h(L)$: the class number of the number field L;
- $h_2(L)$: the 2-class number of the number field L;
- $h_2(m)$: the 2-class number of the quadratic field $\mathbb{Q}(\sqrt{m})$;
- E_L: the unit group of any number field L;
- ε_m: the fundamental unit of $\mathbb{Q}(\sqrt{m})$;
- W_L: the set of roots of unity contained in a field L;
- ω_L: the cardinality of W_L;
- L^+: the maximal real subfield of an imaginary number field L;
- Q_L: the Hasse's index, that is, $[E_L : W_L E_{L^+}]$, if L/L^+ is CM;
- $Q(L/k)$: the unit index of a V_4-extension L/k;
- $q(L) := \left[E_L : \prod_i E_{k_i}\right]$ with k_i the quadratic subfields of L.

2. Preliminaries

Let us first collect some results and definitions that we will need in the sequel. Recall that a field K is said to be a CM-field if it is a totally complex quadratic extension of a totally real number field. Note also that a V_4-extension K/k (i.e., a normal extension of number fields with $\mathrm{Gal}(K/k) = V_4$, where V_4 is the Klein four-group) is called V_4-extension of CM-fields if exactly two of its three quadratic subextensions are CM-fields. Let us next recall the class number formula for a V_4-extension of CM-fields:

Proposition 1 ([20]). *Let L/K be a V_4-extension of* CM*-fields, then*

$$h(L) = \frac{Q_L}{Q_{K_1}Q_{K_2}} \cdot \frac{\omega_L}{\omega_{K_1}\omega_{K_2}} \cdot \frac{h(K_1)h(K_2)h(L^+)}{h(K)^2}.$$

Here K_1, K_2, L^+ are the three subextensions of L/K, with K_1 and K_2 are CM-fields.

The following class number formula for a multiquadratic number field is usually attributed to S. Kuroda [21], but it goes back to G. Herglotz, see [16] (cf. [13, page 27]).

Proposition 2. *Let K be a multiquadratic number field of degree 2^n, $n \in \mathbb{N}$, and k_i the $s = 2^n - 1$ quadratic subfields of K. Then*

$$h(K) = \frac{1}{2^v}\left[E_K : \prod_{i=1}^{s} E_{k_i}\right] \prod_{i=1}^{s} h(k_i),$$

with

$$v = \begin{cases} n(2^{n-1}-1) & \text{if } K \text{ is real,} \\ (n-1)(2^{n-2}-1) + 2^{n-1} - 1 & \text{if } K \text{ is imaginary.} \end{cases}$$

Continue with the next formula called Kuroda's class number formula for a V_4-extension K/k.

Proposition 3 ([22], page 247)**.** *Let K/k be a V_4-extension. Then we have:*

$$h(K) = \begin{cases} \frac{1}{4} \cdot Q(K/k) \cdot \prod_{i=1}^{3} h(k_i) & \text{if } k = \mathbb{Q} \text{ and } K \text{ is real,} \\ \frac{1}{2} \cdot Q(K/k) \cdot \prod_{i=1}^{3} h(k_i) & \text{if } k = \mathbb{Q} \text{ and } K \text{ is imaginary,} \\ \frac{1}{4} \cdot Q(K/k) \cdot \prod_{i=1}^{3} h(k_i)/h(k)^2 & \text{if } k \text{ is an imaginary quadratic} \\ & \text{extension of } \mathbb{Q}. \end{cases}$$

Here k_i are the 3 subextensions of K/k.

3. Fields L_d for which $\mathrm{Cl}_2(L_d)$ is of type $(2,4)$

Our goal in this section is to determine all fields L_d for which $\mathrm{Cl}_2(L_d) \simeq (2,4)$. Recall first the definition of the rational biquadratic residue symbol:

For a prime $p \equiv 1 (\mathrm{mod}\ 4)$ and a quadratic residue $a (\mathrm{mod}\ p)$, $\left(\frac{a}{p}\right)_4$ will denote the rational biquadratic residue symbol defined by

$$\left(\frac{a}{p}\right)_4 = \pm 1 \equiv a^{(p-1)/4} (\mathrm{mod}\ p).$$

Moreover, for an integer $a \equiv 1 (\mathrm{mod}\ 8)$, the symbol $\left(\frac{a}{p}\right)_4$ is defined by

$$\left(\frac{a}{p}\right)_4 = 1 \quad \text{if} \quad \equiv 1 (\mathrm{mod}\ 16), \qquad \left(\frac{a}{p}\right)_4 = -1 \quad \text{if} \quad a \equiv 9 (\mathrm{mod}\ 16).$$

It turns out that $\left(\frac{a}{p}\right)_4 = (-1)^{(a-1)/8}$.

In all this section, let p, p_i, q and q_i be prime integers such that $p \equiv p_i \equiv 1 (\mathrm{mod}\ 4)$ and $q \equiv q_i \equiv 3 (\mathrm{mod}\ 4)$ with $i \in \mathbb{N}^*$. The following lemma is proved in our earlier paper [4, Theorem 5.6].

Lemma 1. *The rank of the 2-class group $\mathrm{Cl}_2(L_d)$ of L_d equals 2 if and only if d takes one of the following forms:*

1. $d = q_1q_2$ *with* $q_1 \equiv q_2 \equiv 3 \pmod 8$.
2. $d = p_1p_2$ *with* $p_1 \equiv p_2 \equiv 5 \pmod 8$.
3. $d = q_1q_2$ *with* $q_1 \equiv 3 \pmod 8$ *and* $q_2 \equiv 7 \pmod 8$.
4. $d = pq$ *with* $p \equiv 5 \pmod 8$ *and* $q \equiv 7 \pmod 8$.
5. $d = p$ *with* $p \equiv 1 \pmod 8$ *and* $\left[p \equiv 9 \pmod{16}\right.$ *or* $\left.\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4\right]$.

We need also the following result.

Lemma 2. *Let* $d = p_1p_2$*, where* p_1, p_2 *are two rational primes such that* $p_i \equiv 5 \pmod 8$*. Then* $h_2(L_d)$ *is divisible by* 16.

Proof: We have $L_d = \mathbb{Q}(\sqrt{p_1p_2}, \sqrt{2}, i)$ is an imaginary multiquadratic number field of degree 2^3. So by Proposition 2 we have

$$h(L_d) = \frac{1}{2^5}q(L_d)h(p_1p_2)h(-p_1p_2)h(2p_1p_2)h(-2p_1p_2)h(2)h(-2)h(-1).$$

It is known that $h_2(2)$, $h_2(-2)$ and $h_2(-1)$ are equal to 1, and by [18, page 350] $h_2(-2p_1p_2) = 4$. On the other hand, [8, Theorem 2.2] implies that $q(L_d)$ is a power of 2. So by passing to the 2-part in the above equation, we get

$$h_2(L_d) = \frac{1}{2^3}q(L_d)h_2(p_1p_2)h_2(-p_1p_2)h_2(2p_1p_2). \tag{1}$$

Note that:

- By [15, Corollary 18.4], $h_2(p_1p_2)$ is divisible by 2.
- By [18, pages 348–349] (Propositions B_1' and B_4'), $h_2(-p_1p_2)$ is divisible by 8.
- By [15, Corollaries 18.4, 19.7 and 19.8], $h_2(2p_1p_2)$ is divisible by 4.

On one hand, $\zeta_8 = (1+i)/\sqrt{2} \in E_{L_d}$. On the other hand, letting k_i be the quadratic subfields of L_d, one gets easily

$$\prod_i E_{k_i} = \langle i, \varepsilon_2, \varepsilon_{-2}, \varepsilon_{p_1p_2}, \varepsilon_{2p_1p_2}, \varepsilon_{-p_1p_2}, \varepsilon_{-2p_1p_2} \rangle.$$

So the 8th root of unity $\zeta_8 \notin \prod_i E_{k_i}$. Thus $\overline{1}$ and $\overline{\zeta_8}$ are two distinct cosets in the quotient $E_{L_d}/\prod_i E_{k_i}$. Thus, $q(L_d)$ is divisible by 2. It follows by the equality (1) above that $h_2(L_d)$ is divisible by $(1/2^3)\cdot 2\cdot 2\cdot 8\cdot 4 = 16$ as we wished to prove. $\square$

Example 1. Let $d = p_1p_2$ be as in the above lemma. Using PARI/GP calculator version 2.9.4 (64bit), December 20, 2017, for $5 \leq p_i \leq 200$, $i = 1, 2$, we could not find a field L_d such that $h_2(L_d) = 16$. We have the following examples.

1. For $p_1 = 13$ and $p_2 = 5$, we have $h_2(L_{13\cdot 5}) = 32$.
2. For $p_1 = 37$ and $p_2 = 53$, we have $h_2(L_{37\cdot 53}) = 64$.

Remark 1. With hypothesis and notations of Lemma 2, we find in [8] a unit group of L_d.

Lemma 3 ([9], Lemma 2)**.** *Let $d \equiv 1 \pmod 4$ be a positive square free integer and $\varepsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\varepsilon_d) = 1$, then:*

1. *$x+1$ and $x-1$ are not squares in $\mathbb{N}$, i.e., $2\varepsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$.*
2. *For all prime p dividing d, $p(x+1)$ and $p(x-1)$ are not squares in $\mathbb{N}$.*

Lemma 4. *Let $d = q_1q_2$, with $q_1 \equiv q_2 \equiv 3 \pmod 8$ two primes such that $\left(\frac{q_1}{q_2}\right) = 1$. Then we have $E_{L_d} = \langle \zeta_8, \varepsilon_2, \varepsilon_{q_1q_2}, \sqrt{\varepsilon_{2q_1q_2}}\rangle$, and thus $q(L_d) = 4$.*

Proof: As $q_1q_2 \equiv 1 \pmod 8$, we claim that the unit $\varepsilon_{q_1q_2}$ can be written as $\varepsilon_{q_1q_2} = a + b\sqrt{q_1q_2}$, where a and b are two integers. Indeed, suppose that $\varepsilon_{q_1q_2} = (\alpha + \beta\sqrt{d})/2$ where α, β are two integers. Since $N(\varepsilon_{q_1q_2}) = 1$, one deduces that $\alpha^2 - 4 = \beta^2 d$, hence $\alpha^2 - 4 \equiv \beta^2 \pmod 8$. On the other hand, if we suppose that α and β are odd, then $\alpha^2 \equiv \beta^2 \equiv 1 \pmod 8$, but this implies the contradiction $-3 \equiv 1 \pmod 8$. Thus α and β are even and our claim is established.

It is known that $N(\varepsilon_{q_1q_2}) = 1$, so by the unique factorization in $\mathbb{Z}$ and Lemma 3 one gets

$$(1): \begin{cases} a+1 = 2q_1b_1^2, \\ a-1 = 2q_2b_2^2, \end{cases} \quad \text{or} \quad (2): \begin{cases} a-1 = 2q_1b_1^2, \\ a+1 = 2q_2b_2^2, \end{cases}$$

for some integers b_1 and b_2 such that $b = 2b_1b_2$.

If the system (2) holds, then

$$-1 = \left(\frac{2q_1}{q_2}\right) = \left(\frac{a-1}{q_2}\right) = \left(\frac{a+1-2}{q_2}\right) = \left(\frac{-2}{q_2}\right) = 1.$$

This is absurd. Therefore $\begin{cases} a+1 = 2q_1b_1^2, \\ a-1 = 2q_2b_2^2. \end{cases}$ Thus, $\sqrt{\varepsilon_{q_1q_2}} = b_1\sqrt{q_1} + b_2\sqrt{q_2}$. So ε_d is not a square in L_d^+.

We have ε_{2d} has a positive norm. Put $\varepsilon_{2d} = x + y\sqrt{2q_1q_2}$. We similarly show that $\sqrt{2\varepsilon_{2d}} = y_1 + y_2\sqrt{2q_1q_2} \in L_d^+$ for some integers y_1 and y_2. Therefore, ε_{2d} is a square in L_d^+ since $\sqrt{2} \in L_d^+$. As ε_2 has a negative norm, then using the algorithm described in [25] (or in [3, page 113]) we have $\{\varepsilon_2, \varepsilon_{q_1q_2}, \sqrt{\varepsilon_{2q_1q_2}}\}$ is a fundamental system of units of L_d^+. Hence the result follows easily by [2, Proposition 2]. $\square$

To continue, consider the following parameters:

- For a rational prime p such that $p \equiv 1 \pmod 8$, set $p = u^2 - 2v^2$ where u and v are two positive integers such that $u \equiv 1 \pmod 8$ (for the existence of u and v, see [23]).

- For two primes q_1 and q_2 such that $q_1 \equiv q_2 \equiv 3 \pmod 8$, $\left(\frac{q_1}{q_2}\right) = 1$, there exist five integers X, Y, k, l and m such that $2q_2 = k^2X^2 + 2lXY + 2mY^2$ and $q_1 = l^2 - 2k^2m$ (cf. [18, page 356]).

Theorem 1. *Let d be an odd positive square free integer. Then $\mathrm{Cl}_2(L_d) \simeq (2,4)$ if and only if d takes one of the two following forms:*

1. $d = p \equiv 9 \pmod{16}$ *is a prime such that* $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$ *and* $\left(\frac{u}{p}\right)_4 = -1$.
2. $d = q_1q_2$, *with* $q_1 \equiv q_2 \equiv 3 \pmod 8$ *primes such that* $\left(\frac{q_1}{q_2}\right)_4 = 1$ *and* $\left(\frac{-2}{|k^2X + lY|}\right) = -1$.

Proof: It suffices to determine for which forms of d appearing in Lemma 1, we have $h_2(L_d) = 8$. Let us firstly eliminate some cases.

- By [4, Propositions 5.13 and 5.14], $h_2(L_d) \neq 8$ whenever d takes forms in the third and the fourth item of Lemma 1.
- The form of d in the second item of Lemma 1 is also eliminated by Lemma 2.
- Note that if $p \equiv 1 \pmod{16}$ and $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$ then by [4, Theorem 5.7], $h_2(L_p) \neq 8$.

Note that by results in the beginning of Section 3, $p \equiv 9 \pmod{16}$ implies that $\left(\frac{p}{2}\right)_4 = -1$. Hence, by Lemma 1, we have to check the following cases.

(I) $d = p$ is a prime such that $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.

(II) $d = p$ is a prime such that $p \equiv 9 \pmod{16}$ and $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.

(III) $d = q_1q_2$ for two primes q_1 and q_2 such that $q_1 \equiv q_2 \equiv 3 \pmod 8$.

Let $p \equiv 1 \pmod 8$ be a prime. Set $L_p^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p})$, $K = \mathbb{Q}(\sqrt{2}, i)$ and $K' = \mathbb{Q}(\sqrt{2}, \sqrt{-p})$. By applying Proposition 1 to the extension $L_p/\mathbb{Q}(\sqrt{2})$, we have

$$h(L_p) = \frac{Q_{L_p}}{Q_KQ_{K'}} \frac{\omega_{L_p}}{\omega_K\omega_{K'}} \frac{h(L_p^+)h(K)h(K')}{h(\mathbb{Q}(\sqrt{2}))^2}.$$

We have $h(\mathbb{Q}(\sqrt{2})) = h(K) = 1$. By [6, Théorème 3], $Q_{L_p} = 1$ and by [4, Lemma 2.5] $Q_K = 1$. Since $\omega_{L_p} = \omega_K = 8$ and $\omega_{K'} = 2$, then by passing to the 2-part in the above equality we get

$$h_2(L_p) = \frac{1}{2Q_{K'}} h_2(L_p^+)h_2(K')\cdot \tag{2}$$

As ε_2 has a negative norm, so by the item (2) of Section 3 of [2] we obtain that $E_{K'} = \langle -1, \varepsilon_2 \rangle$. This in turn implies that $q(K') = Q_{K'} = 1$. From which we infer, by Proposition 2, that $h_2(K') = \frac{1}{2} \cdot 1 \cdot h_2(2)h_2(-p)h_2(-2p) = \frac{1}{2}h_2(-p)h_2(-2p)$. It follows, by the equality (2), that

$$h_2(L_p) = \frac{1}{4}h_2(L_p^+)h_2(-p)h_2(-2p)\cdot \tag{3}$$

Note that from [14, Theorem 2] and the proof of [14, Theorem 1], one deduces easily that $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.

∘ Suppose that d takes the form (3). So 8 divides $h_2(-p)$. Note also that, by [17, page 596], $h_2(-2p)$ is divisible by 4, and by [19, Theorem 2] $h_2(L_p^+)$ is even. It follows by the equality (3) that 16 divides $h_2(L_p)$. So this case is eliminated.

∘ Suppose that d takes the form (3). Thus $\left(\frac{2}{p}\right)_4$ and $\left(\frac{p}{2}\right)_4 = -1$. So, by [19, Theorem 2], $h_2(L_p^+) = 1$. Therefore, by the equality (3) and the note below it, we have $h_2(L_p) = h_2(-2p)$.

Keep the notations of [17, page 601] and let $p = a^2 + b^2 = 2e^2 - d^2$, with $e \geq 1$. By using notations in the proof of [14, Theorem 1] and [17, page 601] we easily deduce that $\left(\frac{2}{p}\right)_4 = (-1)^{b/4}$, so $b \equiv 0 \pmod 8$. Therefore, by [17, Théroème 3], $h_2(-2p) \equiv 0 \pmod 8$. It follows, by [23, Theorem 2], $h_2(-2p) = 8$ if and only if $\left(\frac{u}{p}\right)_4 = -1$. So the first item of our theorem.

∘ Suppose that d takes the third form (3). Without loss of generality we may assume that $\left(\frac{q_1}{q_2}\right) = 1$. By Proposition 2 we have

$$h(L_d) = \frac{1}{2^5}q(L_d)h(q_1q_2)h(-q_1q_2)h(2q_1q_2)h(-2q_1q_2)h(2)h(-2)h(-1).$$

By [15, Corollary 18.4] $h_2(q_1q_2) = 1$ and by [18, pages 345, 354] we have $h_2(2q_1q_2) = 2$ and $h_2(-q_1q_2) = 4$, respectively. It is known that $h(2) = h(-2) = h(-1) = 1$. Since by Lemma 4 $q(L_d) = 4$, then by passing to the 2-part in the above equality we get

$$h_2(L_d) = \frac{1}{2^5} \cdot 4 \cdot 1 \cdot 4 \cdot 2 \cdot h_2(-2q_1q_2) = h_2(-2q_1q_2).$$

Therefore, by [18, page 357] and Lemma 1, $\mathrm{Cl}_2(L_d) = (2,4)$ if and only if $h_2(-2q_1q_2) = 8$ which is equivalent to $\left(\frac{-2}{|k^2X + lY|}\right) = -1$. This achieves the proof. □

Let l be a positive integer. For a finite abelian group G, its 2^l-rank is defined as $r_{2^l}(G) = \dim_{\mathbb{F}_2}(2^{l-1}G/2^lG)$. Or equivalently looking at the decomposition of the group G into cyclic groups as $G=\prod_i C_{n_i}$, the $2^{\,l}$-rank of G equals the number of n_i's divisible by $2^{\,l}$.

Corollary 1. *Let p be a rational prime such that $p\equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4=-1$, then the 8-rank of $\mathrm{Cl}_2(L_p)$ equals 1.*

PROOF: By [6, Théorème 10], we deduce that the 4-rank of $\mathrm{Cl}_2(L_p)$ equals 1, and by the proof of Theorem 1, $h_2(L_p)$ is divisible by 16, so the result follows. □

From the proof of Theorem 1 we get

Corollary 2. *Let q_1 and q_2 be two rational primes such that $q_1\equiv q_2\equiv 3 \pmod{\ 8}$, then $h_2(L_{q_1q_2})=h_2(-2q_1q_2)$.*

Example 2. For all the examples below, we used PARI/GP calculator version 2.9.4 (64bit), December 20, 2017.

1. For $p=89$, $u=17$ and $v=10$, we have $p=u^2-2v^2$, $\left(\frac{2}{p}\right)_4=-\left(\frac{u}{p}\right)_4=1$ and $\mathrm{Cl}_2(L_{89})\simeq(2,4)$.
2. For $q_1=11$, $q_2=19$, $k=1$, $l=3$, $m=-1$, $X=4$ and $Y=1$, we have $q_1=l^2-2k^2m$, $2q_2=k^2X^2+2lXY+2mY^2$, $\left(\frac{-2}{|k^2X+lY|}\right)=\left(\frac{-2}{7}\right)=-1$ and $\mathrm{Cl}_2(L_{11\cdot 19})\simeq(2,4)$.

4. Fields L_d for which $\mathrm{Cl}_2(L_d)$ is of type $(2,2,2)$

In this section we determine all fields L_d such that $\mathrm{Cl}_2(L_d)\simeq(2,2,2)$. Keep the above notations: p, p_i, q and q_i are prime integers satisfying $p\equiv p_i\equiv 1 \pmod 4$ and $q\equiv q_i\equiv 3 \pmod 4$ with $i\in\mathbb{N}^*$. From Section 4 of [4], it is easy to deduce the following result:

Lemma 5. *The rank of the 2-class group $\mathrm{Cl}_2(L_d)$ of L_d equals 3 if and only if d takes one of the following forms.*

1. *$d=p$ with $p\equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4=\left(\frac{p}{2}\right)_4=1$.*
2. *$d=q_1q_2$ with $q_1\equiv q_2\equiv 7 \pmod 8$.*
3. *$d=qp$ with $q\equiv 3 \pmod 8$ and $p\equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4=-1$.*
4. *$d=p_1p_2$ with $p_1\equiv 5 \pmod 8$, $p_2\equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4\neq\left(\frac{p}{2}\right)_4$.*
5. *$d=q_1q_2p$ with $q_1\equiv q_2\equiv 3 \pmod 8$ and $p\equiv 5 \pmod 8$.*
6. *$d=qp_1p_2$ with $q\equiv 3 \pmod 8$ and $p_1\equiv p_2\equiv 5 \pmod 8$.*

We will need the following lemmas.

Lemma 6. *Let $d = q_1q_2$, with $q_1 \equiv q_2 \equiv 7 \pmod 8$ two primes such that $\left(\frac{q_1}{q_2}\right) = 1$. Then we have $E_{L_d} = \langle \zeta_8, \varepsilon_2, \varepsilon_{q_1q_2}, \sqrt{\varepsilon_{2q_1q_2}}$ or $\sqrt{\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\rangle$, and thus $q(L_d) = 4$.*

Proof: Similar to the proof of Lemma 4. □

Lemma 7.

1. *Let $d = p$ for a prime $p \equiv 1 \pmod 8$ such that $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = 1$. Then, $h_2(L_p) \equiv 0 \pmod{16}$, and thus $\mathrm{Cl}_2(L_p)$ is not elementary.*
2. *Let $d = q_1q_2$ with q_1 and q_2 two rational primes such that $q_1 \equiv q_2 \equiv 7 \pmod 8$. Then, $h_2(L_{q_1q_2}) \equiv 0 \pmod{32}$, and thus $\mathrm{Cl}_2(L_{q_1q_2})$ is not elementary.*
3. *Let $d = p_1p_2$ with $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 1 \pmod 8$ two primes such that $\left(\frac{p_2}{p_1}\right) = 1$. Then, $h_2(L_{p_1p_2}) \equiv 0 \pmod{32}$.*

Proof: 1. By equality (3) in the proof of Theorem 1 we have

$$h_2(L_p) = \frac{1}{4} h_2(L_p^+) h_2(-p) h_2(-2p).$$

Note that by [17, page 596], $h_2(-2p)$ is divisible by 4. Since for a prime $p' \equiv 1 \pmod 8$, $h_2(-p') = 4$ if and only if $\left(\frac{2}{p'}\right)_4 \neq \left(\frac{p'}{2}\right)_4$ (see the proof of Theorem 1), then $h_2(-p)$ is divisible by 8 (in fact $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4$ and by [15, Corollaries 18.4 and 19.6] $h_2(-p)$ is divisible by 4). Thus, $h_2(L_p)$ is divisible by $8 \cdot h_2(L_p^+)$. By [19, Theorem 2], $h_2(L_p^+)$ is even. From which we infer that $h_2(L_p)$ is divisible by 16. So we have the first item.

2. By Proposition 2 we have

$$h(L_{q_1q_2}) = \frac{1}{2^5} q(L_{q_1q_2}) h(q_1q_2) h(-q_1q_2) h(2q_1q_2) h(-2q_1q_2) h(2) h(-2) h(-1) \cdot$$

On one hand, by [18, page 345], $h_2(2q_1q_2)$ is divisible by 4, by [18, pages 354, 356], $h_2(-q_1q_2)$ and $h_2(-2q_1q_2)$ are both divisible by 8, and by [15, Corollary 18.4], $h_2(q_1q_2) = 1$. On the other hand, by Lemma 6, $q(L_{q_1q_2}) = 4$. From all these results, it follows by passing to the 2-part in the above equality that $h_2(L_d)$ is divisible by $(1/2^5) \cdot 4 \cdot 8 \cdot 4 \cdot 8 = 32$. And the second item follows.

3. We proceed as in the proof the second item. □

Example 3. Let $d = p$ be as in the first item of the above lemma. Using PARI/GP calculator version 2.9.4 (64bit), December 20, 2017, for $3 \leq p \leq 10^4$, we could not find a field L_d such that $h_2(L_d) = 16$. We have the following examples.

1. For $p=113$, we have $\left(\frac{2}{p}\right)_4=\left(\frac{p}{2}\right)_4=1$ and $h_2(L_{113})=64$.
2. For $p=337$, we have $\left(\frac{2}{p}\right)_4=\left(\frac{p}{2}\right)_4=1$ and $h_2(L_{337})=32$.
3. For $q_1=7$ and $q_2=31$, we have $h_2(L_{7\cdot 31})=32$.
4. For $q_1=7$ and $q_2=23$, we have $h_2(L_{7\cdot 23})=64$.

Lemma 8.

1. *If $d=q_1q_2p$ with $q_1\equiv q_2\equiv 3 (\mathrm{mod}\ 8)$ and $p\equiv 5 (\mathrm{mod}\ 8)$, then $h_2(L_{q_1q_2p})\equiv 0 (\mathrm{mod}\ 32)$. So $\mathrm{Cl}_2(L_{q_1q_2p})$ is not elementary.*
2. *If $d=qp_1p_2$ with $q\equiv 3\,(\mathrm{mod}\ 8)$ and $p_1\equiv p_2\equiv 5\,(\mathrm{mod}\ 8)$, then $h_2(L_{qp_1p_2})\equiv 0 (\mathrm{mod}\ 32)$. So $\mathrm{Cl}_2(L_{qp_1p_2})$ is not elementary.*

Proof: 1. Consider the following diagram (Figure 1 below):

$$\begin{tikzcd}
 & L_d=\mathbb{Q}(\sqrt{q_1q_2p},\sqrt{2},i) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
K_2=\mathbb{Q}(\sqrt{q_1q_2p},i) \arrow[dr, dash] & K_1=\mathbb{Q}(\sqrt{2},i) \arrow[d, dash] & K_3=\mathbb{Q}(\sqrt{2q_1q_2p},i) \arrow[dl, dash] \\
 & \mathbb{Q}(i) &
\end{tikzcd}$$

Figure 1. $\frac{L_{q_1q_2p}}{\mathbb{Q}(i)}$.

By applying Proposition 3 to the V_4-extension $L_{q_1q_2p}/\mathbb{Q}(i)$ we get:

$$h(L_d)=\frac{1}{4}Q\Big(\frac{L_d}{\mathbb{Q}(i)}\Big)\frac{h(K_1)h(K_2)h(K_3)}{h(\mathbb{Q}(i))^2}.$$

It is known that $h(K_1)=h(\mathbb{Q}(i))=1$ (in fact, $K_1=\mathbb{Q}(\zeta_8)$), so

$$h(L_d)=\frac{1}{4}Q\Big(\frac{L_d}{\mathbb{Q}(i)}\Big)h(K_2)h(K_3). \tag{4}$$

Note that by [24, Proposition 2], the ranks of the 2-class groups of K_2 and K_3 equal 2 and 3, respectively. The author of [1] determined all fields $\mathbb{Q}(\sqrt{d},i)$ for which the 2-class group is of type $(2,2)$. By checking all the results of this last reference, we deduce that 2-class group of K_2 is not of type $(2,2)$. So 8 divides the class number of K_2.

On the other hand, by [7, Théorème 5.3] the 2-class group of K_3 is not of type $(2,2,2)$. So the class number of K_2 is divisible by 16. Hence by equality (4) $h(L_d)$ is divisible by $\frac{1}{4}\cdot 8\cdot 16=32$. So the first item follows.

2. We similarly prove the second item by using [7, Théorème 5.3] and [24, Proposition 2]. $\square$

Example 4. Let $d = qp_1p_2$ be as in the second item of the above Lemma. Using PARI/GP calculator version 2.9.4 (64bit), December 20, 2017, for $3 \leq q, p_i \leq 100$, we could not find a field L_d such that $h_2(L_d) = 32$. We have the following examples.

1. For $q_1 = 3$, $q_2 = 11$ and $p = 5$, we have $h_2(L_{3\cdot 11\cdot 5}) = 32$.
2. For $q_1 = 3$, $q_2 = 11$ and $p = 13$, we have $h_2(L_{3\cdot 11\cdot 13}) = 128$.
3. For $q = 3$, $p_1 = 5$ and $p_2 = 13$, we have $h_2(L_{3\cdot 5\cdot 13}) = 64$.
4. For $q = 3$, $p_1 = 5$ and $p_2 = 29$, we have $h_2(L_{3\cdot 5\cdot 29}) = 128$.

Lemma 9. *Let $d = qp$ with $q \equiv 3 \pmod 8$, $p \equiv 1 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$. Then $E_{L_d} = \langle \zeta_8, \varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}} \rangle$, and thus $q(L_d) = 8$.*

PROOF: We proceed as in the proof of Lemma 4. □

Now we are able to state the main theorem of this subsection.

Theorem 2. *Let d be a square free integer, then $\mathrm{Cl}_2(L_d) \simeq (2,2,2)$ if and only if d takes one of the two following forms:*

1. $d = p_1p_2$ *with* $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 1 \pmod 8$, $\left(\frac{2}{p_2}\right)_4 \neq \left(\frac{p_2}{2}\right)_4$ *and* $\left(\frac{p_2}{p_1}\right) = -1$.
2. $d = qp$ *with* $q \equiv 3 \pmod 8$, $p \equiv 1 \pmod 8$, $\left(\frac{2}{p}\right)_4 = -1$ *and* $\left(\frac{p}{q}\right) = -1$.

PROOF: It suffices to determine for which forms of d appearing in Lemma 5, we have $h_2(L_d) = 8$. Let us start, as above, by eliminating certain inconvenient cases.

- ∘ The forms of d in the first and the second items of Lemma 5 are eliminated by Lemma 7.
- ∘ The forms of d in the two last items of Lemma 5 are eliminated by Lemma 8.

It follows that it suffices to check the two following cases:

(I) $d = p_1p_2$ with $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 1 \pmod 8$ and $\left(\frac{2}{p_2}\right)_4 \neq \left(\frac{p_2}{2}\right)_4$.

(II) $d = qp$ with $q \equiv 3 \pmod 8$ and $p \equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 = -1$.

∘ Suppose that d takes the form I, then [10, Theorem 4.2] and the last assertion of Lemma 7 give the first item.

∘ Suppose now that d takes the form II. Consider the following diagram (Figure 2 below):

$$\begin{tikzcd}[column sep=small]
 & L_d = \mathbb{Q}(\sqrt{2}, \sqrt{d}, i) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
L_d^+ = \mathbb{Q}(\sqrt{d}, \sqrt{2}) \arrow[dr, dash] & K_2 = \mathbb{Q}(\sqrt{2}, i) \arrow[d, dash] & K_1 = \mathbb{Q}(\sqrt{2}, \sqrt{-d}) \arrow[dl, dash] \\
 & \mathbb{Q}(\sqrt{2}) &
\end{tikzcd}$$

FIGURE 2. $\dfrac{L_d}{\mathbb{Q}(\sqrt{2})}$.

By Proposition 1 we have

$$h(L_d) = \frac{Q_{L_d}}{Q_{K_1} Q_{K_2}} \frac{\omega_{L_d}}{\omega_{K_1}\omega_{K_2}} \frac{h(L_d^+)h(K_1)h(K_2)}{h(\mathbb{Q}(\sqrt{2}))^2}. \tag{5}$$

Note that $h(K_2) = h(\mathbb{Q}(\sqrt{2})) = 1$. We have, $d\varepsilon_2$ is not a square in $\mathbb{Q}(\sqrt{2})$. Otherwise we will get for some α in $\mathbb{Q}(\sqrt{2})$, $d\varepsilon_2 = \alpha^2$, then $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(d\varepsilon_2) = -d^2 = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\alpha)^2$, which is false. It follows by [2, Proposition 3], that $\{\varepsilon_2\}$ is a fundamental system of units of K_1, i.e., $Q_{K_1} = 1$. Since $Q_{K_2} = 1$ (cf. [4, Lemma 2.5]), $\omega_{L_d} = \omega_{K_2} = 8$ and $\omega_{K_1} = 2$, then by passing to the 2-part in the equality (5), we get $h_2(L_d) = \frac{1}{2}Q_{L_d} h_2(L_d^+)h_2(K_1)$. By Proposition 3, we have $h_2(K_1) = \frac{1}{2}h_2(-2d)h_2(-d)h_2(2)$. So

$$h_2(L_d) = \frac{1}{4}Q_{L_d} h_2(L_d^+)h_2(-2d)h_2(-d). \tag{6}$$

Note that, by Lemma 9, if $\left(\frac{p}{q}\right) = -1$, then $Q_{L_d} = 1$. Note also that,

(1) from [5, Théorème 2] and its proof, we deduce that $h_2(L_d^+)$ is divisible by 4 and $h_2(L_d^+) = 4$ if and only if $\left(\frac{p}{q}\right) = -1$,
(2) from [15, Corollaries 18.5 and 19.6], we deduce that $h_2(-d)$ is even and $h_2(-d) = 2$ if and only if $\left(\frac{p}{q}\right) = -1$,
(3) from [15, Corollaries 18.5 and 19.6] and [18, page 353], we deduce that $h_2(-2d)$ is divisible by 4 and $h_2(-2d) = 4$ if and only if $\left(\frac{p}{q}\right) = -1$.

Hence plugging all of these results into equality (6), one gets the second item, which completes the proof. □

Example 5. For all the examples below, we used PARI/GP calculator version 2.9.4 (64bit), December 20, 2017.

1. Let $p_1 = 29$ and $p_2 = 17$. We have $p_2 \equiv 1 (\mathrm{mod}\ 8)$, $\left(\frac{2}{p_2}\right)_4 \neq \left(\frac{p_2}{2}\right)_4$, $\left(\frac{p_2}{p_1}\right) = -1$ and $\mathrm{Cl}_2(L_{29\cdot 17}) \simeq (2,2,2)$.
2. Let $q = 11$ and $p = 17$. We have $\left(\frac{2}{p}\right)_4 = -1$, $\left(\frac{p}{q}\right) = -1$ and $\mathrm{Cl}_2(L_{11\cdot 17}) \simeq (2,2,2)$.

Acknowledgment. We would like to thank the unknown referee for his/her several helpful suggestions that helped us to improve our paper, and for calling our attention to the missing details. Many thanks are also due to Professor Elliot Benjamin for his important remarks on the paper.

References

[1] Azizi A., *Sur le 2-groupe de classes d'idéaux de* $\mathbb{Q}(\sqrt{d}, i)$, Rend. Circ. Mat. Palermo (2) **48** (1999), no. 1, 71–92 (French. English summary).

[2] Azizi A., *Unités de certains corps de nombres imaginaires et abéliens sur* $\mathbb{Q}$, Ann. Sci. Math. Québec **23** (1999), no. 1, 15–21 (French. English, French summary).

[3] Azizi A., *Sur les unités de certains corps de nombres de degré 8 sur* $\mathbb{Q}$, Ann. Sci. Math. Québec **29** (2005), no. 2, 111–129 (French. English, French summary).

[4] Azizi A., Chems-Eddin M. M., Zekhnini A., *On the rank of the 2-class group of some imaginary triquadratic number fields*, Rend. Circ. Mat. Palermo, II. Ser. (2021), 19 pages.

[5] Azizi A., Mouhib A., *Capitulation des 2-classes d'idéaux de* $\mathbb{Q}(\sqrt{2}, \sqrt{d})$ *où d est un entier naturel sans facteurs carrés*, Acta. Arith. **109** (2003), no. 1, 27–63 (French).

[6] Azizi A., Taous M., *Capitulation des 2-classes d'idéaux de* $k = \mathbb{Q}(\sqrt{2p}, i)$, Acta Arith. **131** (2008), no. 2, 103–123 (French).

[7] Azizi A., Taous M., *Déterminations des corps* $K = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ *dont les 2-groupes de classes sont de type* $(2,4)$ *ou* $(2,2,2)$, Rend. Istit. Mat. Univ. Trieste **40** (2008), 93–116 (French. English, French summary).

[8] Azizi A., Zekhnini A., Taous M., *On the strongly ambiguous classes of* $\mathbb{k}/\mathbb{Q}(i)$ *where* $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$, Asian-Eur. J. Math. **7** (2014), no. 1, 1450021, 26 pages.

[9] Azizi A., Zekhnini A., Taous M., *The generators of the 2-class group of some fields* $\mathbb{Q}(\sqrt{pq_1q_2}, i)$*: Correction to Theorem 3 of* [5], IJPAM. **103** (2015), no. 1, 99–107.

[10] Azizi A., Zekhnini A., Taous M., *On the unit index of some real biquadratic number fields*, Turkish J. Math. **42** (2018), no. 2, 703–715.

[11] Benjamin E., Lemmermeyer F., Snyder C., *Imaginary quadratic fields k with* $\mathrm{Cl}_2(k) \simeq (2, 2^m)$ *and rank* $\mathrm{Cl}_2(k^1) = 2$, Pacific J. Math. **198** (2001), no. 1, 15–31.

[12] Benjamin E., Lemmermeyer F., Snyder C., *Imaginary quadratic fields k with* $\mathrm{Cl}_2(k) \simeq (2,2,2)$, J. Number Theory **103** (2003), no. 1, 38–70.

[13] Benjamin E., Lemmermeyer F., Snyder C., *On the unit group of some multiquadratic number fields*, Pacific J. Math. **230** (2007), no. 1, 27–40.

[14] Brown E., *The class number of* $Q(\sqrt{-p})$ *for* $P \equiv 1 (\mathrm{mod}\ 8)$ *a prime*, Proc. Amer. Math. Soc. **31** (1972), 381–383.

[15] Conner P. E., Hurrelbrink J., *Class Number Parity*, Series in Pure Mathematics, 8, World Scientific Publishing, Singapore, 1988.

[16] Herglotz G., *Über einen Dirichletschen Satz*, Math. Z. **12** (1922), no. 1, 255–261 (German).

[17] Kaplan P., *Divisibilité par 8 du nombre des classes des corps quadratiques dont le 2-groupe des classes est cyclique, et réciprocité biquadratique*, J. Math. Soc. Japan **25** (1973), 596–608 (French).
[18] Kaplan P., *Sur le 2-groupe des classes d'idéaux des corps quadratiques*, J. Reine Angew. Math. **283(284)** (1976), 313–363 (French).
[19] Kučera R., *On the parity of the class number of a biquadratic field*, J. Number theory **52** (1995), no. 1, 43–52.
[20] Lemmermeyer F., *Ideal class groups of cyclotomic number fields. I*, Acta Arith. **72** (1995), no. 4, 347–359.
[21] Kuroda S., *Über die Klassenzahlen algebraischer Zahlkörper*, Nagoya Math. J. **1** (1950), 1–10 (German).
[22] Lemmermeyer F., *Kuroda's class number formula*, Acta Arith. **66** (1994), no. 3, 245–260.
[23] Leonard P. A., Williams K. S., *On the divisibility of the class numbers of* $Q(\sqrt{-p})$ *and* $Q(\sqrt{-2p})$ *by* 16, Canad. Math. Bull. **25** (1982), no. 2, 200–206.
[24] McCall T. M., Parry C. J., Ranalli R., *Imaginary bicyclic biquadratic fields with cyclic 2-class group*, J. Number Theory **53** (1995), no. 1, 88–99.
[25] Wada H., *On the class number and the unit group of certain algebraic number fields*, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 201–209.

A. Azizi, M. M. Chems-Eddin:
Mohammed First University, Mathematics Department, Sciences Faculty, Mohammed V avenue, P. O. Box 524, Oujda 60040, Morocco

E-mail: abdelmalekazizi@yahoo.fr

E-mail: 2m.chemseddin@gmail.com

A. Zekhnini:
Mohammed First University, Mathematics Department,
Pluridisciplinary Faculty, B. P. 300, Selouane, Nador 62700, Morocco

E-mail: zekha1@yahoo.fr

(*Received* June 18, 2019, *revised* September 29, 2020)