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# $\alpha$-filters and $\alpha$-order-ideals in distributive quasicomplemented semilattices

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*Abstract.* We introduce some particular classes of filters and order-ideals in distributive semilattices, called $\alpha$-filters and $\alpha$-order-ideals, respectively. In particular, we study $\alpha$-filters and $\alpha$-order-ideals in distributive quasicomplemented semilattices. We also characterize the filters-congruence-cokernels in distributive quasicomplemented semilattices through $\alpha$-order-ideals.

*Keywords:* bounded distributive semilattice; quasicomplement; relative annihilator; order-ideal; filter

*Classification:* 06A12, 03G10

## 1. Introduction

If $\mathbf{A} = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $a, b \in A$, then the *annihilator of a relative to b* is the set $\langle a, b\rangle = \{x \in A\colon a \wedge x \leq b\}$. In [13] M. Mandelker studied the properties of relative annihilators and prove that a lattice $\mathbf{A}$ is distributive if and only if $\langle a, b\rangle$ is an ideal of $\mathbf{A}$ for all $a, b \in A$. These results were generalized by J. Varlet in [21] to the class of distributive semilattices. In particular, the annihilator of $a$ relative to 0 is the set $a^{\circ} = \langle a, 0\rangle = \{x \in A\colon a \wedge x = 0\}$, called *annihilator of a* or *annulet of a*. This concept was studied by W. Cornish in [8], [9], where introduces the notion of $\alpha$*-ideal* in distributive lattices. A generalization of the concept of $\alpha$-ideal in 0-distributive semilattices and 0-distributive lattices were studied in [16], [17].

On the other hand, in [20], [19], [10] the class of *distributive quasicomplemented lattices* was studied as a generalization of the variety of distributive pseudocomplemented lattices. A bounded distributive lattice $\mathbf{A}$ is *quasicomplemented* if for each $a \in A$, there is $b \in A$ such that $a^{\circ\circ} = b^{\circ}$, where $a^{\circ\circ} = \{x \in A\colon \forall\, y \in a^{\circ}(x \wedge y = 0)\}$. Clearly, this class of lattices is not a variety, and in general, the element $b$ is non-unique. This concept can be generalized to bounded semilattices in [12], [15], [18].

The main aim of this paper is to introduce and study the notions of $\alpha$*-filter* and $\alpha$*-order-ideal* in bounded distributive semilattices, which generalizes the results

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given in [8], [9]. In particular, we investigate $\alpha$-filters and $\alpha$-order-ideals in the class of distributive quasicomplemented semilattices and we give some results on filters-congruence-cokernels through $\alpha$-order-ideals.

The paper is organized as follows. In Section 2 we recall some necessary definitions and results to make the paper self-contained. We recall the notion of annihilator and its properties in terms of irreducible and maximal filters given in [4], [5]. In Section 3 we present the concepts of $\alpha$-filter and $\alpha$-order-ideal in bounded distributive semilattices and give some properties. We prove a separation theorem between filters and $\alpha$-order-ideals by means of irreducible $\alpha$-filters. In Section 4 we study $\alpha$-filters and $\alpha$-order-ideals in distributive quasicomplemented semilattices. We prove that the set of all $\alpha$-filters is a Heyting algebra isomorphic to the Heyting algebra of $\alpha$-order-ideals. Finally, in Section 5, we study the filters-congruence-cokernels in distributive quasicomplemented semilattices. We prove that a subset $I$ is an $\alpha$-filter-congruence-cokernel if and only if it is an $\alpha$-order-ideal.

## 2. Preliminaries

We give some necessary notations and definitions. Let $\langle X, \leq\rangle$ be a poset. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*, respectively), if for all $x, y \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$). The set of all subsets of $X$ is denoted by $\mathcal{P}(X)$ and the set of all increasing subsets of $X$ is denoted by $\mathcal{P}_i(X)$. For each $Y \subseteq X$, the increasing (decreasing) set generated by $Y$ is $[Y) = \{x \in X \colon \exists y \in Y(y \leq x)\}$ ($(Y] = \{x \in X \colon \exists y \in Y(x \leq y)\}$, respectively). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively.

A *meet-semilattice with greatest element*, or simply *semilattice*, is an algebra $\mathbf{A} = \langle A, \wedge, 1\rangle$ of type $(2, 0)$ such that the operation "$\wedge$" is idempotent, commutative, associative and $a \wedge 1 = a$ for all $a \in A$. So, the binary relation "$\leq$" defined by $a \leq b$ if and only if $a \wedge b = a$ is an order. A *bounded* semilattice is an algebra $\mathbf{A} = \langle A, \wedge, 0, 1\rangle$ of type $(2, 0, 0)$ such that $\langle A, \wedge, 1\rangle$ is a semilattice and $a \wedge 0 = 0$ for all $a \in A$.

Let $\mathbf{A}$ be a semilattice. A *filter* is a subset $F$ of $A$ such that $1 \in F$, $F$ is increasing, and if $a, b \in F$, then $a \wedge b \in F$. The *filter generated by a subset* $X \subseteq A$ is the set $F(X) = \{a \in A \colon \exists x_1, \ldots, x_n \in X(x_1 \wedge \ldots \wedge x_n \leq a)\}$. If $X = \{a\}$, then $F(\{a\}) = [a)$. Denote by $\mathrm{Fi}(A)$ the set of all filters of $A$. Since $1 \in A$, it follows that $\langle \mathrm{Fi}(A), \subseteq\rangle$ is a lattice. A proper filter $P$ is *irreducible* if for every $F_1, F_2 \in \mathrm{Fi}(A)$ such that $P = F_1 \cap F_2$, then $P = F_1$ or $P = F_2$. Note that a filter $F$ is irreducible if and only if for every $a, b \notin F$, there exists $c \notin F$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$. The set of all irreducible filters of $A$

is denoted by $\mathrm{X}(A)$. A proper filter $U$ is *maximal* if for any $G \in \mathrm{Fi}(A)$ such that $U \subseteq G$, we have $G = U$ or $G = A$. Denote by $\mathrm{X}_m(A)$ the set of all maximal filters of $A$. A subset $I$ of $A$ is an *order-ideal* if $I$ is decreasing, and for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$. We denote by Id(A) the set of all order-ideals of $A$. Finally, a proper order-ideal $I$ is *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. The following result was proved in [2].

**Theorem 1.** *Let* $\mathbf{A}$ *be a semilattice. Let* $F \in \mathrm{Fi}(A)$ *and* $I \in \mathrm{Id}(A)$ *such that* $F \cap I = \emptyset$. *Then there exists* $P \in \mathrm{X}(A)$ *such that* $F \subseteq P$ *and* $P \cap I = \emptyset$.

We are interested in a particular class of semilattices.

**Definition 2.** Let $\mathbf{A}$ be a semilattice. We say that $\mathbf{A}$ is *distributive* if for every $a, b, c \in A$ such that $a \wedge b \leq c$, there exist $a_1, b_1 \in A$ such that $a \leq a_1$, $b \leq b_1$ and $c = a_1 \wedge b_1$.

We denote by $\mathcal{DS}$ and $\mathcal{DS}_{01}$ the class of distributive semilattices and the class of bounded distributive semilattices, respectively. Note that $\mathcal{DS}$ is not a variety. A lattice is distributive if and only if it is distributive as a semilattice, see [7], [11]. The next theorem was proved by G. Grätzer in [11].

**Theorem 3.** *Let* $\mathbf{A}$ *be a semilattice. The following conditions are equivalent:*

(1) *The semilattice* $\mathbf{A}$ *is distributive.*
(2) $\langle \mathrm{Fi}(A), \subseteq \rangle$, *considered as a lattice, is distributive.*

In [6] it was proved that if $\mathbf{A}$ is a distributive semilattice, then the structure $\mathrm{Fi}(\mathbf{A}) = \langle \mathrm{Fi}(A), \veebar, \barwedge, \rightarrow, \{1\}, A \rangle$ is a Heyting algebra where the least element is $\{1\}$, the greatest element is $A$, $G \veebar H = F(G \cup H)$, $G \barwedge H = G \cap H$ and

$$G \rightarrow H = \{a \in A \colon [a) \cap G \subseteq H\}$$

for all $G, H \in \mathrm{Fi}(A)$.

**Remark 4.** If $\mathbf{A} \in \mathcal{DS}$, then every maximal filter $U$ is irreducible. Indeed, let $F_1, F_2 \in \mathrm{Fi}(A)$ be such that $U = F_1 \cap F_2$. Suppose that $U \subset F_1$ and $U \subset F_2$. Since $U$ is maximal, we have $F_1 = F_2 = A$ and $U = A$, which is a contradiction because $U$ is proper. Thus, $\mathrm{X}_m(A) \subseteq \mathrm{X}(A)$.

Let $\mathbf{A}$ be a bounded semilattice and $a \in A$. The *annihilator of* $a$ is the set

$$a^\circ = \{x \in A \colon a \wedge x = 0\}.$$

The annihilators were studied by several authors in [13], [4], [17], [5], [9], [10]. In general, $a^\circ$ is a decreasing subset, but not an order-ideal. Later, $\mathbf{A}$ is distributive if and only if $a^\circ$ is an order-ideal for all $a \in A$. For more details see [21], [7]. We note that if $\mathbf{A}$ is pseudocomplemented, i.e., if for every $a \in A$ there exists $a^* \in A$ such that $a \wedge x = 0$ if and only if $x \leq a^*$, then $a^\circ = (a^*]$.

If $X \subseteq A$, then we define the *annihilator of* $X$ as the set

$$X^\circ = \{x \in A \colon \forall\, y \in X (x \wedge y = 0)\} = \bigcap \{x^\circ \colon x \in X\}.$$

In particular, $a^{\circ\circ} = \{x \in A \colon \forall\, y \in a^\circ (x \wedge y = 0)\} = \bigcap \{x^\circ \colon x \in a^\circ\}$ for all $a \in A$. In the following result we remember some properties of the annihilators in bounded distributive semilattices.

**Lemma 5.** *Let* $\mathbf{A} \in \mathcal{DS}_{01}$*. Let* $a, b \in A$ *and* $P \in \mathrm{X}(A)$*. We have the following properties:*

(1) *If* $a \in b^\circ$*, then* $b^{\circ\circ} \subseteq a^\circ$*.*
(2) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \cap b^{\circ\circ}$*.*
(3) $a^\circ \cap P = \emptyset$ *if and only if there exists* $Q \in \mathrm{X}(A)$ *such that* $P \subseteq Q$ *and* $a \in Q$*.*
(4) $a^\circ \cap P = \emptyset$ *if and only if there exists* $U \in \mathrm{X}_m(A)$ *such that* $P \subseteq U$ *and* $a \in U$*.*
(5) $U \in \mathrm{X}_m(A)$ *if and only if* $U \in \mathrm{Fi}(A)$ *and* $\forall\, a \in A\, (a \notin U \iff a^\circ \cap U \neq \emptyset)$*.*
(6) *If* $U \in \mathrm{X}_m(A)$*, then* $\forall\, a \in A$ $(a \notin U \iff a^{\circ\circ} \cap U = \emptyset)$*.*

Proof: We only prove (1) and (2). The rest can be seen in [4], [5].

(1) Let $x \in b^{\circ\circ} = \bigcap \{y^\circ \colon y \in b^\circ\}$. In particular, $a \in b^\circ$ and $x \in a^\circ$. Then $b^{\circ\circ} \subseteq a^\circ$.

(2) Let $x \in (a \wedge b)^{\circ\circ}$. Let $y \in a^\circ$. Since $a \wedge b \leq a$, we have $a^\circ \subseteq (a \wedge b)^\circ$. Thus, $y \in (a \wedge b)^\circ$ and as $x \in (a \wedge b)^{\circ\circ}$, it follows that $y \wedge x = 0$, i.e., $x \in a^{\circ\circ}$. Similarly, $x \in b^{\circ\circ}$ and $(a \wedge b)^{\circ\circ} \subseteq a^{\circ\circ} \cap b^{\circ\circ}$.

Conversely, let $x \in a^{\circ\circ} \cap b^{\circ\circ}$. If $y \in (a \wedge b)^\circ$, then $y \wedge (a \wedge b) = (y \wedge a) \wedge b = 0$ and $y \wedge a \in b^\circ$. Since $x \in b^{\circ\circ}$, $x \wedge (y \wedge a) = (x \wedge y) \wedge a = 0$ and $x \wedge y \in a^\circ$. Again, as $x \in a^{\circ\circ}$, we have $x \wedge (x \wedge y) = x \wedge y = 0$ and $x \in (a \wedge b)^{\circ\circ}$. So, $a^{\circ\circ} \cap b^{\circ\circ} \subseteq (a \wedge b)^{\circ\circ}$. □

Let $\mathbf{A} \in \mathcal{DS}_{01}$. Consider $\langle \mathrm{X}(A), \subseteq \rangle$ and the map $\varphi_{\mathbf{A}} \colon A \to \mathcal{P}_i(\mathrm{X}(A))$ given by $\varphi_{\mathbf{A}}(a) = \{P \in \mathrm{X}(A) \colon a \in P\}$. It follows that $\mathbf{A}$ is isomorphic to the subalgebra $\varphi_{\mathbf{A}}[\mathbf{A}] = \{\varphi_{\mathbf{A}}(a) \colon a \in A\}$ of $\mathcal{P}_i(\mathrm{X}(A))$. Later, for each $F \in \mathrm{Fi}(A)$, we define

$$\phi[F] = \{P \in \mathrm{X}(A) \colon F \subseteq P\}.$$

So, $\phi[F] = \bigcap \{\varphi_{\mathbf{A}}(a) \colon a \in F\}$. Also, if $Y \subseteq A$, then

$$\psi[Y] = \{P \in \mathrm{X}(A) \colon Y \cap P \neq \emptyset\} = \bigcup \{\varphi_{\mathbf{A}}(a) \colon a \in Y\}.$$

**Remark 6.** Let $\mathbf{A} \in \mathcal{DS}_{01}$. By Lemma 5, we have that:

(1) $\varphi_{\mathbf{A}}(a)^c \cap \mathrm{X}_m(A) = \psi[a^\circ] \cap \mathrm{X}_m(A)$.
(2) $\varphi_{\mathbf{A}}(a) \cap \mathrm{X}_m(A) = \psi[a^{\circ\circ}] \cap \mathrm{X}_m(A)$.

## 3. $\alpha$-filters and $\alpha$-order-ideals in $\mathcal{DS}_{01}$

In this section we introduce the classes of $\alpha$-filters and $\alpha$-order-ideals in bounded distributive semilattices. In particular, we give a separation theorem between filters and $\alpha$-order-ideals by means of irreducible $\alpha$-filters.

**Definition 7.** Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $F \in \mathrm{Fi}(A)$. We say that $F$ is an *$\alpha$-filter* if $a^{\circ\circ} \cap F \neq \emptyset$ implies that $a \in F$ for all $a \in A$.

Denote by $\mathrm{Fi}_\alpha(A)$ and $\mathrm{X}_\alpha(A)$ the set of all $\alpha$-filters and irreducible $\alpha$-filters of $A$, respectively.

**Example 8.** Let $\mathbf{A} \in \mathcal{DS}_{01}$. It follows that the set $D(A) = \{a \in A \colon a^\circ = \{0\}\}$, called the set of *dense elements*, is a filter of $A$. Then, $D(A)$ is an $\alpha$-filter. Indeed, if $a^{\circ\circ} \cap D(A) \neq \emptyset$, then there exists $x \in a^{\circ\circ}$ such that $x^\circ = \{0\}$. So, by Lemma 5, $a^\circ \subseteq x^\circ = \{0\}$ and $a \in D(A)$. Moreover, the $\alpha$-filter $D(A)$ is the smallest $\alpha$-filter of $A$. Let $F \in \mathrm{Fi}_\alpha(A)$. If $a \in D(A)$, then $a^\circ = \{0\}$ and $a^{\circ\circ} = A$. Then $a^{\circ\circ} \cap F \neq \emptyset$ and since $F$ is an $\alpha$-filter, we have $a \in F$. Therefore, $D(A) \subseteq F$ for all $F \in \mathrm{Fi}_\alpha(A)$.

**Lemma 9.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $I \in \mathrm{Id}(A)$. Then*

$$F_I = \{a \in A \colon \exists\, x \in I (a^\circ \subseteq x^{\circ\circ})\}$$

*is an $\alpha$-filter.*

Proof: It is clear that $1 \in F_I$ and $F_I$ is an increasing subset of $A$. Let $a, b \in F_I$. Then there exist $x, y \in I$ such that $a^\circ \subseteq x^{\circ\circ}$ and $b^\circ \subseteq y^{\circ\circ}$. So, $x^\circ \subseteq a^{\circ\circ}$ and $y^\circ \subseteq b^{\circ\circ}$. By Lemma 5, $x^\circ \cap y^\circ \subseteq a^{\circ\circ} \cap b^{\circ\circ} = (a \wedge b)^{\circ\circ}$. As $I$ is an order-ideal, there is $z \in I$ such that $x \leq z$ and $y \leq z$. It follows that $z^\circ \subseteq x^\circ \cap y^\circ \subseteq (a \wedge b)^{\circ\circ}$, i.e., $(a \wedge b)^\circ \subseteq z^{\circ\circ}$ and $a \wedge b \in F_I$. Thus, $F_I \in \mathrm{Fi}(A)$. We see that $F_I$ is an $\alpha$-filter. Let $a \in A$ be such that $a^{\circ\circ} \cap F_I \neq \emptyset$. Then there is $b \in a^{\circ\circ}$ such that $b \in F_I$. Then $a^\circ \subseteq b^\circ$ and there is $x \in I$ such that $b^\circ \subseteq x^{\circ\circ}$. So, $a^\circ \subseteq x^{\circ\circ}$ and $a \in F_I$. Therefore, $F_I$ is an $\alpha$-filter. □

**Lemma 10.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $F \in \mathrm{Fi}(A)$. We have the following properties:*

(1) *If $\phi[F] \subseteq \mathrm{X}_m(A)$, then $F$ is an $\alpha$-filter.*
(2) $\mathrm{X}_m(A) \subseteq \mathrm{X}_\alpha(A)$.

Proof: (1) Let $a \in A$ be such that $a^{\circ\circ} \cap F \neq \emptyset$. If $a \notin F$, by Theorem 1 there exists $P \in \mathrm{X}(A)$ such that $F \subseteq P$ and $a \notin P$. So, $P \in \phi[F]$ and $P \in \mathrm{X}_m(A)$. Since $a \notin P$, by Lemma 5, we have $a^{\circ\circ} \cap P = \emptyset$ and $a^{\circ\circ} \cap F = \emptyset$, which is a contradiction. Then $a \in F$ and $F$ is an $\alpha$-filter.

(2) It follows by Lemma 5. □

**Remark 11.** Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $P \in \mathrm{X}(A)$. If $\mathbf{A}$ is pseudocomplemented, then $a^\circ = (a^*]$ and $a^{\circ\circ} = (a^{**}]$. Thus, the condition $a^{\circ\circ} \cap P \neq \emptyset$ is equivalent to

$a^{**} \in P$ and by the results developed in [3], $P$ is maximal if and only if $a^{**} \in P$ implies that $a \in P$ for all $a \in A$. Then, if $\mathbf{A}$ is pseudocomplemented, we have that $\mathrm{X}_m(A) = \mathrm{X}_\alpha(A)$. In particular, this result is also valid if $\mathbf{A}$ is a distributive pseudocomplemented lattice, see [1].

**Theorem 12.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $X \subseteq A$. Then*

$$(\star) \qquad F_\alpha(X) = \{a \in A \colon a^{\circ\circ} \cap F(X) \neq \emptyset\}$$

*is the smallest $\alpha$-filter containing $X$.*

Proof: Since $1^{\circ\circ} = A$, we have $1^{\circ\circ} \cap F(X) \neq \emptyset$ and $1 \in F_\alpha(X)$. Let $a, b \in A$ be such that $a \leq b$ and $a \in F_\alpha(X)$. It follows that $a^{\circ\circ} \subseteq b^{\circ\circ}$ and $a^{\circ\circ} \cap F(X) \neq \emptyset$. Thus, $b^{\circ\circ} \cap F(X) \neq \emptyset$ and $b \in F_\alpha(X)$. Let $a, b \in F_\alpha(X)$. So, $a^{\circ\circ} \cap F(X) \neq \emptyset$ and $b^{\circ\circ} \cap F(X) \neq \emptyset$, i.e., there exist $f_1, f_2 \in F(X)$ such that $f_1 \in a^{\circ\circ}$ and $f_2 \in b^{\circ\circ}$. Then, by Lemma 5, we have $f_1 \wedge f_2 \in a^{\circ\circ} \cap b^{\circ\circ} \cap F(X) = (a \wedge b)^{\circ\circ} \cap F(X)$, i.e., $(a \wedge b)^{\circ\circ} \cap F(X) \neq \emptyset$ and $a \wedge b \in F_\alpha(X)$. Then $F_\alpha(X) \in \mathrm{Fi}(A)$. We see that $F_\alpha(X)$ is an $\alpha$-filter. Let $a \in A$ be such that $a^{\circ\circ} \cap F_\alpha(X) \neq \emptyset$. So, there exists $x \in a^{\circ\circ}$ such that $x^{\circ\circ} \cap F(X) \neq \emptyset$. Then $x^{\circ\circ} \subseteq a^{\circ\circ}$ and $a^{\circ\circ} \cap F(X) \neq \emptyset$. Therefore, $a \in F_\alpha(X)$ and $F_\alpha(X)$ is an $\alpha$-filter. Since $a \in a^{\circ\circ}$, it follows that $X \subseteq F_\alpha(X)$. Finally, let $H \in \mathrm{Fi}_\alpha(A)$ be such that $X \subseteq H$. If $a \in F_\alpha(X)$, then $a^{\circ\circ} \cap F(X) \neq \emptyset$ and $a^{\circ\circ} \cap H \neq \emptyset$. As $H$ is an $\alpha$-filter, $a \in H$ and $F_\alpha(X) \subseteq H$. □

It is easy to see that $(\star)$ is equivalent to

$$F_\alpha(X) = \{a \in A \colon \exists\, f \in F(X)(a^\circ \subseteq f^\circ)\}.$$

Throughout this paper we will use the two characterizations. Moreover, by Theorem 12, a filter $F$ is an $\alpha$-filter if and only if $F_\alpha(F) = F$. If $X = \{a\}$, we write simply $F_\alpha(\{a\}) = [a)_\alpha$. Note that

$$[a)_\alpha = \{b \in A \colon b^{\circ\circ} \cap [a) \neq \emptyset\} = \{b \in A \colon a \in b^{\circ\circ}\} = \{b \in A \colon b^\circ \subseteq a^\circ\}.$$

**Lemma 13.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $a, b \in A$. We have the following properties:*

1. *If $a \leq b$, then $[b)_\alpha \subseteq [a)_\alpha$.*
2. *$[a)_\alpha = A$ if and only if $a = 0$.*
3. *$[a)_\alpha = D(A)$ if and only if $a \in D(A)$.*
4. *If $a \wedge b = 0$, then $[a)_\alpha \veebar [b)_\alpha = A$.*
5. *$[a)_\alpha = [b)_\alpha$ if and only if $a^\circ = b^\circ$.*
6. *If $[a)_\alpha = [b)_\alpha$, then $[a \wedge c)_\alpha = [b \wedge c)_\alpha$ for all $c \in A$.*

Proof: It is left to the reader. □

We have the following result that characterizes the $\alpha$-filters.

**Proposition 14.** *Let* $\mathbf{A} \in \mathcal{DS}_{01}$ *and* $F \in \mathrm{Fi}(A)$*. The following conditions are equivalent:*

(1) $F \in \mathrm{Fi}_\alpha(A)$.
(2) *If* $a \in F$*, then* $[a)_\alpha \subseteq F$.
(3) *If* $[a)_\alpha = [b)_\alpha$ *and* $a \in F$*, then* $b \in F$.
(4) $F = \bigcup\{[a)_\alpha \colon a \in F\}$.

Proof: (1) $\Rightarrow$ (2) If $b \in [a)_\alpha$, then $b^\circ \subseteq a^\circ$ and $a^{\circ\circ} \subseteq b^{\circ\circ}$. Since $a \in a^{\circ\circ}$, we have $a^{\circ\circ} \cap F \neq \emptyset$. So, $b^{\circ\circ} \cap F \neq \emptyset$ and as $F$ is an $\alpha$-filter, $b \in F$. Then $[a)_\alpha \subseteq F$.

(2) $\Rightarrow$ (3) It is immediate.

(3) $\Rightarrow$ (4) Since $[a) \subseteq [a)_\alpha$ for all $a \in A$, we have $F \subseteq \bigcup\{[a)_\alpha \colon a \in F\}$. Conversely, if $b \in \bigcup\{[a)_\alpha \colon a \in F\}$, then there exists $a \in F$ such that $b \in [a)_\alpha$, i.e., $b^\circ \subseteq a^\circ$ and $a^{\circ\circ} \subseteq b^{\circ\circ}$. So, $a^{\circ\circ} = a^{\circ\circ} \cap b^{\circ\circ} = (a \wedge b)^{\circ\circ}$ and $[a)_\alpha = [a \wedge b)_\alpha$. By hypothesis, $a \wedge b \in F$ and $b \in F$. Therefore, $F = \bigcup\{[a)_\alpha \colon a \in F\}$.

(4) $\Rightarrow$ (1) Let $b \in A$ be such that $b^{\circ\circ} \cap F \neq \emptyset$. Then there is $f \in F$ such that $f \in b^{\circ\circ}$. So, $b^\circ \subseteq f^\circ$. As $F = \bigcup\{[a)_\alpha \colon a \in F\}$, there exists $a \in F$ such that $f \in [a)_\alpha$, i.e., $f^\circ \subseteq a^\circ$. Thus, $b^\circ \subseteq a^\circ$ and $b \in [a)_\alpha \subseteq F$. We conclude that $F$ is an $\alpha$-filter. $\square$

Now, we define the notion of $\alpha$-order-ideal.

**Definition 15.** Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $I \in \mathrm{Id}(A)$. We say that $I$ is an *$\alpha$-order-ideal* if $a^{\circ\circ} \subseteq I$ for all $a \in I$.

Denote by $\mathrm{Id}_\alpha(A)$ the set of all $\alpha$-order-ideals of $A$.

**Example 16.** If $\mathbf{A} \in \mathcal{DS}_{01}$, then $a^\circ$ is an $\alpha$-order-ideal for all $a \in A$.

**Example 17.** Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $F \in \mathrm{Fi}(A)$. We consider the set

$$I_F = \{a \in A \colon \exists\, f \in F(a \in f^\circ)\}.$$

Then $I_F$ is an $\alpha$-order-ideal. It is easy to see that $I_F$ is decreasing. Let $a, b \in I_F$. So, there exist $f_1, f_2 \in F$ such that $a \in f_1^\circ$ and $b \in f_2^\circ$. As $F$ is a filter, $f = f_1 \wedge f_2 \in F$. It follows that $a, b \in f^\circ$. Since $\mathbf{A}$ is distributive, $f^\circ$ is an order-ideal and there exists $c \in f^\circ$ such that $a \leq c$ and $b \leq c$. It is clear that $c \in I_F$ and $I_F$ is an order-ideal. Now, we prove that $I_F$ is an $\alpha$-order-ideal. Let $a \in I_F$. Then there is $f \in F$ such that $a \in f^\circ$. If $x \in a^{\circ\circ}$, then $x \in \bigcap\{y^\circ \colon y \in a^\circ\}$. By Lemma 5, $f^{\circ\circ} \subseteq a^\circ$ and $f \in a^\circ$. Thus, $x \in f^\circ$ and $x \in I_F$. Therefore, $a^{\circ\circ} \subseteq I_F$. In addition, note that if $F$ is proper, then $F \cap I_F = \emptyset$. If $a \in F \cap I_F$, then there is $f \in F$ such that $a \in f^\circ$. By Lemma 5, $f^{\circ\circ} \subseteq a^\circ$ and $f \in a^\circ$, i.e., $f \wedge a = 0 \in F$, which is a contradiction.

**Lemma 18.** *Let* $\mathbf{A} \in \mathcal{DS}_{01}$ *and* $I \in \mathrm{Id}(A)$*. If* $\mathrm{X}(A) - \psi[I] \subseteq \mathrm{X}_m(A)$*, then* $I$ *is an $\alpha$-order-ideal.*

Proof: Let $a \in I$ and suppose that $a^{\circ\circ} \nsubseteq I$, i.e., there is $x \in a^{\circ\circ}$ such that $x \notin I$. By Theorem 1, there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $P \cap I = \emptyset$. Thus, $P \in \mathrm{X}(A) - \psi[I]$ and $P \in \mathrm{X}_m(A)$. It follows that $x \in a^{\circ\circ} \cap P$, i.e., $a^{\circ\circ} \cap P \neq \emptyset$. Then, by Lemma 5, we have $a \in P$ which is a contradiction. Therefore, $a^{\circ\circ} \subseteq I$ and $I$ is an $\alpha$-order-ideal. □

**Proposition 19.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $I \in \mathrm{Id}(A)$. The following conditions are equivalent:*

(1) *$I \in \mathrm{Id}_\alpha(A)$.*
(2) *If $a^\circ = b^\circ$ and $a \in I$, then $b \in I$.*
(3) *$I = \bigcup\{a^{\circ\circ} : a \in I\}$.*

Proof: (1) ⇒ (2) Let $a, b \in A$ be such that $a^\circ = b^\circ$ and $a \in I$. So, $a^{\circ\circ} = b^{\circ\circ}$ and since $I$ is an $\alpha$-order-ideal, we have $a^{\circ\circ} \subseteq I$. Then $b^{\circ\circ} \subseteq I$ and $b \in I$.

(2) ⇒ (3) As $a \in a^{\circ\circ}$ for all $a \in A$, it is immediate that $I \subseteq \bigcup\{a^{\circ\circ} : a \in I\}$. Inversely, if $x \in \bigcup\{a^{\circ\circ} : a \in I\}$, then there is $b \in I$ such that $x \in b^{\circ\circ}$. Thus, by Lemma 5, $x^{\circ\circ} \subseteq b^{\circ\circ}$ and $x^{\circ\circ} = (x \wedge b)^{\circ\circ}$, i.e., $x^\circ = (x \wedge b)^\circ$. Since $b \in I$, $x \wedge b \in I$ and by hypothesis, $x \in I$. Therefore, $I = \bigcup\{a^{\circ\circ} : a \in I\}$.

(3) ⇒ (1) Let $b \in I$ and $x \in b^{\circ\circ}$. Then $x \in \bigcup\{a^{\circ\circ} : a \in I\} = I$, i.e., $x \in I$. So, $b^{\circ\circ} \subseteq I$ and $I$ is an $\alpha$-order-ideal. □

In [14] the author develops a theorem of separation in 0-distributive posets. Now, we prove a separation theorem between filters and $\alpha$-order-ideals by means of irreducible $\alpha$-filters in the class of bounded distributive semilattices. The following theorem will be used in Theorem 24.

**Theorem 20.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$. Let $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}_\alpha(A)$ such that $F \cap I = \emptyset$. Then there exists $P \in \mathrm{X}_\alpha(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof: Let us consider the set $\mathcal{F} = \{H \in \mathrm{Fi}(A) : F \subseteq H \text{ and } H \cap I = \emptyset\}$. Since $F \in \mathcal{F}$, we have $\mathcal{F} \neq \emptyset$. The union of a chain of elements of $\mathcal{F}$ is also in $\mathcal{F}$. Then, by Zorn's lemma, there exists a filter $P$ maximal in $\mathcal{F}$. We prove that $P \in \mathrm{X}_\alpha(A)$. Let $F_1, F_2 \in \mathrm{Fi}(A)$ be such that $P = F_1 \cap F_2$. So, $F_1, F_2 \notin \mathcal{F}$, i.e., $F_1 \cap I \neq \emptyset$ and $F_2 \cap I \neq \emptyset$. Then there exist $x, y \in I$ such that $x \in F_1$ and $y \in F_2$. As $I$ is an order-ideal, there is $z \in I$ such that $x \leq z$ and $y \leq z$. Thus, $z \in P \cap I$, which is a contradiction. Then $P$ is irreducible. Now, we prove that $P$ is an $\alpha$-filter. Let $a \in A$ be such that $a^{\circ\circ} \cap P \neq \emptyset$ and suppose that $a \notin P$. Let $F = F(P \cup \{a\})$. Then $F \notin \mathcal{F}$ and $F \cap I \neq \emptyset$, i.e., there exists $p \in P$ such that $p \wedge a \in I$. Since $I$ is an $\alpha$-order-ideal, $(p \wedge a)^{\circ\circ} \subseteq I$. On the other hand, as $a^{\circ\circ} \cap P \neq \emptyset$, there is $b \in A$ such that $b \in a^{\circ\circ} \cap P$. It follows that

$$p \wedge b \in p^{\circ\circ} \cap a^{\circ\circ} = (p \wedge a)^{\circ\circ} \subseteq I.$$

So, $p \wedge b \in P \cap I$ which is a contradiction. Therefore, $P$ is an irreducible $\alpha$-filter. $\square$

**Corollary 21.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$. Then every proper $\alpha$-order-ideal is the intersection of prime $\alpha$-order-ideals.*

PROOF: Let $I$ be a proper $\alpha$-order-ideal of $A$. For each $a \notin I$, we have $[a) \cap I = \emptyset$. By Theorem 20, there exists $P_a \in \mathrm{X}_\alpha(A)$ such that $a \in P_a$ and $P_a \cap I = \emptyset$. Since $\mathbf{A}$ is distributive, $P_a^c$ is an order-ideal. As $P_a$ is an $\alpha$-filter, we have that $P_a^c$ is a prime $\alpha$-order-ideal. Thus, $I = \bigcap\{P_a^c \colon P_a \in \mathrm{X}_\alpha(A) \text{ and } a \notin I\}$. $\square$

## 4. Distributive quasicomplemented semilattices

The concept of quasicomplement in bounded distributive lattices was studied by T. Speed in [19] and W. Cornish in [10] as a generalization of the class of distributive pseudocomplemented lattices. In this section we give a characterization of distributive quasicomplemented semilattices and study the concepts of $\alpha$-filter and $\alpha$-order-ideal in distributive quasicomplemented semilattices.

**Definition 22.** Let $\mathbf{A} \in \mathcal{DS}_{01}$. We say that $\mathbf{A}$ is *quasicomplemented* if for each $a \in A$, there exists $b \in A$ such that $a^{\circ\circ} = b^\circ$.

We denote by $\mathcal{QDS}$ the class of distributive quasicomplemented semilattices.

**Theorem 23.** *Let $\mathbf{A} \in \mathcal{DS}_{01}$. The following conditions are equivalent:*

1. $\mathbf{A} \in \mathcal{QDS}$.
2. *For every $a \in A$, there exists $b \in A$ such that*

$$\psi[a^{\circ\circ}] \cap \mathrm{X}_m(A) = \psi[b^\circ] \cap \mathrm{X}_m(A).$$

PROOF: (1) $\Rightarrow$ (2) If $a \in A$, then there exists $b \in A$ such that $a^{\circ\circ} = b^\circ$. It is immediate to see that $\psi[a^{\circ\circ}] \cap \mathrm{X}_m(A) = \psi[b^\circ] \cap \mathrm{X}_m(A)$.

(2) $\Rightarrow$ (1) Let $a \in A$. Then, by hypothesis, there exists $b \in A$ such that $\psi[a^{\circ\circ}] \cap \mathrm{X}_m(A) = \psi[b^\circ] \cap \mathrm{X}_m(A)$. We prove that $a^{\circ\circ} = b^\circ$. Let $x \in a^{\circ\circ}$ and suppose that $x \notin b^\circ$. By Theorem 1, there exists $P \in \mathrm{X}(A)$ such that $b^\circ \cap P = \emptyset$ and $x \in P$. So, by Lemma 5, there exists $U \in \mathrm{X}_m(A)$ such that $P \subseteq U$ and $b \in U$. Then $x \in a^{\circ\circ} \cap U$, i.e., $U \in \psi[a^{\circ\circ}] \cap \mathrm{X}_m(A) = \psi[b^\circ] \cap \mathrm{X}_m(A)$. It follows that $b^\circ \cap U \neq \emptyset$ and $b \notin U$, which is a contradiction. Thus, $x \in b^\circ$ and $a^{\circ\circ} \subseteq b^\circ$. Now, let $x \in b^\circ$ and suppose that $x \notin a^{\circ\circ} = \bigcap\{y^\circ \colon y \in a^\circ\}$. Then there is $y \in a^\circ$ such that $x \notin y^\circ$. By Theorem 1, there exists $P \in \mathrm{X}(A)$ such that $y^\circ \cap P = \emptyset$ and $x \in P$. So, by Lemma 5, there exists $U \in \mathrm{X}_m(A)$ such that $P \subseteq U$ and $y \in U$. Then $x \in b^\circ \cap U$, i.e., $U \in \psi[b^\circ] \cap \mathrm{X}_m(A) = \psi[a^{\circ\circ}] \cap \mathrm{X}_m(A)$ and $a^{\circ\circ} \cap U \neq \emptyset$. Thus, there exists $z \in a^{\circ\circ}$ such that $z \in U$. On the other hand, $y \in a^\circ \cap U$ and

$z \wedge y = 0 \in U$. Then $U = A$, which is a contradiction. It follows that $x \in a^{\circ\circ}$ and $b^{\circ} \subseteq a^{\circ\circ}$. Therefore, $a^{\circ\circ} = b^{\circ}$ and $\mathbf{A}$ is quasicomplemented. $\square$

Now, we will see some consequences of Theorem 20. We define operations of infimum "$\sqcap$", supremum "$\sqcup$", and implication "$\Rightarrow$" in the set of all $\alpha$-filters $\mathrm{Fi}_{\alpha}(A)$ as follows:

$$G \sqcap H = G \cap H,$$
$$G \sqcup H = F_{\alpha}(G \veebar H),$$
$$G \Rightarrow H = F_{\alpha}(G \to H)$$

for each pair $G, H \in \mathrm{Fi}_{\alpha}(A)$. By Theorem 12, $G \sqcap H, G \sqcup H, G \Rightarrow H \in \mathrm{Fi}_{\alpha}(A)$ for all $G, H \in \mathrm{Fi}_{\alpha}(A)$. By Example 8, we consider the structure

$$\mathrm{Fi}_{\alpha}(\mathbf{A}) = \langle \mathrm{Fi}_{\alpha}(A), \sqcup, \sqcap, \Rightarrow, D(A), A \rangle.$$

**Theorem 24.** *Let* $\mathbf{A} \in \mathcal{QDS}$. *Then* $\mathrm{Fi}_{\alpha}(\mathbf{A})$ *is a Heyting algebra.*

Proof: Let $G, H \in \mathrm{Fi}_{\alpha}(A)$. It is immediate that $G \sqcap H$ is the infimum of $G$ and $H$. We prove that $G \sqcup H = F_{\alpha}(G \veebar H)$ is the supremum of $G$ and $H$. Note that $F_{\alpha}(G \veebar H) = F_{\alpha}(G \cup H)$. It is clear that $G \sqcup H$ is an upper bound of $G$ and $H$. Let $K \in \mathrm{Fi}_{\alpha}(A)$ be such that $G \subseteq K$ and $H \subseteq K$. If $a \in G \sqcup H = F_{\alpha}(G \veebar H)$, then $a^{\circ\circ} \cap F(G \cup H) \neq \emptyset$, i.e., there are $g \in G$ and $h \in H$ such that $g \wedge h \in a^{\circ\circ}$. On the other hand, $g, h \in K$ and $g \wedge h \in K$. So, $a^{\circ\circ} \cap K \neq \emptyset$ and since $K$ is an $\alpha$-filter, we have $a \in K$. Then $G \sqcup H$ is the supremum of $G$ and $H$.

We see that $G \sqcap H \subseteq K$ if and only if $G \subseteq H \Rightarrow K$ for all $G, H, K \in \mathrm{Fi}_{\alpha}(A)$. Suppose that $G \sqcap H \subseteq K$. If $x \in G$, then $[x) \cap H \subseteq G \cap H \subseteq K$, i.e., $[x) \cap H \subseteq K$. So, $x \in H \to K$. Thus, $x \in F_{\alpha}(H \to K) = H \Rightarrow K$ and $G \subseteq H \Rightarrow K$. Reciprocally, we assume that $G \subseteq H \Rightarrow K$. Let $x \in G \sqcap H$. Then $x \in G$ and by hypothesis, $x \in H \Rightarrow K = F_{\alpha}(H \to K)$ and there exists $f \in H \to K$, i.e., $[f) \cap H \subseteq K$, such that $x^{\circ} \subseteq f^{\circ}$. Suppose that $x \notin K$. Since $K$ is an $\alpha$-filter, we have $x^{\circ\circ} \cap K = \emptyset$. As $\mathbf{A}$ is quasicomplemented, there exists $y \in A$ such that $x^{\circ\circ} = y^{\circ}$. So, $y^{\circ} \in \mathrm{Id}_{\alpha}(A)$ and by Theorem 20 there exists $P \in \mathrm{X}_{\alpha}(A)$ such that $K \subseteq P$ and $P \cap y^{\circ} = \emptyset$, i.e., $P \cap x^{\circ\circ} = \emptyset$. It follows that $f^{\circ\circ} \subseteq x^{\circ\circ}$ and $x, f \in x^{\circ\circ}$. Then $x, f \in P^{c}$ and since $\mathbf{A}$ is distributive, $P^{c}$ is an order-ideal. Then there is $p \in P^{c}$ such that $x \leq p$ and $f \leq p$. On the other hand, $x \in H$ and $p \in H$. Thus, $p \in [f) \cap H \subseteq K \subseteq P$ and $p \in P$, which is a contradiction. Then $x \in K$ and $G \sqcap H \subseteq K$. Therefore, $\mathrm{Fi}_{\alpha}(\mathbf{A})$ is a Heyting algebra. $\square$

**Remark 25.** Let $\mathbf{A} \in \mathcal{DS}_{01}$. If $\mathbf{A}$ is pseudocomplemented, $\mathrm{R}(A) = \{a^{*} : a \in A\}$ is the set of all *regular elements of* $A$. So, $\mathrm{R}(A) = \{a \in A : a = a^{**}\}$ and $a^{**} \in \mathrm{R}(A)$ for all $a \in A$. If we consider the binary operation $a \curlyvee b = (a^{*} \wedge b^{*})^{*}$

for each $a, b \in A$, then we have that

$$\mathrm{R}(\mathbf{A}) = \langle \mathrm{R}(A), \curlyvee, \wedge, {}^*, 0, 1 \rangle$$

is a Boolean algebra, see [7]. On the other hand, recall that the set of all filters of a Boolean algebra has a structure of Heyting algebra, see [1]. We denote by $\mathrm{Fi}(\mathrm{R}(\mathbf{A})) = \langle \mathrm{Fi}(\mathrm{R}(A)), \veebar, \barwedge, \rightarrow, \{1\}, \mathrm{R}(A) \rangle$ the Heyting algebra of filters of the Boolean algebra $\mathrm{R}(\mathbf{A})$. Then, $\mathrm{Fi}_\alpha(\mathbf{A})$ is isomorphic to the Heyting algebra $\mathrm{Fi}(\mathrm{R}(\mathbf{A}))$. If we define $\lambda\colon \mathrm{Fi}_\alpha(A) \to \mathrm{Fi}(\mathrm{R}(A))$ given by $\lambda(F) = F \cap \mathrm{R}(A)$, then $\lambda$ is well-defined, $\lambda(F \sqcap G) = \lambda(F) \barwedge \lambda(G)$, $\lambda(F \sqcup G) = \lambda(F) \veebar \lambda(G)$ and $\lambda(F \Rightarrow G) = \lambda(F) \to \lambda(G)$ for all $F, G \in \mathrm{Fi}_\alpha(A)$. Let $F, G \in \mathrm{Fi}_\alpha(A)$ such that $\lambda(F) = \lambda(G)$ and $a \in F$. Since $a \leq a^{**}$, we have $a^{**} \in F$. So, $a^{**} \in F \cap \mathrm{R}(A) = \lambda(F) = \lambda(G)$, i.e., $a^{**} \in G$. As $G$ is an $\alpha$-filter, $a \in G$ and $F \subseteq G$. The other inclusion is similar and $\lambda$ is 1-1. Let $H \in \mathrm{Fi}(\mathrm{R}(A))$. Then $F_\alpha(H) \in \mathrm{Fi}_\alpha(A)$ and

$$\begin{aligned} a \in \lambda(F_\alpha(H)) &\iff a \in F_\alpha(H) \cap \mathrm{R}(A) \\ &\iff a^{**} \in H \quad \text{and} \quad a \in \mathrm{R}(A) \\ &\iff a \in H. \end{aligned}$$

Thus, $\lambda$ is onto and therefore $\lambda$ is an isomorphism.

In every distributive pseudocomplemented lattice the filter of dense elements is the intersection of maximal filters, see [1]. We see that the filter $D(A)$ is the intersection of irreducible $\alpha$-filters in any distributive quasicomplemented semilattice.

**Lemma 26.** *Let $\mathbf{A} \in \mathcal{QDS}$. Then $D(A) = \bigcap\{P\colon P \in \mathrm{X}_\alpha(A)\}$.*

Proof: By Example 8, $D(A) \subseteq F$ for all $F \in \mathrm{Fi}_\alpha(A)$. In particular, we have $D(A) \subseteq \bigcap\{P\colon P \in \mathrm{X}_\alpha(A)\}$. We prove the other inclusion. Suppose there is $a \in \bigcap\{P\colon P \in \mathrm{X}_\alpha(A)\}$ such that $a \notin D(A)$. Since $D(A)$ is an $\alpha$-filter, $a^{\circ\circ} \cap D(A) = \emptyset$. As $\mathbf{A}$ is quasicomplemented, there exists $b \in A$ such that $a^{\circ\circ} = b^\circ$. So, $b^\circ \in \mathrm{Id}_\alpha(A)$ and by Theorem 20 there exists $Q \in \mathrm{X}_\alpha(A)$ such that $D(A) \subseteq Q$ and $Q \cap b^\circ = \emptyset$, i.e., $Q \cap a^{\circ\circ} = \emptyset$. It follows that $a \notin Q$, which is a contradiction because $a \in \bigcap\{P\colon P \in \mathrm{X}_\alpha(A)\}$. Therefore, we have $D(A) = \bigcap\{P\colon P \in \mathrm{X}_\alpha(A)\}$. □

In Lemma 10 we proved that every maximal filter is an $\alpha$-filter. Now, we see that in the class $\mathcal{QDS}$ the reciprocal is also valid.

**Lemma 27.** *Let $\mathbf{A} \in \mathcal{QDS}$. Then $\mathrm{X}_m(A) = \mathrm{X}_\alpha(A)$.*

Proof: By Lemma 10, $\mathrm{X}_m(A) \subseteq \mathrm{X}_\alpha(A)$. We prove the other inclusion. Let $P \in \mathrm{X}_\alpha(A)$ and $a \in A$ be such that $a \notin P$. Since $\mathbf{A}$ is quasicomplemented, there is $b \in A$ such that $a^{\circ\circ} = b^\circ$. So, $b \in a^\circ$. Note that $[a) \cap [b) \subseteq D(A)$. Indeed, if

$x \in [a) \cap [b)$ and $y \in x^{\circ}$, then $a \wedge y = 0$ and $b \wedge y = 0$, i.e., $y \in a^{\circ}$ and $y \in b^{\circ} = a^{\circ\circ}$. Thus, $y = 0$ and $x^{\circ} = \{0\}$. So, $[a) \cap [b) \subseteq D(A)$ and by Lemma 26, $[a) \cap [b) \subseteq P$. Since $P$ is irreducible and $a \notin P$, we have $b \in P$. It follows that $b \in a^{\circ} \cap P$, i.e., $a^{\circ} \cap P \neq \emptyset$. Conversely, it is easy to see that if $a^{\circ} \cap P \neq \emptyset$, then $a \notin P$. Therefore, by Lemma 5, $P \in \mathrm{X}_m(A)$. $\square$

Let $\mathbf{A} \in \mathcal{DS}_{01}$ and $I \in \mathrm{Id}(A)$. We consider

$$I_{\alpha}(I) = \{a \in A \colon \exists\, x \in I (a \in x^{\circ\circ})\} = \{a \in A \colon \exists\, x \in I (x^{\circ} \subseteq a^{\circ})\}.$$

It is clear that $I_{\alpha}((a]) = a^{\circ\circ}$ for all $a \in A$.

**Theorem 28.** *Let $\mathbf{A} \in \mathcal{QDS}$ and $I \in \mathrm{Id}(A)$. Then $I_{\alpha}(I)$ is the smallest $\alpha$-order-ideal containing $I$.*

Proof: It is easy to see that $I \subseteq I_{\alpha}(I)$ and that $I_{\alpha}(I)$ is decreasing. Let $a, b \in I_{\alpha}(I)$. Then there exist $x, y \in I$ such that $a \in x^{\circ\circ}$ and $b \in y^{\circ\circ}$. As $I$ is an order-ideal, there is $z \in I$ such that $x \leq z$ and $y \leq z$. So, $x^{\circ\circ} \subseteq z^{\circ\circ}$ and $y^{\circ\circ} \subseteq z^{\circ\circ}$. Then $a, b \in z^{\circ\circ}$. On the other hand, since $\mathbf{A}$ is quasicomplemented, there is $w_z \in A$ such that $z^{\circ\circ} = w_z^{\circ}$. Thus, $a, b \in w_z^{\circ}$ and as $w_z^{\circ}$ is an order-ideal, there is $c \in w_z^{\circ}$ such that $a \leq c$ and $b \leq c$. It follows that $z \in I$ and $c \in z^{\circ\circ}$, i.e., $c \in I_{\alpha}(I)$. Then $I_{\alpha}(I)$ is an order-ideal. We prove that $I_{\alpha}(I)$ is an $\alpha$-order-ideal. Let $a \in I_{\alpha}(I)$. Then there is $x \in I$ such that $x^{\circ} \subseteq a^{\circ}$. So, $a^{\circ\circ} \subseteq x^{\circ\circ}$. If $y \in a^{\circ\circ}$, then $y \in x^{\circ\circ}$ and $y \in I_{\alpha}(I)$, i.e., $a^{\circ\circ} \subseteq I_{\alpha}(I)$ and $I_{\alpha}(I)$ is an $\alpha$-order-ideal. Finally, let $H \in \mathrm{Id}_{\alpha}(A)$ such that $I \subseteq H$. If $a \in I_{\alpha}(I)$, then there is $x \in I$ such that $a \in x^{\circ\circ}$. Thus, $x \in H$ and since $H$ is an $\alpha$-order-ideal, $x^{\circ\circ} \subseteq H$. Therefore, $a \in H$ and $I_{\alpha}(I) \subseteq H$. We proved that $I_{\alpha}(I)$ is the smallest $\alpha$-order-ideal containing $I$. $\square$

The following technical result will be useful.

**Lemma 29.** *Let $\mathbf{A} \in \mathcal{QDS}$. Let $P \in \mathrm{X}_m(A)$ and $a, b \in A$. If $(a^{\circ} \cap b^{\circ})^{\circ} \cap P \neq \emptyset$, then $a \in P$ or $b \in P$.*

Proof: As $\mathbf{A}$ is quasicomplemented, there exist $\tilde{a}, \tilde{b} \in A$ such that $a^{\circ} = (\tilde{a})^{\circ\circ}$ and $b^{\circ} = (\tilde{b}\,)^{\circ\circ}$. So,

$$(a^{\circ} \cap b^{\circ})^{\circ} \cap P = ((\tilde{a})^{\circ\circ} \cap (\tilde{b}\,)^{\circ\circ})^{\circ} \cap P = ((\tilde{a} \wedge \tilde{b})^{\circ\circ})^{\circ} \cap P = (\tilde{a} \wedge \tilde{b})^{\circ} \cap P \neq \emptyset.$$

Then, by Lemma 5, $\tilde{a} \wedge \tilde{b} \notin P$ and $\tilde{a} \notin P$ or $\tilde{b} \notin P$. If $\tilde{a} \notin P$, then $(\tilde{a})^{\circ} \cap P \neq \emptyset$ and $a^{\circ\circ} \cap P \neq \emptyset$. It follows by Lemma 5 that $a \in P$. Analogously, if $\tilde{b} \notin P$, then $b \in P$. Therefore, we conclude that $a \in P$ or $b \in P$. $\square$

On $\mathrm{Id}_{\alpha}(A)$ we define the binary operations

$$I \Cup J = \{a \in A \colon \exists\, (x, y) \in I \times J \;[a \in (x^{\circ} \cap y^{\circ})^{\circ}]\}$$

and $I \Cap J = I \cap J$ for all $I, J \in \mathrm{Id}_\alpha(A)$. We consider

$$\mathrm{Id}_\alpha(\mathbf{A}) = \langle \mathrm{Id}_\alpha(A), \Cup, \Cap, \{0\}, A\rangle.$$

**Theorem 30.** *Let $\mathbf{A} \in \mathcal{QDS}$. Then $\mathrm{Id}_\alpha(\mathbf{A})$ is a bounded distributive lattice.*

Proof: Let $I, J \in \mathrm{Id}_\alpha(A)$. It is clear that $I \Cap J \in \mathrm{Id}_\alpha(A)$ and $I \Cap J$ is the infimum of $I$ and $J$. We prove that $I \Cup J \in \mathrm{Id}_\alpha(A)$. By definition, $I \Cup J$ is decreasing. Let $a, b \in I \Cup J$. Then there exist $(x_1, y_1), (x_2, y_2) \in I \times J$ such that $a \in (x_1^\circ \cap y_1^\circ)^\circ$ and $b \in (x_2^\circ \cap y_2^\circ)^\circ$. Since $I$ and $J$ are order-ideals, there is $(x, y) \in I \times J$ such that $x_1, x_2 \leq x$ and $y_1, y_2 \leq y$. So, $(x_1^\circ \cap y_1^\circ)^\circ \subseteq (x^\circ \cap y^\circ)^\circ$ and $(x_2^\circ \cap y_2^\circ)^\circ \subseteq (x^\circ \cap y^\circ)^\circ$. It follows that $a, b \in (x^\circ \cap y^\circ)^\circ$. Since $\mathbf{A}$ is quasicomplemented, there exist $\tilde{x}, \tilde{y} \in A$ such that $x^\circ = (\tilde{x})^{\circ\circ}$ and $y^\circ = (\tilde{y})^{\circ\circ}$. Then

$$(x^\circ \cap y^\circ)^\circ = ((\tilde{x})^{\circ\circ} \cap (\tilde{y})^{\circ\circ})^\circ = ((\tilde{x} \wedge \tilde{y})^{\circ\circ})^\circ = (\tilde{x} \wedge \tilde{y})^\circ$$

and $a, b \in (\tilde{x} \wedge \tilde{y})^\circ$. As $(\tilde{x} \wedge \tilde{y})^\circ$ is an order-ideal, there is $c \in (\tilde{x} \wedge \tilde{y})^\circ$ such that $a \leq c$ and $b \leq c$, i.e., $I \Cup J$ is an order-ideal. Let $a \in I \Cup J$. Then there exists $(x, y) \in I \times J$ such that $a \in (x^\circ \cap y^\circ)^\circ$. So, $a^{\circ\circ} \subseteq (x^\circ \cap y^\circ)^\circ$. It follows that $a^{\circ\circ} \subseteq I \Cup J$ and $I \Cup J$ is an $\alpha$-order-ideal. Now, we see that $I \Cup J$ is the supremum of $I$ and $J$. Let $H \in \mathrm{Id}_\alpha(A)$ such that $I \subseteq H$ and $J \subseteq H$. If $a \in I \Cup J$, then there exists $(x, y) \in I \times J$ such that $a \in (x^\circ \cap y^\circ)^\circ$. In particular, $x, y \in H$. Suppose that $a \notin H$. Then, by Theorem 20, there exists $P \in \mathrm{X}_\alpha(A)$ such that $a \in P$ and $P \cap H = \emptyset$. By Lemma 27, $\mathrm{X}_\alpha(A) = \mathrm{X}_m(A)$ and $P \in \mathrm{X}_m(A)$. Also, $a \in (x^\circ \cap y^\circ)^\circ \cap P$, i.e., $(x^\circ \cap y^\circ)^\circ \cap P \neq \emptyset$ and by Lemma 29 we have $x \in P$ or $y \in P$. In both cases, $P \cap H \neq \emptyset$ which is a contradiction. Then $I \Cup J \subseteq H$ and $I \Cup J$ is the supremum of $I$ and $J$.

Finally, we see that $\mathrm{Id}_\alpha(\mathbf{A})$ is distributive. Let $I, J, H \in \mathrm{Id}_\alpha(A)$. We prove that $I \Cap (J \Cup H) \subseteq (I \Cap J) \Cup (I \Cap H)$. Let $a \in I \Cap (J \Cup H)$. If we suppose that $a \notin (I \Cap J) \Cup (I \Cap H)$, then by Theorem 20 and Lemma 27 there exists $P \in \mathrm{X}_m(A)$ such that $a \in P$ and $P \cap [(I \cap J) \Cup (I \cap H)] = \emptyset$. So, $P \cap (I \cap J) = \emptyset$ and $P \cap (I \cap H) = \emptyset$. On the other hand, since $a \in J \Cup H$, there exists $(x, y) \in J \times H$ such that $a \in (x^\circ \cap y^\circ)^\circ$. Then $(x^\circ \cap y^\circ)^\circ \cap P \neq \emptyset$ and by Lemma 29, $x \in P$ or $y \in P$. Thus, $x \wedge a \in P \cap (I \cap J)$ or $y \wedge a \in P \cap (I \cap H)$ which is a contradiction. Therefore, $a \in (I \Cap J) \Cup (I \Cap H)$ and $\mathrm{Id}_\alpha(\mathbf{A})$ is a bounded distributive lattice. □

Actually, $\mathrm{Id}_\alpha(\mathbf{A})$ is a Heyting algebra as we will see later. Following Lemma 9 and Example 17, we have the following result.

**Lemma 31.** *Let $\mathbf{A} \in \mathcal{QDS}$. We have the following properties:*

(1) *If $F \in \mathrm{Fi}_\alpha(A)$, then $F = F_{I_F}$.*
(2) *If $I \in \mathrm{Id}_\alpha(A)$, then $I = I_{F_I}$.*

Proof: (1) If $a \in F_{I_F}$, then there is $x \in I_F$ such that $a^\circ \subseteq x^{\circ\circ}$. So, there exists $f \in F$ such that $x \in f^\circ$. By Lemma 5, $f^{\circ\circ} \subseteq x^\circ$ and $x^{\circ\circ} \subseteq f^\circ$. It follows that $f^{\circ\circ} \subseteq a^{\circ\circ}$. Since $f \in f^{\circ\circ}$, we have $f \in a^{\circ\circ}$. As $a^{\circ\circ} \cap F \neq \emptyset$ and $F$ is an $\alpha$-filter, $a \in F$. Thus, $F_{I_F} \subseteq F$. Reciprocally, let $a \in F$. We take $a^{\circ\circ}$. Since $\mathbf{A}$ is quasicomplemented, there is $x \in A$ such that $a^{\circ\circ} = x^\circ$, i.e., $a^\circ = x^{\circ\circ}$. As $x \in x^{\circ\circ}$, then $x \in a^\circ$ and $a \in F$. So, $x \in I_F$. On the other hand, since $a^\circ \subseteq x^{\circ\circ}$ and $x \in I_F$, we have $a \in F_{I_F}$ and $F \subseteq F_{I_F}$. Thus, $F = F_{I_F}$.

(2) Let $a \in I_{F_I}$. Then there is $f \in F_I$ such that $a \in f^\circ$. So, there exists $x \in I$ such that $f^\circ \subseteq x^{\circ\circ}$. Since $x \in I$ and $I$ is an $\alpha$-order-ideal, we have $x^{\circ\circ} \subseteq I$. It follows that $a \in f^\circ \subseteq x^{\circ\circ}$, i.e., $a \in I$ and $I_{F_I} \subseteq I$. Conversely, let $a \in I$ and we take $a^{\circ\circ}$. As $\mathbf{A}$ is quasicomplemented, there exists $x \in A$ such that $a^{\circ\circ} = x^\circ$. Then $x^\circ \subseteq a^{\circ\circ}$ and $x \in F_I$. Later, $a \in x^\circ$ and $x \in F_I$, i.e., $a \in I_{F_I}$. Therefore, $I = I_{F_I}$. □

The following observation will be useful.

**Remark 32.** Let $\mathbf{L} = \langle L, \vee_L, \wedge_L, 0, 1\rangle$ be a bounded distributive lattice. Let $\mathbf{H} = \langle H, \vee_H, \wedge_H, \to_H, 0, 1\rangle$ be a Heyting algebra and $h\colon L \to H$ an isomorphism between bounded distributive lattices. If for each $a, b \in L$ we define the binary operation $a \to_L b = h^{-1}(h(a) \to_H h(b))$, then we have

$$\begin{aligned} a \leq b \to_L c &\iff a \leq h^{-1}(h(b) \to_H h(c)) \\ &\iff h(a) \leq h(b) \to_H h(c) \\ &\iff h(a) \wedge_H h(b) \leq h(c) \\ &\iff h(a \wedge_L b) \leq h(c) \\ &\iff a \wedge_L b \leq c \end{aligned}$$

for all $a, b, c \in L$. Then $\mathbf{L} = \langle L, \vee_L, \wedge_L, \to_L, 0, 1\rangle$ is a Heyting algebra.

**Theorem 33.** *Let $\mathbf{A} \in \mathcal{QDS}$. Then the Heyting algebras $\mathrm{Id}_\alpha(\mathbf{A})$ and $\mathrm{Fi}_\alpha(\mathbf{A})$ are isomorphic.*

Proof: Let $f\colon \mathrm{Id}_\alpha(A) \to \mathrm{Fi}_\alpha(A)$ be the mapping given by $f(I) = F_I$. For Lemma 9, $f$ is well defined. On the other hand, by Lemma 31, $f$ is 1-1 and onto. We prove that $f$ is an isomorphism between bounded distributive lattices.

Let $I_1, I_2 \in \mathrm{Id}_\alpha(A)$. We see that $F_{I_1 \Cap I_2} = F_{I_1} \sqcap F_{I_2}$. If $a \in F_{I_1 \Cap I_2}$, then there is $x \in I_1 \cap I_2$ such that $a^\circ \subseteq x^{\circ\circ}$. Since $x \in I_1$, $a \in F_{I_1}$. Analogously, $a \in F_{I_2}$. So, $a \in F_{I_1} \cap F_{I_2}$ and $F_{I_1 \Cap I_2} \subseteq F_{I_1} \sqcap F_{I_2}$. Reciprocally, if $a \in F_{I_1} \sqcap F_{I_2}$, then there is $x \in I_1$ such that $a^\circ \subseteq x^{\circ\circ}$ and there is $y \in I_2$ such that $a^\circ \subseteq y^{\circ\circ}$. By Lemma 5, we have $a^\circ \subseteq x^{\circ\circ} \cap y^{\circ\circ} = (x \wedge y)^{\circ\circ}$ and $x \wedge y \in I_1 \cap I_2$, i.e., $a \in F_{I_1 \Cap I_2}$. Therefore, $f(I_1 \Cap I_2) = f(I_1) \sqcap f(I_2)$.

Now, we prove that $F_{I_1 \Cup I_2} = F_{I_1} \sqcup F_{I_2}$. If $a \in F_{I_1 \Cup I_2}$, then there exists $x \in I_1 \Cup I_2$ such that $a^\circ \subseteq x^{\circ\circ}$. So, there is $y \in I_1$ and there is $z \in I_2$ such that $x \in (y^\circ \cap z^\circ)^\circ$. We take $y^{\circ\circ}$ and $z^{\circ\circ}$. Since $\mathbf{A}$ is quasicomplemented, there exist $f_1, f_2 \in A$ such that $y^{\circ\circ} = f_1^\circ$ and $z^{\circ\circ} = f_2^\circ$. It follows that $f_1 \in F_{I_1}$ and $f_2 \in F_{I_2}$. If we consider $f = f_1 \wedge f_2$, then $f \in F_{I_1} \veebar F_{I_2}$. On the other hand, by Lemma 5,

$$(y^\circ \cap z^\circ)^\circ = (f_1^{\circ\circ} \cap f_2^{\circ\circ})^\circ = ((f_1 \wedge f_2)^{\circ\circ})^\circ = f^\circ$$

and since $x \in (y^\circ \cap z^\circ)^\circ$, we have $x \in f^\circ$ and $x^{\circ\circ} \subseteq f^\circ$. Then $a^\circ \subseteq f^\circ$ and $a \in F_{I_1} \sqcup F_{I_2}$. Thus, $F_{I_1 \Cup I_2} \subseteq F_{I_1} \sqcup F_{I_2}$. Conversely, if $a \in F_{I_1} \sqcup F_{I_2}$, then there is $f \in F_{I_1} \veebar F_{I_2}$ such that $a^\circ \subseteq f^\circ$. So, there exists $f_1 \in F_{I_1}$ and there exists $f_2 \in F_{I_2}$ such that $f_1 \wedge f_2 \leq f$, i.e., there exists $(y, z) \in I_1 \times I_2$ such that $f_1^\circ \subseteq y^{\circ\circ}$ and $f_2^\circ \subseteq z^{\circ\circ}$. Then, by Lemma 5, $y^\circ \cap z^\circ \subseteq (f_1 \wedge f_2)^{\circ\circ}$. Again, as $\mathbf{A}$ is quasicomplemented, there is $g \in A$ such that $(f_1 \wedge f_2)^{\circ\circ} = g^\circ$. Thus, $g \in g^{\circ\circ} \subseteq (y^\circ \cap z^\circ)^\circ$ and $g \in I_1 \Cup I_2$. However, since $a^\circ \subseteq f^\circ \subseteq (f_1 \wedge f_2)^\circ$, it follows that $a^\circ \subseteq g^{\circ\circ}$. Then $a \in F_{I_1 \Cup I_2}$ and $F_{I_1} \sqcup F_{I_2} \subseteq F_{I_1 \Cup I_2}$. We conclude that $f(I_1 \Cup I_2) = f(I_1) \sqcup f(I_2)$.

Finally, by Remark 32, we define for each $I_1, I_2 \in \mathrm{Id}_\alpha(A)$ the operation $I_1 \rightsquigarrow I_2 = f^{-1}(f(I_1) \Rightarrow f(I_2))$. Then the structure

$$\mathrm{Id}_\alpha(\mathbf{A}) = \langle \mathrm{Id}_\alpha(A), \Cup, \Cap, \rightsquigarrow, \{0\}, A \rangle$$

is a Heyting algebra and $f(I_1 \rightsquigarrow I_2) = f(I_1) \Rightarrow f(I_2)$ for all $I_1, I_2 \in \mathrm{Id}_\alpha(A)$. Therefore, $f$ is an isomorphism between Heyting algebras. □

**Remark 34.** If $\mathbf{L}$ is a bounded distributive lattice, then we know that the set of all ideals $\mathrm{Id}(L)$ of $L$ is a Heyting algebra, see [1], where

$$(\bullet) \qquad I \rightsquigarrow J = \{x \in A \colon \forall i \in I (x \wedge i \in J)\},$$

for all $I, J \in \mathrm{Id}(L)$. Let $a \in I \rightsquigarrow J$. If $x \in a^{\circ\circ}$ and $i \in I$, then $x^{\circ\circ} \subseteq a^{\circ\circ}$ and $a \wedge i \in J$. By Lemma 5 and since $J$ is an $\alpha$-order-ideal, we have

$$(x \wedge i)^{\circ\circ} = x^{\circ\circ} \cap i^{\circ\circ} \subseteq a^{\circ\circ} \cap i^{\circ\circ} = (a \wedge i)^{\circ\circ} \subseteq J,$$

i.e., $x \wedge i \in J$. So, $x \in I \rightsquigarrow J$ and $a^{\circ\circ} \subseteq I \rightsquigarrow J$. Therefore, $I \rightsquigarrow J$ is an $\alpha$-order-ideal and we have a characterization of the implication given by $(\bullet)$.

## 5. Filters-congruence-cokernels in $\mathcal{QDS}$

In this section we study filters-congruence-cokernels of a distributive quasicomplemented semilattice $\mathbf{A}$ and we shall also describe the smallest filter-congruence $\theta$ in $\mathbf{A}$ such that $I = |0|_\theta$ for some $\alpha$-order-ideal $I$.

Let $\mathbf{A} \in \mathcal{DS}_{01}$. If $F \in \mathrm{Fi}(A)$, it is easy to see that the relation

$$\theta(F) = \{(a,b) \in A \times A \colon \exists f \in F(a \wedge f = b \wedge f)\}$$

is a congruence on $\mathbf{A}$. We say that a congruence $\theta$ on $\mathbf{A}$ is a *filter-congruence* if there is $F \in \mathrm{Fi}(A)$ such that $\theta = \theta(F)$. In particular, if $F \in \mathrm{Fi}_\alpha(A)$, we say that $\theta$ is an *$\alpha$-filter-congruence*. There are congruences that are not filter-congruences, see [5]. If $\theta$ is a congruence on $\mathbf{A}$, then the equivalence class $|0|_\theta = \{a \in A \colon (a,0) \in \theta\}$ is called the *cokernel of* $\theta$. A subset $X \subseteq A$ is called *congruence-cokernel* if there exists a congruence $\theta$ on $\mathbf{A}$ such that $X = |0|_\theta$.

Now, we prove that for each $\alpha$-order-ideal $I$ of $A$, there exists an $\alpha$-filter $F_I$ such that $I = |0|_{\theta(F_I)}$.

**Theorem 35.** *Let $\mathbf{A} \in \mathcal{QDS}$ and $I \in \mathrm{Id}_\alpha(A)$. Then $\theta(F_I)$ is the smallest $\alpha$-filter-congruence such that $I = |0|_{\theta(F_I)}$.*

Proof: We consider the $\alpha$-filter $F_I$ of Lemma 9. First, we see that $a^\circ \cap F_I \neq \emptyset$ for all $a \in I$. Suppose there is $a \in I$ such that $a^\circ \cap F_I = \emptyset$. Since $a^\circ$ is an order-ideal, by Theorem 1 there exists $P \in \mathrm{X}(A)$ such that $F_I \subseteq P$ and $a^\circ \cap P = \emptyset$. By Lemma 5, there exists $Q \in \mathrm{X}_m(A)$ such that $P \subseteq Q$ and $a \in Q$. It is easy to prove that $a^\circ \cap Q = \emptyset$. On the other hand, since $\mathbf{A}$ is quasicomplemented, there is $b \in A$ such that $a^{\circ\circ} = b^\circ$. In particular, $b^\circ \subseteq a^{\circ\circ}$ and $a \in I$, i.e., $b \in F_I$ and $b \in Q$. So, $b \in b^{\circ\circ} = a^\circ$. Then $a, b \in Q$ and $a \wedge b = 0 \in Q$, which is a contradiction because $Q$ is maximal. Therefore, $a^\circ \cap F_I \neq \emptyset$ for all $a \in I$.

Now, we see that $I = |0|_{\theta(F_I)}$. If $a \in I$, then $a^\circ \cap F_I \neq \emptyset$. So, there is $f \in F_I$ such that $a \wedge f = 0$. It follows that $(a,0) \in \theta(F_I)$ and $a \in |0|_{\theta(F_I)}$. Then $I \subseteq |0|_{\theta(F_I)}$. Reciprocally, if $a \in |0|_{\theta(F_I)}$, then there is $f \in F_I$ such that $a \wedge f = 0$. Thus, there exists $x \in I$ such that $f^\circ \subseteq x^{\circ\circ}$ and $a \in f^\circ$. It follows that $a \in x^{\circ\circ}$ and as $I$ is an $\alpha$-order-ideal, $x^{\circ\circ} \subseteq I$. Then $a \in I$ and $|0|_{\theta(F_I)} \subseteq I$.

Let $F \in \mathrm{Fi}_\alpha(A)$ be such that $I = |0|_{\theta(F)}$. It is enough to show that $F_I \subseteq F$. If $a \in F_I$, then there is $x \in I$ such that $a^\circ \subseteq x^{\circ\circ}$. Also, since $I = |0|_{\theta(F)}$ we have $x \in |0|_{\theta(F)}$ and there exists $f \in F$ such that $x \wedge f = 0$, i.e., $f \in x^\circ$. So, $x^{\circ\circ} \subseteq f^\circ$. Then $a^\circ \subseteq f^\circ$ and $f \in a^{\circ\circ} \cap F$. Thus, as $a^{\circ\circ} \cap F \neq \emptyset$ and $F$ is an $\alpha$-filter, $a \in F$ and $F_I \subseteq F$. We conclude that $\theta(F_I) \subseteq \theta(F)$ and $\theta(F_I)$ is the smallest $\alpha$-filter-congruence such that $I = |0|_{\theta(F_I)}$. □

**Theorem 36.** *Let $\mathbf{A} \in \mathcal{QDS}$ and $I \subseteq A$. The following conditions are equivalent:*

(1) *$I$ is an $\alpha$-order-ideal.*
(2) *$I$ is an $\alpha$-filter-congruence-cokernel.*

Proof: (1) $\Rightarrow$ (2) By Lemma 9, $F_I$ is an $\alpha$-filter and by Theorem 35 we have $I = |0|_{\theta(F_I)}$. So, $I$ is an $\alpha$-filter-congruence-cokernel.

(2) $\Rightarrow$ (1) If $I$ is an $\alpha$-filter-congruence-cokernel, then there is $F \in \mathrm{Fi}_\alpha(A)$ such that $I = |0|_{\theta(F)}$. We prove that $I$ is an $\alpha$-order-ideal. It follows that $I$ is decreasing. Let $a, b \in I$. Then $(a, 0), (b, 0) \in \theta(F)$, i.e., there exist $f_1, f_2 \in F$ such that $a \wedge f_1 = 0$ and $b \wedge f_2 = 0$. Let $f = f_1 \wedge f_2 \in F$. So, $a, b \in f^\circ$ and since $f^\circ$ is an order-ideal, there is $c \in f^\circ$ such that $a \leq c$ and $b \leq c$. Thus, $(c, 0) \in \theta(F)$ and $c \in |0|_{\theta(F)} = I$. Hence, $I$ is an order-ideal. Let $a \in I$. We see that $a^{\circ\circ} \subseteq I$. Since $a \in I = |0|_{\theta(F)}$, there is $f \in F$ such that $a \wedge f = 0$, i.e., $f \in a^\circ$. If $x \in a^{\circ\circ}$, then $a^\circ \subseteq x^\circ$ and $f \in x^\circ$. So, $x \wedge f = 0$ and $x \in |0|_{\theta(F)} = I$. Therefore, $a^{\circ\circ} \subseteq I$ and $I$ is an $\alpha$-order-ideal. $\square$

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