A characterization of symplectic groups related to Fermat primes

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Abstract. We proved that the symplectic groups $\mathrm{PSp}(4, 2^n)$, where $2^{2n} + 1$ is a Fermat prime number is uniquely determined by its order, the first largest element orders and the second largest element orders.

Keywords: element order; the largest element order; prime graph; symplectic group

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1. Introduction

Throughout this paper, all groups considered are finite and p is a Fermat prime number. The set of prime divisors of $|G|$ is denoted by $\pi(G)$, the set of orders of elements of G is denoted by $\pi_e(G)$, the largest element orders and the second largest element orders of $\pi_e(G)$ are denoted by $k_1(G)$ and $k_2(G)$, respectively. Also, a Sylow p-subgroup of G is denoted by G_p. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Furthermore, assume that $\Gamma(G)$ has $t(G)$ connected components π_i for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, let π_1 be the connected component containing 2.

In the past 30 years, there has been a lot of attention to the problem, whether a finite simple group G is fully determined by $|G|$ and $\pi_e(G)$. The problem was completely solved by V. D. Mazurov et al. in [9]. After that, several researchers tried to characterize some finite simple groups with less information. It turns out that the largest element orders can be used to characterize $L_3(q)$ $(q \leq 8)$ and $U_3(q)$ $(q \leq 11)$, $L_2(q)$ where $q < 125$, the sporadic simple groups, K_4-group of type $L_2(p)$, $\mathrm{PGL}(2, q)$, the Suzuki group $\mathrm{Sz}(q)$, where $q - 1$ or $q \pm \sqrt{2q} + 1$ is a prime number by using the largest orders, see [6], [3], [5], [1], [4], [8].

In this paper, we show that the simple symplectic groups $\mathrm{PSp}(4, 2^n)$, where $2^{2n} + 1$ is a Fermat prime number, is uniquely determined by its order, the first

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largest element orders and the second largest element orders. The main theorem is as follows.

Main theorem. *Let G be a group such that $|G| = |\mathrm{PSp}(4, 2^n)|$, $k_1(G) = k_1(\mathrm{PSp}(4, 2^n)) = 2^{2n}+1$ and $k_2(G) = k_2(\mathrm{PSp}(4, 2^n)) = 2^{2n}-1$, where $(2^{2n}+1) > 5$ is a Fermat prime number. Then $G \cong \mathrm{PSp}(4, 2^n)$.*

2. Preliminaries

Lemma 2.1 ([2])**.** *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(1) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(2) *$|H|$ divides $|K| - 1$;*
(3) *kernel K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

Lemma 2.3 ([9])**.** *Let G be a 2-Frobenius group of evenorder. Then*

(1) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(2) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathrm{Aut}(K/H)|$.*

Lemma 2.4 ([10])**.** *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(1) *group G is a Frobenius group;*
(2) *group G is a 2-Frobenius group;*
(3) *group G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group.*

3. Proof of the main theorem

We denote the simple symplectic groups $\mathrm{PSp}(4, 2^n)$ by C and prime number $2^{2n} + 1 = q^2 + 1$ by p. Recall that G is a group such that $|G| = |C| = q^4 \times (q^4 - 1)(q^2 - 1)$, $k_1(G) = k_1(C) = q^2 + 1$ and $k_2(G) = k_2(C) = q^2 - 1$.

Lemma 3.1. *Vertex p is an isolated vertex of $\Gamma(G)$.*

Proof: Let there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So, $tp \geq 2p = 2(q^2 + 1) > q^2 + 1$ and so $k_1(G) = q^2 + 1 < tp$, which is a contradiction. $\square$

By the above lemma, we have $t(G) \geq 2$. By Lemma 2.4, G is a Frobenius group, or a 2-Frobenius group, or G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, and K/H is a non-abelian simple group. In the following two lemmas, we show that G is neither a Frobenius group nor a 2-Frobenius group.

Lemma 3.2. *Group G is not a Frobenius group.*

Proof: Let G be a Frobenius group with kernel K and complement H. By Lemma 3.2, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and also $|H|$ divides $|K| - 1$. Now, by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus, we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, the last case cannot occur. So, $|H| = p$ and $|K| = |G|/p$, and hence $q^2 + 1 \mid q^4(q^4 - 1)(q^2 - 1)/(q^2 + 1) - 1$. Therefore, $(q^2 + 1)$ divides $(q^2 + 1)(q^6 - 3q^4 + 4q^2 - 4) + 4$. It follows that $p \mid 4$, which is a contradiction. □

Lemma 3.3. *Group G is not a 2-Frobenius group.*

Proof: Let G be a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H, respectively. Set $|G/K| = x$. Since p is an isolated vertex of $\Gamma(G)$, we have $|K/H| = p$ and $|H| = |G|/(xp)$. By Lemma 3.3, $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Thus, $x \mid p-1$. On the other hand, $(q^2, q-1) = 1$. So, $|G/K| \mid (p-1)$, we deduce that $(q-1) \mid |H|$. Since H is nilpotent, $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complement K/H, where t is a prime divisor of $q-1$. So, $|K/H|$ divides $|H_t| - 1$. That implies that $q^2 + 1 \leq q - 2 \leq q$, which is a contradiction. □

Now, we show that G is isomorphic to C. Since G is neither a Frobenius group nor a 2-Frobenius group, by Lemma 2.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and also K/H is a non-abelian simple group. Since $|G| = |G/K| \times |K/H| \times |H|$ and H and G/K are π_1-groups, each odd order component of G is an odd order component of K/H. Since $p \mid |K/H|$, $t(k/H) \geq 2$. According to the classification of finite simple groups, we know that the possibilities for K/H are the alternating group Alt_m, $m \geq 5$, one of the 26 sporadic simple groups, and the simple groups of Lie types.

In the rest of the proof, we shall consider all these possibilities one after another.

Step 1. Let $K/H \cong \mathrm{Alt}_m$, where there is a prime p' such that $m \in \{p', p'+1, p'+2\}$. By [10] $k_2(\mathrm{Alt}_m) = p'$, or $p' - 2$. If $q^2 - 1 = p'$, then $q^2 - 2 = p' - 1$. On the other hand,

$$|\mathrm{Alt}_m| \mid |\mathrm{PSp}(4, q)| = q^4(q^4 - 1)(q^2 - 1).$$

Hence, $q^2-2=p'-1$ divides the order of Alt_m, and that divides the order of $\mathrm{PSp}(4,q)$, which is equal to $q^4(q^4-1)(q^2-1)$. The latter integer is not divisible by q^2-2, and that is a contradiction.

Similarly, if $q^2-1=p'-2$, then $q^2-2=p'-3$. So, $q^2-2=p'-3$ divides the order of Alt_m, and that divides the order of $\mathrm{PSp}(4,q)$, which is a contradiction.

Step 2. If K/H is isomorphic to one of the sporadic simple groups, then by [7], $k_2(S)=\{5,7,11,13,17,19,23,31,37,59\}$. So, $q^2-1=5$, 7, 11, 17, 19, 23, 31, 37, 59. Let $q^2-1=5$. Then $q^2=6$. Now, we can see easily this equation is impossible. Similarly, we can rule out the other cases.

Step 3. Let K/H be isomorphic to a simple group of Lie-type.

3.1. Then $K/H \ncong B_m(\overline{q}\,)$, where $m>2$, $\overline{q}$ is a prime power. By [7], $k_2(B_m(\overline{q}\,))=\overline{q}^{\,m}-\overline{q}$. Also,

$$|B_m(\overline{q}\,)|=\frac{1}{(2,\overline{q}-1)}\overline{q}^{\,m^2}\prod_{i=1}^{m}(\overline{q}^{\,2i}-1).$$

Since $|B_m(\overline{q}\,)| \mid |G|$,

$$\frac{1}{(2,\overline{q}-1)}\overline{q}^{\,m^2}\prod_{i=1}^{m}(\overline{q}^{\,2i}-1) \mid q^4(q^4-1)(q^2-1).$$

On the other hand, $q^2-1=\overline{q}^{\,m}-\overline{q}$. So, $(q-1)(q+1)=\overline{q}\,(\overline{q}^{\,m-1}-1)$. Since $(q-1,q+1)=1$, so we have $q-1=\overline{q}$ and $q+1=\overline{q}^{\,m-1}-1$. As a result, $q=\overline{q}+1$ and $q=\overline{q}^{\,m-1}-2$ and so $|B_m(\overline{q}\,)| \nmid |G|$, which is a contradiction.

3.2. If $K/H\cong {}^3D_4(\overline{q})$, then by [7], $k_2({}^3D_4(\overline{q}\,))=\overline{q}^{\,4}-\overline{q}^{\,2}+1$. Also, we have

$$|{}^3D_4(\overline{q}\,)|=\overline{q}^{12}(\overline{q}^{\,8}+\overline{q}^{\,4}+1)(\overline{q}^{\,6}-1)(\overline{q}^{\,2}-1).$$

Since $|{}^3D_4(\overline{q}\,)| \mid |G|$,

$$\overline{q}^{12}(\overline{q}^{\,8}+\overline{q}^{\,4}+1)(\overline{q}^{\,6}-1)(\overline{q}^{\,2}-1) \mid q^4(q^4-1)(q^2-1).$$

On the other hand, $q^2-1=\overline{q}^{\,4}-\overline{q}^{\,2}+1$. It follows that

$$q^2-2=\overline{q}^{\,2}(\overline{q}^{\,2}-1).$$

Thus,

$$2(2^{2n-1}-1)=\overline{q}^{\,2}(\overline{q}^{\,2}-1)$$

and so $\overline{q}^{\,2}=2$, or 3. In both cases, we get a contradiction.

3.3. Let $K/H\cong E_6(\overline{q}\,)$, $E_7(\overline{q}\,)$, $E_8(\overline{q}\,)$, $F_4(\overline{q}\,)$. For example, if $K/H\cong F_4(\overline{q}\,)$, then by [7], $k_2(F_4(\overline{q}\,))=\overline{q}^{\,4}+1$. Also,

$$|F_4(\overline{q}\,)|=\overline{q}^{\,24}(\overline{q}^{\,2}-1)(\overline{q}^{\,6}-1)(\overline{q}^{\,8}-1)(\overline{q}^{12}-1).$$

Since $|F_4(\overline{q}\,)| \mid |G|$,

$$\overline{q}^{\,24}(\overline{q}^{\,2}-1)(\overline{q}^{\,6}-1)(\overline{q}^{\,8}-1)(\overline{q}^{12}-1) \mid q^{\,4}(q^{\,4}-1)(q^{\,2}-1).$$

Therefore, $q^2-1=\overline{q}^{\,4}+1$ and then $q^2=\overline{q}^{\,4}+2$, which is a contradiction by $|F_4(\overline{q}\,)| \mid |G|$.

Similarly, when $K/H \cong E_6(\overline{q}\,)$, $E_7(\overline{q}\,)$, or $E_8(\overline{q}\,)$, we get a contradiction.

3.4. If $K/H \cong {}^2E_6(\overline{q}\,)$, then by [7] $k_2({}^2E_6(\overline{q}\,))=(\overline{q}^{\,6}-1)/(3,\overline{q}+1)$. Also,

$$|{}^2E_6(\overline{q}\,)|=\frac{\overline{q}^{\,36}(\overline{q}^{12}-1)(\overline{q}^{\,9}+1)(\overline{q}^{\,8}-1)(\overline{q}^{\,6}-1)(\overline{q}^{\,5}+1)(\overline{q}^{\,2}-1)}{(3,\overline{q}+1)}.$$

So, $q^2-1=(\overline{q}^{\,6}-1)/(3,\overline{q}+1)$. First, if $(3,\overline{q}+1)=1$, then $q^2-1=\overline{q}^{\,6}-1$ and so $q^2=\overline{q}^{\,6}$. It follows that $|{}^2E_6(\overline{q}\,)| \nmid |G|$, which is a contradiction.

Similarly, when $(3,\overline{q}+1)=3$, we get a contradiction.

3.5. If $K/H \cong {}^2G_2(3^{2m+1})$ where $m \geq 1$, then by [7], $k({}^2G_2(3^{2m+1}))=3^{2m+1}-3^{m+1}+1$. Also, if $\overline{q}=3^{2m+1}$, then

$$|{}^2G_2(3^{2m+1}|=\overline{q}^{\,3}(\overline{q}^{\,3}+1)(\overline{q}-1).$$

Since $|{}^2G_2(3^{2m+1})| \mid |G|$, we have

$$\overline{q}^{\,3}(\overline{q}^{\,3}+1)(\overline{q}-1) \mid q^4(q^4-1)(q^2-1).$$

So, $3^{2m+1}-3^{m+1}+1=q^2-1$ and so $q^2-2=3^{2m+1}-3^{m+1}$. Hence,

$$2(2^{2n-1}-1)=3^{m+1}(3^m-1).$$

Then $3^m-1=2$ and $3^{m+1}=2^{2n-1}-1$, which implies that $m=1$ and $n=2$. On the other hand, $|{}^2G_2(27)| \nmid |\mathrm{PSp}(4,4)|$, which is a contradiction.

3.6. If $K/H \cong {}^2B_2(\overline{q}\,)$, where $\overline{q}=2^{2m+1} \geq 8$, then by [7], $k_2({}^2B_2(\overline{q}\,))=\overline{q}-1$. We have

$$|{}^2B_2(\overline{q}\,)|=\overline{q}^{\,2}(\overline{q}^{\,2}+1)(\overline{q}-1).$$

Since $|{}^2B_2(\overline{q}\,)| \mid |G|$, we have

$$\overline{q}^{\,2}(\overline{q}^{\,2}+1)(\overline{q}-1) \mid q^4(q^4-1)(q^2-1).$$

On the other hand, $\overline{q}-1=q^2-1$. So, $\overline{q}=q^2$. Hence,

$$q^4(q^4+1)(q^2-1) \mid q^4(q^4-1)(q^2-1),$$

which is a contradiction.

3.7. If $K/H \cong G_2(\overline{q}\,)$, then by [7], $k_2(G_2(\overline{q}\,))=\overline{q}'^2-\overline{q}+1$ and also

$$|G_2(\overline{q}\,)|=\overline{q}^{\,6}(\overline{q}^{\,6}-1)(\overline{q}^{\,2}-1).$$

Since $|G_2(\overline{q}\,)| \mid |G|$,

$$\overline{q}^{\,6}(\overline{q}^{\,6}-1)(\overline{q}^{\,2}-1) \mid q^4(q^4-1)(q^2-1).$$

On the other hand, $q^2-1=\overline{q}^{\,2}-\overline{q}+1$. So, $2(2^{2n-1}-1)=\overline{q}\,(\overline{q}-1)$ and then $\overline{q}=2$. As a result, $2=2^{2n-1}$, which is a contradiction.

3.8. Let $K/H \cong {}^2A_m(\overline{q}\,)$, where $m \geq 2$. If $m=2$, then by [7], $k_2({}^2A_2(\overline{q}\,))=(\overline{q}^{\,2}-\overline{q}+1)/(3,\overline{q}+1)$. Also,

$$|{}^2A_2(\overline{q}\,)|=\frac{1}{(3,\overline{q}+1)}\overline{q}^{\,3}\prod_{i=1}^{m}(\overline{q}^{\,i+1}-(-1)^{i+1}).$$

Since $|{}^2A_2(\overline{q}\,)| \mid |G|$, we have $(\overline{q}^{\,2}-\overline{q}+1)/(3,\overline{q}+1)=q^2-1$. If $(3,\overline{q}+1)=1$, then $\overline{q}^{\,2}-\overline{q}+1=q^2-1$. Thus,

$$2(2^{2n-1}-1)=\overline{q}\,(\overline{q}-1).$$

So, $\overline{q}=2$, which is a contradiction.

If $m>2$, then

$$k_2({}^2A_m(\overline{q}\,))=\frac{\overline{q}^{\,2m}-1}{(2n'+1,\overline{q}+1)}.$$

Now, $q^2-1=(\overline{q}^{\,2m}-1)/(2n'+1,\overline{q}+1)$. So,

$$q^4=\Big(\frac{\overline{q}^{\,2m}-1}{(2m+1,\overline{q}+1)}+1\Big)^2 \leq \overline{q}^{\,4m}.$$

Also,

$$\frac{\overline{q}^{\,m(m+1)/2}\prod_{i=1}^{m}(\overline{q}^{\,i+1}-(-1)^{i+1})}{(m+1,\overline{q}+1)} \mid |G|.$$

Thus,

$$\overline{q}^{\,m(m+1)/2}<\frac{\overline{q}^{\,m(m+1)/2}\prod_{i=1}^{m}(\overline{q}^{\,i+1}-(-1)^{i+1})}{(m+1,\overline{q}+1)} \leq q^4(q^4-1)(q^2-1) \leq q^4$$

and so $\overline{q}^{\,m(m+1)/2} \leq \overline{q}^{\,4m}$. As a result, $m \leq 7$, which is a contradiction.

3.9. Let $K/H \cong D_m(\overline{q})$, where $m \geq 4$. Similar to Case 3.8, we get a contradiction.

3.10. If $K/H \cong {}^2F_4(\overline{q}\,)$, where $\overline{q}=2^{2m+1} \geq 8$, then by [7],

$$k_2({}^2F_4(\overline{q}\,))=\overline{q}^{\,2}-\sqrt{2\overline{q}^{\,3}}+\overline{q}-\sqrt{2\overline{q}}+1.$$

So, $\overline{q}^{\,2}-\sqrt{2\overline{q}^{\,3}}+\overline{q}-\sqrt{2\overline{q}}+1=q^2-1$. As a result,

$$2(2^{2n-1}-1)=2^{m+1}(2^{3m+1}-2^{2m+1}+2^m-1).$$

Hence, $2^{m+1}=2$ and $2^{3m+1}-2^{2m+1}+2^m=2^{2n-1}$.

If $2^{m+1} = 2$, then $m = 0$, which is a contradiction.

If $2^{3m+1} - 2^{2m+1} + 2^m = 2^{2n-1}$, then

$$2^m(2^{2m+1} - 2^{m+1} + 1) = 2^{2n-1}.$$

So, $2^m = 2$ and $2^{2m+1} - 2^{m+1} + 1 = 2^{2n-2}$. Then $m = 1$ and so $5 = 2^{2n-2}$, which is a contradiction.

3.11. If $K/H \cong L_{m+1}(\overline{q})$, where $m \geq 1$, then by [7]

$$k_2(L_{m+1}(\overline{q})) = \frac{(\overline{q}^{m+1} - 1)(\overline{q}^m - 1)}{(\overline{q} - 1)(\overline{q} - 1, m+1)}.$$

Also,

$$|L_{m+1}(\overline{q})| = \frac{1}{(m+1, \overline{q} - 1)} \overline{q}^{m(m+1)/2}(\overline{q}^m - 1) \prod_{i=1}^{m} (\overline{q}^{i+1} - 1).$$

We have

$$\frac{1}{(m+1, \overline{q} - 1)} \overline{q}^{m(m+1)/2}(\overline{q}^m - 1) \prod_{i=1}^{m} (\overline{q}^{i+1} - 1) \mid q^4(q^4 - 1)(q^2 - 1)$$

so,

$$\frac{(\overline{q}^{m+1} - 1)(\overline{q}^m - 1)}{(\overline{q} - 1)(\overline{q} - 1, m+1)} = q^2 - 1$$

and so

$$q^4 = \Big(\frac{(\overline{q}^{m+1} - 1)(\overline{q}^m - 1)}{(\overline{q} - 1)(\overline{q} - 1, m+1)} + 1\Big)^2.$$

Since $|L_{m+1}(\overline{q})| \nmid |G|$, we get a contradiction.

3.12. Let $K/H \cong \mathrm{PSp}(m, \overline{q})$, where $m \geq 4$. First, if $m > 4$, then by [7],

$$k_2(\mathrm{PSp}(m, \overline{q})) = \overline{q}^m - \overline{q}.$$

So, $q^2 - 1 = \overline{q}^m - \overline{q}$ and then $(q-1)(q+1) = \overline{q}(\overline{q}^{m-1} - 1)$. Since $(q-1, q+1) = 1$, $q - 1 = \overline{q}$ and $q + 1 = \overline{q}^{m-1} - 1$, $q = \overline{q} + 1$ and $q = \overline{q}^{n-1} - 2$. Therefore, $|\mathrm{PSp}(m, \overline{q})| \nmid |G|$, which is a contradiction.

If $m = 4$, then $k_2(\mathrm{PSp}(m, \overline{q})) = \overline{q}^2 - 1 = q^2 - 1$. Thus, $q = \overline{q}$ and so $K/H \cong C$. It follows that $H = 1$ and $G = K \cong C$. The proof of the main theorem is completed.

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