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# On the hyperspace $C_n(X)/C_{nK}(X)$

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*Abstract.* Let $X$ be a continuum and $n$ a positive integer. Let $C_n(X)$ be the hyperspace of all nonempty closed subsets of $X$ with at most $n$ components, endowed with the Hausdorff metric. For $K$ compact subset of $X$, define the hyperspace $C_{nK}(X) = \{A \in C_n(X) \colon K \subset A\}$. In this paper, we consider the hyperspace $C^n_K(X) = C_n(X)/C_{nK}(X)$, which can be a tool to study the space $C_n(X)$. We study this hyperspace in the class of finite graphs and in general, we prove some properties such as: aposyndesis, local connectedness, arcwise disconnectedness, and contractibility.

*Keywords:* hyperspace; continuum; containment hyperspace; aposyndesis; finite graph; Peano continuum; contractibility

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## 1. Introduction

In this paper, the set of positive integers is denoted by $\mathbb{N}$ and a *map* is a continuous function. A *continuum* is a nonempty compact connected metric space. Given a continuum $X$, a *subcontinuum* of $X$ is a subset of $X$ which is a continuum. Given $n \in \mathbb{N}$, we consider the following hyperspaces of $X$:

- $2^X = \{A \subset X \colon A$ is nonempty and closed$\}$;
- $C(X) = \{A \in 2^X \colon A$ is connected$\}$;
- $F_n(X) = \{A \in 2^X \colon A$ has at most $n$ points$\}$;
- $C_n(X) = \{A \in 2^X \colon A$ has at most $n$ components$\}$.

All are endowed with the Hausdorff metric, see the definition below. Note that $C(X) = C_1(X)$.

The hyperspace $C_n(X)$ is called *n-fold hyperspace* of $X$, his topological structure is different to other hyperspaces, see [15] and [16].

On the other hand, in 1979 S. B. Nadler Jr., see [21], began the study of hyperspace suspension when the author considered the quotient space $HS(X) = C(X)/F_1(X)$, which he called the *hyperspace suspension* of $X$. Later, in 2004,

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R. Escobedo, M. de J. López and S. Macías extended the study of hyperspace suspension in [9]. Subsequently, in the same year, S. Macías generalized the study of hyperspace suspension, considering the quotient space $HS_n(X) = C_n(X)/F_n(X)$, which he called the *n-fold hyperspace suspension* of $X$, see [17], continuing with the study in 2006, see [18]. In the year 2008, J. C. Macías analyzes the quotient space $PHS_n(X) = C_n(X)/F_1(X)$, which he called the *n-fold pseudo-hyperspace suspension* of $X$, see [13]. J. Camargo and S. Macías in 2016 considered the quotient space $C_1^n(X) = C_n(X)/C_1(X)$, and they show several of their properties, see [4].

Following this line of research, given a closed subset $K$ of a continuum $X$ and a hyperspace $\mathcal{H}(X) \in \{2^X, C(X), F_n(X), C_n(X)\}$. We consider the quotient space

$$\frac{\mathcal{H}(X)}{\mathcal{H}_K(X)},$$

where $\mathcal{H}_K(X)$ is the containment hyperspace for $K$ in $\mathcal{H}(X)$ defined by $\{A \in \mathcal{H}(X) : K \subset A\}$ and considered as a subspace of $\mathcal{H}(X)$.

The fact that $\mathcal{H}(X)/\mathcal{H}_K(X)$ is a continuum follows from [22, Theorem 3.10, page 40]. Let $\pi_K$ denote the quotient map $\pi_K \colon \mathcal{H}(X) \to \mathcal{H}(X)/\mathcal{H}_K(X)$, and $\mathcal{H}_K = \pi_K(\mathcal{H}_K(X))$.

In this paper we study some topological properties of the quotient space $C_K^n(X) = C_n(X)/C_{nK}(X)$ such as local connectedness, arcwise disconnectedness, contractibility, unicoherence, homogeneity and aposyndesis. Throughout the article you can see how this space be a good tool to study hyperspace $C_n(X)$. The paper is divided into six sections. In Section 2 we provide the basic definitions, notation and some basic results of cut points. In Section 3, for the class of finite graphs, we calculate the dimension of space $C_K^n(X)$, and the relationship it has with cones, suspensions and space $C_n(X)$. In Section 4 we give conditions on $K$ under which the space $C_K^n(X)$ is aposyndetic and finitely aposyndetic, in particular, we prove that $X$ is aposyndetic if and only if $C_K^n(X)$ is aposyndetic for each $K \in F_1(X)$, see Theorem 4.6. Section 5 is devoted to the study of the connectedness of $C_K^n(X)$ as well as the local connectedness, in particular, we characterize the local connectedness of $X$ in terms of the local connectedness of $C_K^n(X)$, see Theorem 5.3. Also, we present results of the arcwise disconnectedness of $C_K^n(X) - \{C_{nK}\}$, which allow to give a characterization of the indecomposable continua, see Theorem 5.8. Finally, in Section 6 we analyze the non-contractibility, homogeneity and when $C_K^n(X)$ contains $n$-cells.

**Remark 1.1.** Let $n \in \mathbb{N}$. If $K = X$, then $C_K^n(X)$ is homeomorphic to $C_n(X)$.

**Remark 1.2.** Let $n \in \mathbb{N}$ and $K \in 2^X$. Then

$$\pi_K|_{C_n(X)-C_{nK}(X)}\colon C_n(X) - C_{nK}(X) \longrightarrow C_K^n(X) - \{C_{nK}\}$$

is a homeomorphism.

## 2. Preliminaries

An *arc* is any topological space homeomorphic to $I = [0, 1]$ and a *simple closed curve* is any topological space homeomorphic to the unit circle $S^1$. We will denote by $\mathrm{Cl}_Z(A)$, $\mathrm{Int}_Z(A)$ and $\mathrm{Bd}(A)$, the closure, interior and boundary of $A$ in $Z$, respectively. When two topological spaces $Y$ and $Z$ are homeomorphic is denoted as $Y \approx Z$.

Let $X$ be a continuum with metric $d$ and $\varepsilon > 0$. For any $x \in X$ and any $A \in 2^X$, we define the open ball in $X$ of radius $\varepsilon$ and center $x$ as

$$B_\varepsilon^d(x) = \{y \in X\colon d(x, y) < \varepsilon\},$$

and the generalized open $d$-ball in $X$ about $A$ of radius $\varepsilon$,

$$N_d(A, \varepsilon) = \bigcup_{x\in A} B_\varepsilon^d(x).$$

The hyperspace $2^X$ is considered with the Hausdorff metric $H_d$ induced by $d$, see [12, Definition 2.1, page 11], defined as follows: For any $A, B \in 2^X$,

$$H_d(A, B) = \inf\{\varepsilon > 0\colon A \subset N_d(B, \varepsilon) \text{ and } B \subset N_d(A, \varepsilon)\}.$$

A map $f\colon X \to Y$ between continua induces a natural map $f^*\colon 2^X \to 2^Y$ defined by

$$f^*(A) = f(A) \qquad \text{for each } \ A \in 2^X.$$

Thus, if $\mathcal{H}(X) \in \{2^X, C(X), F_n(X), C_n(X)\}$, then the induced map $\mathcal{H}(f)\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ is the map $\mathcal{H}(f) = f^*|_{\mathcal{H}(X)}$, see [12, Theorem 13.3, page 106].

Let $A$, $B \in 2^X$. An *order arc* from $A$ to $B$ is a mapping $\alpha\colon I \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$, and $\alpha(r)$ is a proper subset of $\alpha(s)$ whenever $r < s$, see [20, 1.2–1.8, page 57–59] for the definition and existence.

A *Whitney map for* $C(X)$ is a map $\mu\colon C(X) \to [0, \infty)$ that satisfies the following two conditions:

(1) $\mu(A) < \mu(B)$ for any $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$;
(2) $\mu(A) = 0$ if and only if $A \in F_1(X)$.

Given a continuum $X$, for any finitely many subsets $U_1, \dots, U_r$ of $X$, we define $\langle U_1, \dots, U_r \rangle$ as

$$\{A \in 2^X : A \subset \bigcup_{i=1}^{r} U_i,\ A \cap U_i \neq \emptyset \text{ for each } i = 1, \dots, r\}.$$

The set

$$\{\langle U_1, \dots, U_r \rangle : \text{ for each } i \in \{1, \dots, r\},\ U_i \text{ is an open set of } X,\ r \in \mathbb{N}\}$$

is a base for some topology for $2^X$, called the *Vietoris topology*. This topology matches with the topology induced by the Hausdorff metric, see [12, Theorem 3.2, page 18]. Given $n \in \mathbb{N}$ we write $\langle U_1, \dots, U_r \rangle_n$ instead of $\langle U_1, \dots, U_r \rangle \cap C_n(X)$.

A *Peano continuum* is a locally connected continuum. If $p \in X$, then $X$ is said to be *connected im kleinen* at $p$ provided that every neighborhood of $p$ contains a connected neighborhood of $p$. A continuum $X$ is *unicoherent* provided that whenever $A$ and $B$ are subcontinua of $X$, such that $A \cup B = X$, $A \cap B$ is connected. By [16, Theorem 4.8, page 244], $C_n(X)$ is unicoherent. Note that, given $K \in 2^X$, $\pi_K$ is monotone and by [22, Proposition 3.7, page 39], $\pi_K$ is a closed map. Thus, by [6, Corollary 7, page 211] we have following result.

**Theorem 2.1.** *Let $n \in \mathbb{N}$. If $X$ is a continuum and $K \in 2^X$, then $C_K^n(X)$ is unicoherent.*

**2.1 Cut points.** Given a connected topological space $Z$, a *cut set* of $Z$ is a subset $S$ of $Z$ such that $Z - S$ is not connected. A *cut point* of $Z$ is a point $p \in Z$ such that $\{p\}$ is a cut set of $Z$.

**Proposition 2.2.** *Let $X$ be a continuum, $n \in \mathbb{N}$ and $p \in X$. Then, $p$ is a cut point of $X$ if and only if $C_{n\{p\}}(X)$ is a cut set of $C_n(X)$.*

Proof: If $p \in X$ is a cut point, then there are disjoint nonempty open subsets $U$, $V$ such that $X - \{p\} = U \cup V$. Since $\langle U \rangle_n$ and $\langle X - \{p\}, V \rangle_n$ are disjoint nonempty open subsets of $C_n(X)$, it is enough to prove that $C_n(X) - C_{n\{p\}}(X) = \langle U \rangle_n \cup \langle X - \{p\}, V \rangle_n$.

Let $A \in C_n(X) - C_{n\{p\}}(X)$. Then $p \notin A$ and $A \subset X - \{p\} = U \cup V$. If $A \cap V \neq \emptyset$ then $A \in \langle X - \{p\}, V \rangle_n$, otherwise, $A \in \langle U \rangle_n$. Thus $C_n(X) - C_{n\{p\}}(X) \subset \langle U \rangle_n \cup \langle X - \{p\}, V \rangle_n$.

On the other hand, if $A \in \langle U \rangle_n \cup \langle X - \{p\}, V \rangle_n$, then $A \subset U$, or, $A \subset X - \{p\} = U \cup V$ and $A \cap V \neq \emptyset$. Thus, $A \subset X - \{p\}$, i.e. $p \notin A$. Then $\langle U \rangle_n \cup \langle X - \{p\}, V \rangle_n \subset C_n(X) - C_{n\{p\}}(X)$. Therefore, $C_{n\{p\}}(X)$ is a cut set of $C_n(X)$.

Now, suppose that $X - \{p\}$ is connected then $C_n(X) - C_{n\{p\}}(X) = C_n(X - \{p\})$ is connected. This is a contradiction. Therefore, $p$ is a cut point of $X$. □

**Corollary 2.3.** *Let $X$ be a continuum, $n \in \mathbb{N}$ and $p \in X$. Then $p$ is a cut point of $X$ if and only if $C_{n\{p\}}$ is a cut point of $C^n_{\{p\}}(X)$.*

A continuum $X$ is said to be *colocally connected at* $p \in X$, provided that $p$ has a local base of open sets whose complements are connected. A continuum $X$ is said to be *colocally connected*, if $X$ is colocally connected at each of its points.

**Proposition 2.4.** *If $X$ is colocally connected at $p \in X$, then $p$ is not a cut point of $X$.*

Proof: Suppose that $p$ is a cut point of $X$, then there exist $U, W$ nonempty open subsets of $X$ such that $X - \{p\} = U \cup W$ and $U \cap W = \emptyset$. Let $u \in U$ and $w \in W$. Let $\delta = \frac{1}{2}\min\{d(p,u), d(p,w)\} > 0$. Since $X$ is colocally connected at $p$, there exists $\{V_\alpha\}_{\alpha \in J}$ a local base of open subsets at $p$, where $J$ is an index set. Then there exists $\alpha \in J$ such that $p \in V_\alpha \subset B^d_\delta(p)$. Thus $X - B^d_\delta(p) \subset X - V_\alpha \subset X - \{p\}$. Since $X - V_\alpha$ is connected, we may assume that $X - V_\alpha \subset U$. Note that $u, w \in X - B^d_\delta(p)$. So, $u, w \in U$ and $w \in U \cap W$. This is a contradiction. Therefore $p$ is not a cut point of $X$. □

## 3. Finite graphs

A *free arc* in a continuum $X$ is an arc $A$ in $X$ such that $A$ without its end points is an open set in $X$. A *Hilbert cube* is any space homeomorphic to $\prod_{j=1}^{\infty} I_j$ with the product topology, where $I_j = I$ for each $j \in \mathbb{N}$.

A *finite graph* $X$ is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. If $X$ is a finite graph, the arcs and the end points of the arcs are called *edges* and *vertices*, respectively. Given $m \in \mathbb{N}$, $m \geq 3$, a *simple $m$-od* $Y$ is a finite graph which is the union of $m$ arcs $J_1, \ldots, J_m$ such that there exists a point $v \in Y$ with the property $J_i \cap J_j = \{v\}$, if $i \neq j$, and $v$ is an end point of $J_i$ for each $i = 1, \ldots, m$. The point $v$ is called the *core of* $Y$. A simple 3-od is called a *simple triod.*

The order of a point $p$ in a finite graph $X$, will be defined using the classic Menger definition. Given a point $p \in X$ and $m, n \in \mathbb{N}$, the order of $p$ in $X$, denoted by $\mathrm{ord}_X(p)$, is defined as $\mathrm{ord}_X(p) \leq n$, if for every $\varepsilon > 0$ there exists an open set $G$ containing $p$ with diameter of $G$ less than $\varepsilon$ such that $\mathrm{Bd}(G)$ consists of at most $n$ points. Define $\mathrm{ord}_X(p) = n$ if $\mathrm{ord}_X(p) \leq n$ and $\mathrm{ord}_X(p)$ is not less than or equal to $m$ for each $m < n$. A point $q \in X$ is called an *end point of* $X$ provided that $\mathrm{ord}_X(q) = 1$. A point $q \in X$ is called a *ramification point of* $X$ provided that $\mathrm{ord}_X(q) \geq 3$. The set of ramification points of $X$ is denoted by $R(X)$ and the set of end points of $X$ is denoted by $E(X)$. For an arc or a simple closed curve, the set of ramification points is empty. In any other case we assume that each vertex of $X$ is either an end point of $X$ or a ramification point of $X$. With this restriction the two end points of an edge of $X$ may coincide in this case the edge is a simple closed curve.

We write $\dim(X)$ to denote the *dimension of the space* $X$; and for $p \in X$, $\dim_p(X)$ stands for the *dimension of the space* $X$ *at the point* $p$. First, define $\dim(X) = -1$ when $X = \emptyset$. Now, assume inductively that we have defined $\dim_p(X) \leq n-1$ and $\dim(X) \leq n-1$ for some integer $n \geq 0$. Then define $\dim_p(X) \leq n$ when $p$ has arbitrarily small open neighborhoods in $X$ whose boundaries have dimension less than or equal to $n-1$, and define $\dim(X) \leq n$ when $\dim_p(X) \leq n$ for all $p \in X$. Now define $\dim_p(X) = n$ when $\dim_p(X) \leq n$ and $\dim_p(X) \nleq n-1$, and we define $\dim(X) = n$ when $\dim(X) \leq n$ and $\dim(X) \nleq n-1$. Finally, define $\dim(X) = \infty$ when $\dim(X) \nleq n$ for any integer $n$.

The following result is a consequence of [16, Theorem 7.1, page 250].

**Corollary 3.1.** *Let $X$ be a locally connected continuum such that $X$ is not a finite graph and let $K \in 2^X$. If $X - \{x\}$ contains a subcontinuum without free arcs for some $x \in K$, then $C_K^n(X)$ contains a Hilbert cube for every $n \in \mathbb{N}$.*

**Lemma 3.2.** *Let $X$ be a continuum and $n \in \mathbb{N}$. If $C_n(X) \approx C_K^n(X)$ for all $K \in F_1(X)$, then $X$ does not contain cut points.*

Proof: Let $p \in X$. Since $C_n(X)$ is homeomorphic to $C_{\{p\}}^n(X)$, by [16, Theorem 5.1, page 245], $C_{\{p\}}^n(X)$ is colocally connected at each point. By Proposition 2.4, $C_{n\{p\}}$ is not a cut point of $C_{\{p\}}^n(X)$. Then, by Corollary 2.3, $p$ is not a cut point of $X$. $\square$

The following lemma is a consequence of [23, Exercise 7.4, page 36].

**Lemma 3.3.** *Let $X$ be a continuum, $K \in 2^X$ and $n \in \mathbb{N}$. Then*

$$\dim(C_K^n(X)) = \dim(C_n(X) - C_{nK}(X)).$$

Given a finite graph $X$ and $n \in \mathbb{N}$, by [19, Theorem 2.4, page 791], for $A \in C_n(X)$, $\dim_A(C_n(X)) = 2n + \sum_{p \in (R(X) \cap A)} (\mathrm{ord}_X(p) - 2)$. Let $x \in X$. We consider

$$D_x^n = \{\dim_A(C_n(X)) \colon A \in \langle X - \{x\} \rangle_n\}.$$

Since $X$ is a finite graph, by [15, Theorem 5.1, page 270], $\dim_A(C_n(X)) < \infty$ for every $A \in C_n(X)$ and $\emptyset \neq D_x^n \subset \mathbb{N}$. Let $\mathcal{O}_x^n = \max D_x^n$. Now, if $K \in 2^X$, define $M_K^n = \max\{\mathcal{O}_x^n \colon x \in K\}$ and we have the following lemma.

**Lemma 3.4.** *Let $X$ be a finite graph and $n \in \mathbb{N}$. If $K \in 2^X$, then*

$$\dim(C_K^n(X)) = M_K^n \leq \dim(C_n(X)).$$

PROOF: By Lemma 3.3, we will prove that $\dim(C_n(X) - C_{nK}(X)) = M_K^n$. Let $A \in C_n(X) - C_{nK}(X)$. Since $K \not\subset A$, there exists $x \in K - A$ such that $A \subset X - \{x\}$. Thus $\dim_A(C_n(X)) \in D_x^n$. We have that $\dim_A(C_n(X)) \leq \mathcal{O}_x^n \leq M_K^n \leq \dim_X(C_n(X))$. On the other hand, assume that $\mathcal{O}_{x_0}^n = \max\{\mathcal{O}_x^n : x \in K\}$, then there exists $B \in \langle X - \{x_0\}\rangle_n$ such that $\mathcal{O}_{x_0}^n = \dim_B(C_n(X))$. Since $x_0 \in K$, $B \in C_n(X) - C_{nK}(X)$. Therefore $\dim(C_K^n(X)) = M_K^n$. □

The following corollary is a consequence of [19, Theorem 2.4, page 791] and Lemma 3.4.

**Corollary 3.5.** *Let $X$ be a finite graph and $n \in \mathbb{N}$.*

(1) *If $K \subset R(X)$, then* $\dim(C_K^n(X)) < \dim(C_n(X))$.
(2) *If $K \subset E(X)$, then* $\dim(C_K^n(X)) = \dim(C_n(X))$.

The following example shows what happens if $R(X)$ is contained in $K$. Let $X$ be a continuum homeomorphic to the capital letter $H$. Without loss of generality we may assume that

$$\begin{aligned} X = &\{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\} \cup \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \\ &\cup \{(1, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}. \end{aligned}$$

Let $p = (0, 0)$, $q = (1, 0)$, $r = \left(\frac{1}{2}, 0\right)$, $a_1 = (0, -1)$, $a_2 = (1, -1)$, $a_3 = (0, 1)$, $a_4 = (1, 1)$. Note that $p, q \in R(X)$ and $a_1, a_2, a_3, a_4 \in E(X)$.

**Example 3.6.** If $X$ is homeomorphic to the capital letter $H$, then

a) $\dim(C_K^n(X)) < \dim(C_n(X))$ for $K = \{p, q, r\}$ and $n = 1$.
b) $\dim(C_K^n(X)) = \dim(C_n(X))$ for $K = \{p, q, r\}$ and $n \geq 2$.
c) $\dim(C_K^n(X)) = \dim(C_n(X))$ for $K = \{p, q, a_1\}$ and $n \geq 1$.

PROOF: Note that $D_p^n = D_q^n = \{2n, 2n + 1\}$ and $D_{a_1}^n = \{2n, 2n + 1, 2n + 2\}$ for every $n \geq 1$. We have

$$D_r^n = \begin{cases} \{2, 3\} & \text{if } n = 1, \\ \{2n, 2n + 1, 2n + 2\} & \text{if } n \geq 2. \end{cases}$$

Thus, $\mathcal{O}_p^n = \mathcal{O}_q^n = 2n + 1$ and $\mathcal{O}_{a_1}^n = 2n + 2$ for $n \geq 1$. Also, $\mathcal{O}_r^n = 3$ and $\mathcal{O}_r^n = 2n + 2$ for $n = 1$ and $n \geq 2$, respectively. So that

$$M_{\{p,q,r\}}^n = \begin{cases} 3 & \text{if } n = 1, \\ 2n + 2 & \text{if } n \geq 2, \end{cases}$$

and $M^n_{\{p,q,a_1\}} = 2n+2$ for $n \geq 1$. Therefore, if $K = \{p,q,r\}$, then $\dim(C^1_K(X)) = 3 < \dim(C(X))$ and $\dim(C^n_K(X)) = 2n+2 = \dim(C_n(X))$ for every $n \geq 2$. And, if $K = \{p,q,a_1\}$, then $\dim(C^n_K(X)) = 2n+2 = \dim(C_n(X))$ for every $n \geq 1$. $\square$

Given a continuuum $X$, the set of cut points of $X$ is denoted by $\mathrm{Cut}(X)$. In the following, the set of subsets of $\mathrm{Cut}(X)$ with only one point is denoted by $F_1(\mathrm{Cut}(X))$, in the same way we denote $F_1(E(X))$.

**Theorem 3.7.** *Let $X$ be a finite graph and $n \in \mathbb{N}$. If $C^n_K(X) \approx C_n(X)$ for all $K \in 2^X - F_1(\mathrm{Cut}(X))$, then $|R(X)| \leq 1$.*

PROOF: Suppose that $|R(X)| > 1$. Since $X$ is a finite graph, we have that $R(X)$ is finite. By (1) of Corollary 3.5 for $K = R(X)$, $\dim(C^n_K(X)) < \dim(C_n(X))$. This is a contradiction. $\square$

Let $Y$ be a simple $m$-od with edges $J_1, \ldots, J_m$ and core $v$. Note that

$$C(Y) = C_{\{v\}}(Y) \cup \left( \bigcup_{i=1}^{m} C(J_i) \right).$$

Furthermore, for any $i,j \in \{1,\ldots,m\}$, $C(J_i) \cap C(J_j) = \{\{v\}\}$ with $i \neq j$, $C_{\{v\}}(Y) \cap C(J_i) = C_{\{v\}}(J_i)$, and $C_{\{v\}}(Y) \cap \left(\bigcap_{i=1}^m C(J_i)\right) = \{\{v\}\}$. A *tree* is a finite graph without simple closed curves.

**Corollary 3.8.** *Let $X$ be a tree and $n \in \mathbb{N}$. If $C^n_K(X) \approx C_n(X)$ for all $K \in 2^X - F_1(\mathrm{Cut}(X))$, then $X$ is an arc or a simple $m$-od.*

PROOF: By Theorem 3.7, $|R(X)| \leq 1$. If $X$ has a ramification point, then $X$ is a simple $m$-od for some $m \in \mathbb{N}$. On the other hand, if $\mathrm{ord}_X(p) \leq 2$ for every $p \in X$, since $X$ is a tree, by [22, Proposition 9.5, page 142], $X$ is an arc. $\square$

Given a topological space $Y$, the *cone over* $Y$, which we will denote by $\mathrm{Cone}(Y)$, is the quotient space $Y \times I / Y \times \{1\}$ obtained from $Y \times I$ by shrinking $Y \times \{1\}$ to a point. The point $Y \times \{1\}$ and the subset $Y \times \{0\}$ of $\mathrm{Cone}(Y)$ are called the *vertex* and the *base* of $\mathrm{Cone}(Y)$, respectively. We will denote by $v_Y$ and $B(Y)$ the vertex and the base of $\mathrm{Cone}(Y)$, respectively. The *suspension over* $Y$, which we will denote by $\Sigma(Y)$, is the quotient space obtained from $Y \times [-1,1]$ by shrinking $Y \times \{-1\}$ and $Y \times \{1\}$ to two different points which are called *vertices of* $\Sigma(Y)$. Note that $\Sigma(Y) \approx \mathrm{Cone}(Y)/B(Y)$. We will denote by $q\colon \mathrm{Cone}(Y) \to \mathrm{Cone}(Y)/B(Y)$ the quotient map.

As consequence of the proof of [1, Theorem 5.5, page 356] we have the following proposition.

**Proposition 3.9.** *Let $n \in \mathbb{N}$. Suppose that $X = \text{Cone}(Y)$ for some compact metric space $Y$. Then, $C_n(X)$ is homeomorphic to Cone($Z$), where $Z = \bigcup_{p\in B(Y)} C_{n\{p\}}(X)$.*

**Theorem 3.10.** *Assume that $X$ is an arc or a simple $m$-od. If $K \in F_1(E(X))$, then $C^1_K(X)$ is homeomorphic to $C(X)$.*

Proof: Suppose that $X = I$. By [12, Exercise 14.24, page 118], $C_K(X)$ is an arc. Then $C(X) - C_K(X)$ is homeomorphic to $I \times [0, 1)$. Thus, $C^1_K(X)$ is homeomorphic to $C(X)$. Note that, in this case, $C^1_K(X)$ is a 2-dimensional cell.

Now, suppose that for some $m \in \mathbb{N}$, $X$ is a simple $m$-od with edges $J_1, \ldots, J_m$. Assume that $K = \{e\}$, where $e \in J_1$. Note that $C_K(X) = C_{\{e\}}(J_1) \cup C_{J_1}(X)$ and $C_{\{e\}}(J_1) \cap C_{J_1}(X) = \{J_1\}$. By Proposition 3.9, $C(X)$ is homeomorphic to Cone($Z$), where

$$Z = \bigcup_{p\in E(X)} C_{\{p\}}(X).$$

On the other hand, by [12, Exercise 14.24, page 118], $C_{\{e\}}(J_1)$ is an arc in $C(J_1)$. Also, $C_{\{v\}}(X)$ is an $m$-cell, where $v$ is the core of $X$, see [14, Theorem 3 and Theorem 4, page 3071]. By [8, 5.2, page 271], $C_{J_1}(X)$ is a $(m-1)$-cell in $C_{\{v\}}(X)$. Note that $C_{J_1}(X) \cap C(J_i) = \emptyset$ for each $i \neq 1$. Since $C(J_1)/C_{\{e\}}(J_1)$ is homeomorphic to $C(J_1)$, and $C_{\{v\}}(X)/C_{J_1}(X)$ is homeomorphic to $C_{\{v\}}(X)$, we conclude that $C^1_K(X)$ is homeomorphic to $C(X)$. □

As consequence of Proposition 3.9, we have the following result.

**Corollary 3.11.** *Let $n \in \mathbb{N}$. Then $C^n_K(I)$ is homeomorphic to $\Sigma(C_{nK}(I))$ for $K \in \{\{0\}, \{1\}\}$.*

Proof: We only prove this for $K = \{0\}$. Since $I \approx \text{Cone}(\{0\})$, by Proposition 3.9, there exists a homeomorphism $h\colon C_n(I) \to \text{Cone}(Z)$, where $Z = C_{nK}(I)$. Note that the following diagram is commutative

$$\begin{array}{ccccc} & h & & & \\ C_n(I) & \longrightarrow & \text{Cone}(Z) & & \\ \pi_K \downarrow & & \downarrow q & & \\ C^n_K(I) & \longrightarrow & \text{Cone}(Z)/B(Z) & \approx & \Sigma(Z) \end{array}$$

The fact that $C^n_K(I)$ is homeomorphic to $\Sigma(C_{nK}(I))$ follows from the Transgression lemma, see [22, Exercise 3.22, page 45]. □

**Theorem 3.12.** *Let $X$ be a finite graph and $n \in \mathbb{N}$. If $C_n(X) \approx C^n_K(X)$ for all $K \in 2^X$, then $X$ is a simple closed curve.*

Proof: Since $C_n(X) \approx C^n_K(X)$, by Theorem 3.7, $|R(X)| \leq 1$. By Lemma 3.2, $X$ does not contain cut points. Suppose that $R(X) = \{p\}$. Since $p$ is the only ramification point of $X$, $p$ is a cut point of $X$. This is a contradiction. Thus $R(X) = \emptyset$.

Since $X$ is a finite graph and $\mathrm{ord}_X(x) \leq 2$ for every $x \in X$, by [22, Proposition 9.5, page 142], $X$ is an arc or a simple closed curve. But $X$ does not contain cut points, then $X$ is not an arc. Therefore, $X$ is a simple closed curve. □

To finish this section we focus on case $n = 1$ in the class of finite graphs. In the following theorem, the set of subsets of $(0,1)$ with only one point is denoted by $F_1((0,1))$.

**Theorem 3.13.** *If $X$ is a finite graph, $C^1_K(X)$ is homeomorphic to* $\mathrm{Cone}(X)$ *if only if*

(1) *$X = I$ and $K \in 2^X - F_1((0,1))$, or*
(2) *$X = S^1$ and $K \in 2^X$.*

PROOF: Assume that $h\colon \mathrm{Cone}(X) \to C^1_K(X)$ is a homeomorphism. Since $X$ is a finite graph, $\dim(\mathrm{Cone}(X)) = 2$. Thus, $\dim(C^1_K(X)) = 2$. Note that for every $y \in \mathrm{Cone}(X)$, $y$ is not a cut point. Then $h(y)$ is not a cut point of $C^1_K(X)$, in particular, $C_K$ is not a cut point of $C^1_K(X)$.

Suppose that $K = \{z\}$ for some $z \in X$. Since $C_K$ is not a cut point of $C^1_K(X)$, by Corollary 2.3, $z$ is not a cut point of $X$. Then $z$ is an end point or $z$ belongs to a simple closed curve in $X$.

On the other hand, suppose that $K \in 2^X$ such that $|K| \geq 2$. Let $C \in C(X)$ such that $K \subset C$. Assume that $p \in X$ is a ramification point. If $p \in X - C$, since $X$ is a finite graph, there exists $C_p \in C(X)$ such that $p \in C_p \subset X - C$ and $\dim_{C_p}(C(X)) \geq 3$. Moreover, $C_p \notin C_K(X)$. Thus, $h^{-1}(C_p) \in \mathrm{Cone}(X)$ this is a contradiction because $\dim(\mathrm{Cone}(X)) = 2$.

Now, if $p \in C$, there exists $q \in K$ such that $p \neq q$. Since $X$ is a metric space, there are disjoint open sets $U$ and $V$ such that $p \in U$ and $q \in V$. By locally connectedness, there is $C_p \in C(X)$ such that $p \in C_p \subset U$. Note that $K \not\subset C_p$, thus $C_p \notin C_K(X)$ and $\dim_{C_p}(C(X)) \geq 3$. This is a contradiction. Therefore, $X$ does not contain ramification points.

By both cases and [22, Proposition 9.5, page 142], we have that $X$ is the arc $I$ or $S^1$.

Conversely, let $X = I$ and $K \in 2^X - F_1((0,1))$. Suppose that $K = \{0\}$ or $K = \{1\}$. By [12, Exercise 14.24, page 118], $C_K(X)$ is an arc. Then $C(X) - C_K(X)$ is homeomorphic to $I \times [0,1)$. Thus, $C^1_K(X)$ is homeomorphic to $\mathrm{Cone}(X)$. If $K \in 2^X - F_1(X)$, let $a = \min K$ and $b = \max K$, note that $a \neq b$. First, assume that $a = 0$ or $b = 1$. By [12, Exercise 14.24, page 118], $C_K(X)$ is an arc, then $C^1_K(X)$ is homeomorphic to $\mathrm{Cone}(X)$. Now, assume that $a \neq 0$ and $b \neq 1$. Then

$C(X) - C_K(X) = \langle [0,b) \rangle_1 \cup \langle (a,1] \rangle_1$ which is homeomorphic to $I \times [0,1)$. Thus, $C_K^1(X)$ is homeomorphic to Cone$(X)$.

Finally, let $X = S^1$ and $K \in 2^X$. If $K \in F_1(X)$, by [11, Ejemplo 3.2, page 31], $C_K(X)$ is homeomorphic to a 2-cell in $C(X)$. Thus, $C_K^1(X)$ is homeomorphic to a 2-cell, which is homeomorphic to Cone$(X)$. In the other case, by Theorem 6.3, $C_K(X)$ is contractible in $C(X)$ such that $C_K(X) \cap F_1(X) = \emptyset$. Then $C(X) - C_K(X)$ is homeomorphic to $S^1 \times [0,1)$. Thus, $C_K^1(X)$ is homeomorphic to Cone$(X)$. $\square$

As consequence of [12, Example 5.2, page 35] we have the following result.

**Corollary 3.14.** *For each $K, L \in 2^{S^1}$, $C_K^1(S^1)$ is homeomorphic to $C_L^1(S^1)$. Moreover, $C_K^1(S^1)$ is homeomorphic to a 2-cell.*

**Question 3.15.** If $n \geq 2$, then is $C_A^n(S^1)$ homeomorphic to $C_B^n(S^1)$ for every $A, B \in C(S^1)$ ( or $2^{S^1}$)?

## 4. Aposyndesis

In this section we show that for every $n \in \mathbb{N}$, $C_K^n(X)$ is finitely aposyndetic for some $K \in 2^X$.

**Proposition 4.1.** *Let $X$ be a continuum and $n \in \mathbb{N}$. If $K \in 2^X$, then $C_K^n(X)$ is colocally connected at $\mathcal{A}$ for every $\mathcal{A} \in C_K^n(X) - \{C_{nK}\}$.*

PROOF: Let $\mathcal{A} \in C_K^n(X) - \{C_{nK}\}$ and let $A \in C_n(X)$ such that $\pi_K(A) = \mathcal{A}$. Note that $A \notin C_{nK}(X)$. By [16, Theorem 5.1 page 245], there exists a local base $\{V_\alpha\}$ of open subsets at $A$ whose complements are connected. We may assume that $V_\alpha \subset C_n(X) - C_{nK}(X)$ for all $\alpha$. By Remark 1.2, $\{\pi_K(V_\alpha)\}$ is a local base of open subsets at $\mathcal{A}$. Since $\pi_K(C_n(X) - V_\alpha) = C_K^n(X) - \pi_K(V_\alpha)$ is connected, $C_K^n(X)$ is colocally connected at $\mathcal{A}$. $\square$

Let $p, q \in X$, $p \neq q$. A continuum $X$ is *aposyndetic at $p$ with respect to $q$* provided that there exists a subcontinuum $M$ of $X$ such that $p \in \mathrm{Int}_X(M)$ and $q \in X - M$. If for each $q \in X - \{p\}$, $X$ is aposyndetic at $p$ with respect to $q$, then $X$ is *aposyndetic at $p$*. If $X$ is aposyndetic at each of its points then $X$ is *aposyndetic*. A continuum $X$ is *finitely aposyndetic* provided that for each finite subset $F$ of $X$ and point $x$ of $X$ not in $F$, there exists a subcontinuum $W$ of $X$ such that $x \in \mathrm{Int}_X(W) \subset W \subset X - F$.

**Remark 4.2.** If $X$ is colocally connected at $y$, then $X$ is aposyndetic at $x$ with respect to $y$ for each $x \in X - \{y\}$.

By Proposition 4.1 and Remark 4.2, we have the following result.

**Lemma 4.3.** *Let $n \in \mathbb{N}$. If $X$ is a continuum and $K \in 2^X$, then*

- $C_K^n(X)$ *is aposyndetic at* $C_{nK}$;
- $C_K^n(X)$ *is aposyndetic at* $\mathcal{A} \neq C_{nK}$ *with respect to any* $\mathcal{B} \neq C_{nK}$.

**Theorem 4.4.** *If $X$ is a continuum and $n \in \mathbb{N}$, then $C_K^n(X)$ is aposyndetic for each $K \in 2^X - F_1(X)$.*

Proof: Let $\mathcal{A} \in C_K^n(X)$. By Lemma 4.3, we prove that $C_K^n(X)$ is aposyndetic at $\mathcal{A}$ with respect to $C_{nK}$. Let $A \in C_n(X)$ such that $\pi_K(A) = \mathcal{A}$. Since $\mathcal{A} \neq C_{nK}$, $K \not\subset A$, and there exists $k_0 \in K - A$. We consider $A_0 = \{k_0\}$ and $A = A_1 \cup \cdots \cup A_m$, where $A_i$ is a component of $A$ for every $i = 1, \ldots, m$ with $1 \leq m \leq n$. Since $X$ is a metric space, **1)** holds:

**1)** For each $i = 0, \ldots, m$ there exists $W_i$ an open subset of $X$ such that $A_i \subset W_i$, and $\mathrm{Cl}_X(W_i) \cap \mathrm{Cl}_X(W_j) = \emptyset$ for any $i, j \in \{0, \ldots, m\}$ with $i \neq j$. In consequence **2)** holds:

**2)** $A \in \mathcal{U} = \langle W_1, \ldots, W_m \rangle_n$ and $\mathrm{Cl}_{C_n(X)}(\mathcal{U}) \cap C_{nK}(X) = \emptyset$.

**3)** For every component $\mathcal{D}$ of $\mathcal{U}$, we have $\mathcal{D} \cap F_n(X) \neq \emptyset$. Moreover, $F_n(X) \cap \mathrm{Cl}_{C_n(X)}(\mathcal{U}) \neq \emptyset$.

To prove **3)**, let $\mathcal{D}$ be a component of $\mathcal{U}$. Since $\mathcal{D}$ is a connected subset of $C_n(X)$, by [10, Lemma 1, page 1578], $D_0 = \bigcup\{A \colon A \in \mathcal{D}\}$ has at most $n$ components, we may suppose that $D_0 = D_1 \cup \cdots \cup D_l$ with $1 \leq l \leq n$. Thus, $\{d_1, \ldots, d_l\} \in \mathcal{D} \cap F_n(X)$ where $d_i \in D_i$ for every $i = 1, \ldots, l$. Note that $D_i$ is a component of $W_{j_i}$ for some $j_i \in \{1, \ldots, m\}$. By the boundary bumping theorem, see [22, Theorem 5.6, page 74] for every $i \in \{1, \ldots, l\}$ there exists $d_i \in \mathrm{Cl}_X(D_i) \cap \mathrm{Bd}(W_{j_i})$. We conclude **4)**.

**4)** There exists $\{d_1, \ldots, d_l\} \in \mathrm{Cl}_{C_n(X)}(\mathcal{D})$ such that for each $i = 1, \ldots, l$ there exists $j \in \{1, \ldots, m\}$ with $d_i \in \mathrm{Bd}(W_j)$.

Suppose that $K \in F_n(X)$. The existence of a subcontinuum of $F_n(X) - F_{nK}(X)$, see the proof of Theorem 10 of [5], and **4)**, give the proof of **5)** and **6)**.

**5)** For each $\mathcal{D}$ there exists $M_{\mathcal{D}}$ a subcontinuum of $F_n(X) - F_{nK}(X)$ such that $M_{\mathcal{D}} \cap F_1(X) \neq \emptyset$ and $\{d_1, \ldots, d_l\} \in M_{\mathcal{D}}$.

**6)** $M = \mathrm{Cl}_{C_n(X)}\big(\bigcup\{M_{\mathcal{D}} \colon \mathcal{D} \text{ is component of } \mathcal{U}\}\big) \cup F_1(X)$ is a subcontinuum of $F_n(X) - F_{nK}(X)$.

Thus, $\mathcal{C} = \mathrm{Cl}_{C_n(X)}(\mathcal{U}) \cup M$ is a subcontinuum of $C_n(X) - C_{nK}(X)$ such that $A \in \mathrm{Int}_{C_n(X)}(\mathcal{C})$.

Now, if $K \notin F_n(X)$, $F_n(X) \cap C_{nK}(X) = \emptyset$. By **2)** and **3)** $\mathcal{C} = \mathrm{Cl}_{C_n(X)}(\mathcal{U}) \cup F_n(X)$ is a subcontinuum of $C_n(X) - C_{nK}(X)$ such that $A \in \mathrm{Int}_{C_n(X)}(\mathcal{C})$. This completes the proof. □

By Theorem 2.1, Theorem 4.4 and [3, Corollary 1, page 586] we have the following result.

**Corollary 4.5.** *Let $n \in \mathbb{N}$. If $X$ is a continuum and $K \in 2^X - F_1(X)$, then $C_K^n(X)$ is finitely aposyndetic.*

**Theorem 4.6.** *Let $X$ be a continuum and $n \in \mathbb{N}$. Then $X$ is aposyndetic if and only if $C_K^n(X)$ is aposyndetic for each $K \in F_1(X)$.*

PROOF: Let $K \in F_1(X)$. By Lemma 4.3, we prove that $C_K^n(X)$ is aposyndetic at $\mathcal{A} \in C_K^n(X) - \{C_{nK}\}$ with respect to $C_{nK}$. Let $A \in C_n(X)$ such that $\pi_K(A) = \mathcal{A}$, suppose that $A = A_1 \cup \cdots \cup A_m$, where $A_i \in C(X)$ for each $i = 1, \ldots, m$ and $m \leq n$. Assume that $K = \{x_0\}$ for some $x_0 \in X$. Since $A \notin C_{nK}(X)$, then $A_i \subset X - \{x_0\}$ for every $i = 1, \ldots, m$. Let $i \in \{1, \ldots, m\}$ and let $a \in A_i$. Since $X$ is aposyndetic, there exists $M_a^i$ a subcontinuum of $X$ such that $a \in \mathrm{Int}_X(M_a^i) \subset M_a^i \subset X - \{x_0\}$. Then $\mathcal{C}_i = \{\mathrm{Int}(M_a^i) \colon a \in A_i\}$ is an open cover of $A_i$. Since $A_i$ is compact, there exists $l_i \in \mathbb{N}$ such that $A_i \subset \bigcup_{j=1}^{l_i} \mathrm{Int}_X(M_{a_j}^i)$. For each $i = 1, \ldots, m$, let $U_i = \bigcup_{j=1}^{l_i} \mathrm{Int}_X(M_{a_j}^i)$, note that $U_i \subset X - \{x_0\}$ and $A \in \langle U_1, \ldots, U_m \rangle_n$. Thus, $\mathcal{W} = \mathrm{Cl}_{C_n(X)}(\langle U_1, \ldots, U_m \rangle_n)$ is a subcontinuum of $C_n(X) - C_{nK}(X)$. Then $\pi_K(\mathcal{W})$ is a subcontinuum of $C_K^n(X) - \{C_{nK}\}$ such that $\mathcal{A} \in \mathrm{Int}_{C_K^n(X)}(\pi_K(\mathcal{W}))$. Therefore $C_K^n(X)$ is aposyndetic at $\mathcal{A}$.

Conversely, let $p, q \in X$ with $p \neq q$. We may assume that $K = \{q\}$. Since $C_K^n(X)$ is aposyndetic, there exists $\mathfrak{W}$ a subcontinuum of $C_K^n(X)$ such that $\mathcal{P} = \pi_K(\{p\}) \in \mathrm{Int}_{C_K^n(X)}(\mathfrak{W})$ and $C_{nK} \in C_K^n(X) - \mathfrak{W}$. Then $\pi_K^{-1}(\mathfrak{W})$ is a subcontinuum of $C_n(X) - C_{nK}(X)$. Note that $\{p\} \in \pi_K^{-1}(\mathfrak{W})$. By [10, Lemma 1, page 1578], $M = \bigcup \pi_K^{-1}(\mathfrak{W})$ is a subcontinuum of $X$ such that $p \in \mathrm{Int}_X(M)$. Since $\pi_K^{-1}(\mathfrak{W}) \subset C_n(X) - C_{nK}(X)$, $q \notin M$. Thus, $X$ is aposyndetic at $p$ for every $p \in X$. □

As consequence of Theorem 4.4 and Theorem 4.6 we conclude the following result.

**Corollary 4.7.** *Let $X$ be a continuum and $n \in \mathbb{N}$. Then $X$ is aposyndetic if and only if $C_K^n(X)$ is aposyndetic for each $K \in 2^X$.*

Let $\Sigma(Z)$ be the suspension over $Z$ where $Z = \left\{\frac{1}{n} \in \mathbb{R} \colon n \in \mathbb{N}\right\} \cup \{0\}$. Denote by $v_1$, $v_{-1}$ the vertices of $\Sigma(Z)$, $L_0 = \{0\} \times (-1, 1)$ and $p = (0, 0)$. Now, we consider $Y$ defined by identifying $v_1$, $v_{-1}$ in $\Sigma(Z)$ to one point denoted by $v$. Note that $Y$ is a continuum not aposyndetic at $x \in q(L_0)$.

**Example 4.8.** The continuum $Y$ is not aposyndetic and $C_K^n(Y)$ is aposyndectic for $K = \{p\}$ and $C_L^n(Y)$ is not aposyndetic for $L = \{v\}$.

PROOF: Let $n \in \mathbb{N}$ and $K = \{p\}$. By Lemma 4.3, we prove that $C_K^n(Y)$ is aposyndetic at $\mathcal{A}$ with respect to $C_{nK}$. Let $A \in C_n(X)$ such that $\pi_K(A) = \mathcal{A}$. Note that $Y$ is colocally connected at $p$. Since $\mathcal{A} \neq C_{nK}$, $p \notin A$. There exists $U$ an open subset such that $p \in U \subset \mathrm{Cl}_Y(U) \subset Y - A$ and $Y - U$ is a continuum. Thus, $\langle Y - U\rangle_n$ is a subcontinuum of $C_n(Y) - C_{nK}(Y)$ such that $A \in \langle Y - \mathrm{Cl}_Y(U)\rangle_n \subset \langle Y - U\rangle_n$. Therefore $C^n_K(Y)$ is aposyndetic.

Now, we prove that $C^n_L(Y)$ is not aposyndetic at $\mathcal{P} = \pi_L(\{p\})$ with respect to $C_{nL}$. Let $\mathfrak{W}$ be a subcontinuum of $C^n_L(Y)$ such that $\mathcal{P} \in \mathrm{Int}_{C^n_L(Y)}(\mathfrak{W})$. Suppose that $C_{nL} \notin \mathfrak{W}$, then $\pi_L^{-1}(\mathfrak{W})$ is a subcontinuum of $C_n(Y) - C_{nL}(Y)$. Note that $\{p\} \in \pi_L^{-1}(\mathfrak{W})$. By [10, Lemma 1, page 1578], $M = \bigcup \pi_L^{-1}(\mathfrak{W})$ is a subcontinuum of $Y$ such that $p \in \mathrm{Int}_Y(M)$. Then $v \in M$, this is a contradiction. Therefore, $C^n_L(Y)$ is not aposyndetic. □

## 5. Connectedness and arcwise disconnectedness

**Theorem 5.1.** *If $X$ is a continuum and $n \in \mathbb{N}$, then $C^n_K(X)$ is an arcwise connected continuum for each $K \in 2^X$.*

Proof: Let $K \in 2^X$. By [16, Theorem 3.1, page 240], $C_n(X)$ is an arcwise connected continuum. Since $\pi_K$ is a map, we have that $C^n_K(X)$ is an arcwise connected continuum. □

**Lemma 5.2.** *Let $X$ be a continuum, $K \in 2^X$ and $n \in \mathbb{N}$. If $C_n(X) - C_{nK}(X)$ is locally connected, then $X$ is locally connected.*

Proof: Let $x \in X$. Suppose that $x \notin K$, let $V$ be an open subset of $X$ such that $x \in V$. Thus, $W = V \cap (X - K)$ is an open subset of $X$ containing $x$, then $\{x\} \in \langle W\rangle_n \subset C_n(X) - C_{nK}(X)$. Since $C_n(X) - C_{nK}(X)$ is locally connected, there exists an open connected subset $\mathcal{U}$ of $C_n(X) - C_{nK}(X)$ such that $\{x\} \in \mathcal{U} \subset \mathrm{Cl}_{C_n(X)}(\mathcal{U}) \subset \langle W\rangle_n$. Let $\varepsilon > 0$ be such that $B^H_\varepsilon(\{x\}) \cap C_n(X) \subset \mathcal{U}$. Let $H = \bigcup \mathrm{Cl}_{C_n(X)}(\mathcal{U})$. By [20, Lemma 1.49, page 102], $H$ is a subcontinuum of $X$. Note that $x \in H \subset W \subset V$. For $y \in B_\varepsilon(x)$, we have that $\{y\} \in B^H_\varepsilon(\{x\}) \cap C_n(X) \subset \mathcal{U}$. Thus, $\{y\} \in \mathcal{U}$ and $y \in H$. Then, $x \in \mathrm{Int}_X(H) \subset W \subset V$.

Now, suppose that $x \in K$ and $K \notin F_1(X)$, then there exists $z \in K$ such that $z \neq x$. Let $V$ be an open subset of $X$ such that $x \in V$. Since $W = V \cap (X - \{z\})$ is an open subset of $X$ containing $x$ and $\{x\} \in \langle W\rangle_n \subset C_n(X) - C_{nK}(X)$, we proceed as before.

Finally, if $K = \{x\}$, by [22, Corollary 5.13, page 78] and the previous argument, $X$ cannot be connected im kleinen at only one point. Then, we have that $X$ is connected im kleinen at each of its points. Therefore $X$ is locally connected. □

**Theorem 5.3.** *Let $n \in \mathbb{N}$ and $K \in 2^X$. Then, $X$ is a Peano continuum if only if $C^n_K(X)$ is a Peano continuum.*

Proof: Since $X$ is locally connected, by [16, Theorem 3.2, page 240], $C_n(X)$ is locally connected. By [22, Proposition 3.7, page 39], $\pi_K$ is a closed map. By [22, Proposition 8.16, page 127], $C_K^n(X)$ is locally connected.

Now, suppose that $C_K^n(X)$ is locally connected. Since $C_K^n(X) - \{C_{nK}\}$ is locally connected, by Remark 1.2, $C_n(X) - C_{nK}(X)$ is locally connected. By Lemma 5.2, $X$ is locally connected. $\square$

The following result is a consequence of Theorem 5.3.

**Corollary 5.4.** *Let $X$ be a continuum. Then the following are equivalent:*

(1) *$X$ is a Peano continuum;*
(2) *$C_K^n(X)$ is a Peano continuum for every $n \in \mathbb{N}$ and every $K \in 2^X$;*
(3) *$C_K^n(X)$ is a Peano continuum for some $n \in \mathbb{N}$ and every $K \in 2^X$;*
(4) *$C_K^n(X)$ is a Peano continuum for every $n \in \mathbb{N}$ and some $K \in 2^X$;*
(5) *$C_K^n(X)$ is a Peano continuum for some $n \in \mathbb{N}$ and some $K \in 2^X$.*

**Theorem 5.5.** *Let $X$ be a continuum and $n \in \mathbb{N}$. If $K \in 2^X - F_1(X)$, then $C_n(X) - C_{nK}(X)$ is connected.*

Proof: Let $K \in 2^X - F_1(X)$. Note that $C_{nK}(X) \cap F_1(X) = \emptyset$. Let $A \in C_n(X) - C_{nK}(X)$. Suppose that $A = A_1 \cup \cdots \cup A_m$, where $m \leq n$ and $A_i \in C(X)$ for each $i \in \{1, \ldots, m\}$. Take an element $a_i$ in $A_i$ for each $i \in \{1, \ldots, m\}$. There exists $\alpha\colon I \to C_n(X)$ an order arc such that $\alpha(0) = \{a_1, \ldots, a_m\}$ and $\alpha(1) = A$. Since $K \not\subset A$, there exists $k_0 \in K - A$ and $K \not\subset \alpha(t)$ for every $t \in I$. Assume that $a_1 \in K \cap \alpha(0)$. For each $i \in \{1, \ldots, m-1\}$ let $\gamma_i\colon X \to C_n(X)$ given by $\gamma_i(x) = \{x\} \cup (\alpha(0) - \{a_1, \ldots, a_i\})$. Then $\gamma_i$ is well defined and is a map. Let $\Gamma_i = \gamma_i(X)$. Note that $\{k_0, a_1\} \not\subset \gamma_i(x)$ for every $i \in \{1, \ldots, m-1\}$ and every $x \in X$. Thus, $\Gamma_i$ is a connected subset of $C_n(X) - C_{nK}(X)$. Since $\alpha(0) - \{a_1, \ldots, a_{i-1}\} \in \Gamma_{i-1} \cap \Gamma_i$ for every $i \in \{2, \ldots, m-1\}$, $\Gamma = \bigcup_{i=1}^{m-1} \Gamma_i$ is a connected subset of $C_n(X)$ containing $\alpha(0)$ and $\{a_m\}$. Thus, $A \in \alpha(I) \cup \Gamma \cup F_1(X)$ which is connected subset of $C_n(X) - C_{nK}(X)$. Therefore $C_n(X) - C_{nK}(X)$ is connected. $\square$

**Proposition 5.6.** *Let $X$ be a continuum, $n \in \mathbb{N}$ and $K \in 2^X$. If $A$ is a subcontinuum of $X - \{x\}$ for some $x \in K$, then $C_K^n(X) - \pi_K(C_n(A))$ is arcwise connected.*

Proof: Since $A \in C(X) - C_K(X)$, by [16, Theorem 6.1, page 246], $C_n(X) - C_n(A)$ is arcwise connected. Then, $\pi_K(C_n(X) - C_n(A)) = C_K^n(X) - \pi_K(C_n(A))$ is arcwise connected. $\square$

**Proposition 5.7.** *Let $X$ be a continuum, $n \in \mathbb{N}$ and $K \in 2^X$. If $\mathcal{A} \in C_K^n(X) - \{C_{nK}\}$ is such that $C_K^n(X) - \{\mathcal{A}\}$ is not arcwise connected then $\pi_K^{-1}(\mathcal{A}) \in C(X)$.*

Proof: Let $\mathcal{A} \in C_K^n(X) - \{C_{nK}\}$. Then $\pi_K^{-1}(\mathcal{A}) \in C_n(X)$. Since $\pi_K^{-1}(C_K^n(X) - \{\mathcal{A}\}) = C_n(X) - \{\pi_K^{-1}(\mathcal{A})\}$ and $C_K^n(X) - \{\mathcal{A}\}$ is not arcwise connected, $C_n(X) - \{\pi_K^{-1}(\mathcal{A})\}$ is not arcwise connected. Thus, by [16, Theorem 6.2, page 246], $\pi_K^{-1}(\mathcal{A}) \in C(X)$. $\square$

**Theorem 5.8.** *Let $X$ be a continuum. Then, for any $K \in 2^X$ the following statements are equivalent:*

(1) *$X$ is indecomposable;*
(2) *$2^X/2_K^X - \{2_K\}$ is not arcwise connected;*
(3) *for each $n \in \mathbb{N}$, $C_K^n(X) - \{C_{nK}\}$ is not arcwise connected;*
(4) *$C_K^1(X) - \{C_{1K}\}$ is not arcwise connected.*

Proof: Let $K \in 2^X$. We first prove that (1) $\Rightarrow$ (3). Let $x$ and $y$ be points in different composants of $X$. Since $X$ is an indecomposable continuum, for any function $\alpha\colon I \to C_n(X)$ such that $\alpha(0) = \{x\}$ and $\alpha(1) = \{y\}$, there exists $t_0 \in I$ such that $K \subset \alpha(t_0)$. Then $\alpha(t_0) \in C_{nK}(X)$. Thus, $\alpha(I) \not\subset C_n(X) - C_{nK}(X)$. By Remark 1.2, we conclude that $C_K^n(X) - \{C_{nK}\}$ is not arcwise connected. In the same manner we can see that (1) $\Rightarrow$ (2) and (4).

Now, we will prove that (3) $\Rightarrow$ (1). Assume $X$ is decomposable. Let $X_1$ and $X_2$ be proper subcontinua of $X$ such that $X = X_1 \cup X_2$. Let $r \in X_1 - X_2$ and $q \in X_2 - X_1$. By Remark 1.2, it is enough to prove that $C_n(X) - C_{nK}(X)$ is arcwise connected for $K = \{r, q\}$. Consider $p \in X_1 \cap X_2$, note that $\{p\} \in C_n(X) - C_{nK}(X)$. Now, let $A \in C_n(X) - C_{nK}(X)$. Suppose that $A = A_1 \cup \cdots \cup A_m$, where $A_i \in C(X)$ for each $i = 1, \ldots, m$, without loss of generality we may assume that there exist $1 \leq s \leq t \leq m$ such that:

(a) $A_l \subset X_1$ where $1 \leq l \leq s$;
(b) $A_l \subset X_2$ where $s + 1 \leq l \leq t$; and
(c) $A_l \cap (X_1 \cap X_2) \neq \emptyset$ and $A_l \not\subset X_i$ for $i = 1, 2$, where $t + 1 \leq l \leq m$.

For (a), we will construct an arc from $A$ to $\{p\} \cup \left(\bigcup_{i=s+1}^m A_i\right)$. Since $C_s(X_1)$ is arcwise connected and $A_1 \cup \cdots \cup A_s, \{p\} \in C_s(X_1)$, there exists an arc $\alpha\colon I \to C_s(X_1)$ such that $\alpha(0) = A_1 \cup \cdots \cup A_s$ and $\alpha(1) = \{p\}$. Now, define the function $\gamma_1\colon I \to C_n(X)$ by $\gamma_1(t) = \alpha(t) \cup \left(\bigcup_{i=s+1}^m A_i\right)$. Then $\gamma_1$ is well defined and is a map. Since $q \notin X_1$, $K \not\subset \gamma_1(t)$ for every $t \in I$. Thus, $\mathcal{A}_1 = \gamma_1(I)$ is an arc such that $\mathcal{A}_1 \subset C_n(X) - C_{nK}(X)$ containing $A$ and $\{p\} \cup \left(\bigcup_{i=s+1}^m A_i\right)$.

For (b), we will construct an arc from $\{p\} \cup \left(\bigcup_{i=s+1}^m A_i\right)$ to $\{p\} \cup \left(\bigcup_{i=t+1}^m A_i\right)$. Since $C_{t-s}(X_2)$ is arcwise connected and $A_{s+1} \cup \cdots \cup A_t, \{p\} \in C_{t-s}(X_2)$, there exists an arc $\beta\colon I \to C_{t-s}(X_2)$ such that $\beta(0) = A_{s+1} \cup \cdots \cup A_t$ and $\beta(1) = \{p\}$. Define $\gamma_2\colon I \to C_n(X)$ by $\gamma_2(t) = \beta(t) \cup \{p\} \cup \left(\bigcup_{i=t+1}^m A_i\right)$. Then $\gamma_2$ is well defined and is a map. Moreover, $K \not\subset \gamma_2(t)$ for every $t \in I$. Thus, $\mathcal{A}_2 = \gamma_2(I)$

is an arc such that $\mathcal{A}_2 \subset C_n(X) - C_{nK}(X)$ containing $\{p\} \cup \left(\bigcup_{i=s+1}^m A_i\right)$ and $\{p\} \cup \left(\bigcup_{i=t+1}^m A_i\right)$.

Note that $\{p\} \cup \bigcup_{i=s+1}^m A_i \in \mathcal{A}_1 \cap \mathcal{A}_2$, then $\mathcal{A}_1 \cup \mathcal{A}_2 \subset C_n(X) - C_{nK}(X)$ is arcwise connected.

For (c), we will construct an arc from $\{p\}$ to $\{p\} \cup \left(\bigcup_{i=t+1}^m A_i\right)$. For each $j \in \{t+1, \dots, m\}$ we choose $a_j \in A_j \cap X_1$. There exists $\eta \colon I \to C_n(X)$ an order arc such that $\eta(0) = \{a_{t+1}, \dots, a_m\}$ and $\eta(1) = A_{t+1} \cup \cdots \cup A_m$. Note that $K \not\subset \eta(t)$ for every $t \in I$. Since $C_{m-t}(X_1)$ is arcwise connected and $\{a_{t+1}, \dots, a_m\}, \{p\} \in C_{m-t}(X_1)$, there exists an arc $\eta_0 \colon I \to C_{m-t}(X_1)$ such that $\eta_0(1) = \{a_{t+1}, \dots, a_m\}$ and $\eta_0(0) = \{p\}$. Now, define $\gamma_3 \colon I \to C_n(X)$ by

$$\gamma_3(t) = \begin{cases} \eta_0(2t) \cup \{p\} & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \eta(2t-1) \cup \{p\} & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then $\gamma_3$ is well defined and is a map. Moreover, $K \not\subset \gamma_3(t)$ for every $t \in I$. Thus $\mathcal{A}_3 = \gamma_3(I)$ is an arc such that $\mathcal{A}_3 \subset C_n(X) - C_{nK}(X)$ containing $\{p\}$ and $\{p\} \cup \left(\bigcup_{i=t+1}^m A_i\right)$.

Note that $\{p\} \cup \left(\bigcup_{i=t+1}^m A_i\right) \in \mathcal{A}_3 \cap (\mathcal{A}_1 \cup \mathcal{A}_2)$, then $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ is arcwise connected and $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \subset C_n(X) - C_{nK}(X)$.

Now, if $K \cap A = \emptyset$, then $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ is a subcontinuum arcwise connected of $C_n(X) - C_{nK}(X)$ containing $\{p\}$ and $A$. But, if $A \cap K \neq \emptyset$, then exists $j \in \{1, \dots, m\}$ such that $A_j \cap K \neq \emptyset$. Since $K \not\subset A$, we may assume that $r \in A_j$. Thus, without loss of generality we may assume that $j = 1$, or $j = t+1$. If $j = 1$, we construct $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ as before. And, for $j = t+1$, we construct $\mathcal{A}_3$, $\mathcal{A}_1$, and $\mathcal{A}_2$ to avoid containing $K$. Similar arguments prove that (4) $\Rightarrow$ (1).

In order to prove (2) $\Rightarrow$ (1), we proceed as in the proof of (3) $\Rightarrow$ (1), now assuming that $A \in 2^X - 2^X_K$. Without lost of generality we can assume that $q \notin A$. Set $H_i = A \cap X_i$ for $i = 1, 2$. Since $H_i \in 2^{X_i}$, for each $i = 1, 2$ there is an arc $\alpha_i \colon I \to 2^{X_i}$ such that $\alpha_i(0) = H_i$ and $\alpha_i(1) = \{p\}$. We define $\gamma \colon I \to 2^X$ by

$$\gamma(t) = \begin{cases} \alpha_1(2t) \cup H_2 & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \alpha_2(2t-1) \cup \{p\} & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then $\gamma$ is well defined and is a map. Note that $K \not\subset \gamma(t)$ for every $t \in I$. Thus, $\gamma(I)$ is a continuum arcwise connected and $\gamma(I) \subset 2^X - 2^X_K$. This proves the theorem. $\square$

Let $Z$ be a topological space, the set $\{A \subset Z \colon A \text{ is a component of } Z\}$ is denote by $\mathcal{C}(Z)$.

**Lemma 5.9.** *Let $X$ be a continuum and $n \in \mathbb{N}$. If $U$ is an open set of $X$ and $C$ is a component of $U$, then $\langle C \rangle_n$ is a component of the open set $\langle U \rangle_n$.*

Proof: Let $\mathcal{C}$ be a component of $\langle U\rangle_n$ containing $\langle C\rangle_n$ and $A \in \mathcal{C}$. Since $\langle C\rangle_n \cap C(X) \neq \emptyset$, by [10, Lemma 1, page 1578], $\bigcup \mathcal{C}$ is a connected subset of $U$ which contains $C$. But, since $C$ is a component of $U$, $A \subset \bigcup \mathcal{C} \subset C$. Then, $A \in \langle C\rangle_n$. Therefore $\langle C\rangle_n = \mathcal{C}$. □

**Corollary 5.10.** *Let $X$ be a continuum, $p \in X$ and $n \in \mathbb{N}$. Suppose that $A \subset X$. Then, $A \in \mathcal{C}(X - \{p\})$ if and only if $\langle A\rangle_n \in \mathcal{C}(C_n(X) - C_{n\{p\}}(X))$.*

Proof: Let $A \in \mathcal{C}(X - \{p\})$. By Lemma 5.9, $\langle A\rangle_n$ is a component of $\langle X - \{p\}\rangle_n$. Since $\langle X - \{p\}\rangle_n = C_n(X) - C_{\{p\}}(X)$, $\langle A\rangle_n \in \mathcal{C}(C_n(X) - C_{n\{p\}}(X))$.

Now, let $A \subset X$ such that $\langle A\rangle_n \in \mathcal{C}(C_n(X) - C_{n\{p\}}(X))$. Then $p \notin A$. Let $D$ be a component of $X - \{p\}$ such $A \subset D$. Note that $\langle A\rangle_n \subset \langle D\rangle_n$, but $\langle A\rangle_n \in \mathcal{C}(C_n(X) - C_{n\{p\}}(X))$. Thus, $\langle A\rangle_n = \langle D\rangle_n$ and $A = D$. □

**Corollary 5.11.** *Let $X$ be a continuum and $p \in X$. Then $|\mathcal{C}(X - \{p\})| = |\mathcal{C}(C(X) - C_{\{p\}}(X))|$.*

**Proposition 5.12.** *Let $X$ be a continuum, suppose that $\mu\colon C(X) \to [0, \infty)$ is a Whitney map and let $K \in C(X) - F_1(X)$. If $t_0 < \mu(K)$ is such that $\mu^{-1}(t_0)$ is arcwise connected then $C(X) - C_K(X)$ is arcwise connected.*

Proof: Let $A \in C(X) - C_K(X)$ and $B \in \mu^{-1}(t_0)$. If $\mu(A) = t_0$, since $\mu^{-1}(t_0)$ is arcwise connected, there exists an arc in $\mu^{-1}(t_0)$ joining $A$ and $B$.

Suppose that $\mu(A) < t_0$. There exists $\alpha\colon I \to C(X)$ an order arc such that $\alpha(0) = A$ and $\alpha(1) = X$. Thus, $t_0 \in \mu(\alpha(I)) = [\mu(A), \mu(X)]$. By the intermediate value theorem, there exists $t \in (0, 1)$ such that $\alpha(t) \in \mu^{-1}(t_0)$. Since $\mu^{-1}(t_0)$ is arcwise connected, there exists an arc in $\mu^{-1}(t_0)$ joining $\alpha(t)$ and $B$. Then, there is $\gamma\colon I \to C(X)$ an arc from $A$ to $B$. Note that $\mu(\gamma(s)) \leq t_0$ for every $s \in I$ and $\mu(D) > t_0$ for every $D \in C_K(X)$. Thus, $\gamma(I) \subset C(X) - C_K(X)$.

Now, suppose $\mu(A) > t_0$. Let $a \in A$, there exists $\beta\colon I \to C(X)$ an order arc such that $\beta(0) = \{a\}$ and $\beta(1) = A$. Thus, $t_0 \in \mu(\beta(I)) = [\mu(\{a\}), \mu(A)]$. By the intermediate value theorem, there exists $s \in (0, 1)$ such that $\beta(s) \in \mu^{-1}(t_0)$. Note that $\beta([s, 1]) \cap C_K(X) = \emptyset$, otherwise, $K \subset \beta(s_0)$ and $K \subset A$, which is a contradiction. Since $\mu^{-1}(t_0)$ is arcwise connected, there exists an arc in $\mu^{-1}(t_0)$ joining $\beta(s)$ and $B$. Then, there is an arc in $C(X) - C_K(X)$ joining $A$ and $B$. Therefore $C(X) - C_K(X)$ is arcwise connected. □

A continuum $X$ is *continuum chainable* if for each positive number $\varepsilon$ and each pair of points $p \neq q$ in $X$, there is a finite sequence of subcontinua $\{A_1, \ldots, A_n\}$ of $X$ such that diameter $(A_i) < \varepsilon$, $p \in A_1$, $q \in A_n$ and $A_i \cap A_{i+1} \neq \emptyset$ for every $i < n$.

As consequence of Proposition 5.12 and [12, Theorem 33.4, page 248] we have the following corollary.

**Corollary 5.13.** *If a continuum $X$ is continuum chainable and $K \in C(X) - F_1(X)$, then $C(X) - C_K(X)$ is arcwise connected.*

## 6. Other topological properties

In this section we consider other topological properties of $C_K^n(X)$, some of them are consequences of the properties of $C_n(X)$.

### 6.1 Cells in the hyperspace $C_K^n(X)$.

**Theorem 6.1.** *Let $X$ be a nondegenerate continuum and $n \in \mathbb{N}$. Then $C_K^n(X)$ contains an $n$-cell for every $K \in 2^X$.*

PROOF: Let $K \in 2^X$ and $x \in K$. Let $A_1, \ldots, A_n$ be $n$ pairwise disjoint nondegenerate subcontinua of $X - \{x\}$. For each $j \in \{1, \ldots, n\}$, let $a_j \in A_j$, and let $\alpha_j \colon I \to C(A_j)$ be an order arc such that $\alpha_j(0) = \{a_j\}$ and $\alpha_j(1) = A_j$. Note that $K \not\subset \alpha_j(t)$ for every $j \in \{1, \ldots, n\}$ and each $t \in I$. Then the map $\beta \colon I^n \to C_K^n(X)$ given by $\beta((t_1, \ldots, t_n)) = \pi_K(\alpha_1(t_1) \cup \cdots \cup \alpha(t_n))$ is an embedding of $I^n$ in $C_K^n(X)$. $\square$

**Theorem 6.2.** *Let $X$ be a continuum, $n \in \mathbb{N}$ and let $K \in 2^X$. If $X - \{x\}$ contains $n$ pairwise disjoint decomposable subcontinua for some $x \in K$, then $C_K^n(X)$ contains a $2n$-cell.*

PROOF: Let $M_1, \ldots, M_n$ be $n$ pairwise disjoint decomposable subcontinua of $X - \{x\}$. Suppose that $M_j = A_j \cup B_j$, where $A_j$ and $B_j$ are subcontinua for each $j \in \{1, \ldots, n\}$. By the proof of (1.145) of [20], we may assume that each $A_j \cap B_j$ is connected, $A_j - (A_j \cap B_j) \neq \emptyset$, $B_j - (A_j \cap B_j) \neq \emptyset$, and $[A_j - (A_j \cap B_j)] \cap [B_j - (A_j \cap B_j)] = \emptyset$ for every $j \in \{1, \ldots, n\}$. For each $j \in \{1, \ldots, n\}$, let $\alpha_j \colon I \to C(A_j)$ and $\beta_j \colon I \to C(B_j)$ be order arcs such that $\alpha_j(0) = A_j \cap B_j$, $\alpha_j(1) = A_j$, $\beta_j(0) = A_j \cap B_j$ and $\beta_j(1) = B_j$. Therefore, the map $\gamma \colon I^{2n} \to C_K^n(X)$ given by $\gamma(t_1, \ldots, t_{2n}) = \pi_K\big(\bigcup_{j=1}^n (\alpha_j(t_{2j-1}) \cup \beta_j(t_{2n}))\big)$ is an embedding of $I^{2n}$ in $C_K^n(X)$. $\square$

**6.2 Contractibility.** A topological space $Z$ is *contractible* provided that the identity map of $Z$ is homotopic to a constant map of $Z$ into $Z$.

**Theorem 6.3.** *Let $X$ be a continuum and $n \in \mathbb{N}$. Then $C_{nK}(X)$ is contractible for each $K \in 2^X$.*

PROOF: Let $K \in 2^X$. There exists an order arc $\alpha \colon I \to 2^X$ such that $\alpha(0) = K$ and $\alpha(1) = X$. Let $H \colon C_{nK}(X) \times I \to C_{nK}(X)$ be the function defined by $H(A, t) = \alpha(t) \cup A$. We shall prove that $H(C_{nK}(X) \times I) \subset C_{nK}(X)$. Let $A \in C_{nK}(X)$ and $t \in I$. By [12, Theorem 15.3, page 120], each component of $K$

intersects $\alpha(t)$ for every $t \in I$. Since $K \subset A$, each component of $A$ intersects $\alpha(t)$. Thus, $\alpha(t) \cup A$ has at most $n$ components. Hence $H$ is a map. Note that $H(A,0) = \alpha(0) \cup A = A$ and $H(A,1) = \alpha(1) \cup A = X$ for each $A \in C_{nK}(X)$. Therefore $C_{nK}(X)$ is contractible. $\square$

**Theorem 6.4.** *Let $X$ be a continuum and $n \in \mathbb{N}$. Consider the following statements:*

(1) *$C_n(X)$ is contractible;*
(2) *$C^n_K(X)$ is contractible for each $K \in 2^X$;*
(3) *$C^n_K(X)$ is contractible for some $K \in 2^X$.*

*Then,* (1), (2) *are equivalents, and* (2) *implies* (3).

Proof: Of course, (2) implies (3) is inmediate. Let $K \in 2^X$. Suppose that $C_n(X)$ is contractible, there exists a map $H'\colon C_n(X) \times I \to C_n(X)$ such that $H'(A,0) = A$ and $H'(A,1) = X$ for each $A \in C_n(X)$. Let $H\colon C_n(X) \times I \to C_n(X)$ be the segment homotopy associated with $H'$ defined by

$$H(A,t) = \bigcup\{H'(A,s)\colon 0 \le s \le t\}.$$

Then $H$ is a map, see [20, Lemma 16.3, page 533]. Observe that $H(\{A\} \times I)$ is an order arc from $A$ to $X$ for every $A \in C_n(X)$.

**Claim.** We have $H(C_{nK}(X) \times I) = C_{nK}(X)$.

Since $C_{nK}(X) = H(C_{nK}(X) \times \{0\})$, $C_{nK}(X) \subset H(C_{nK}(X) \times I)$. Now, let $A \in C_{nK}(X)$. Since $H(A,0) = A$ and $H(A,0) \subset H(A,t)$ for every $t \in I$, $K \subset A \subset H(A,t)$. Thus, $H(A,t) \in C_{nK}(X)$. This completes the proof of the claim.

On the other hand, we define $G\colon C^n_K(X) \times I \to C^n_K(X)$ by

$$G(\mathcal{A},t) = \pi_K(H(\pi_K^{-1}(\mathcal{A}),t)),$$

which is a map such that for each $\mathcal{A} \in C^n_K(X)$, $G(\mathcal{A},0) = \mathcal{A}$ and $G(\mathcal{A},1) = C_{nK}$. Hence $C^n_K(X)$ is contractible. Thus, (1) implies (2).

Now, take $K = X$. By Remark 1.1, $C^n_K(X)$ is homeomorphic to $C_n(X)$. Since $C^n_K(X)$ is contractible, $C_n(X)$ is contractible. Thus, (2) implies (1). $\square$

Given a continuum $X$, denote by $C_\infty(X)$ the set $\bigcup_{n=1}^\infty C_n(X)$. By Theorem 6.4, [16, Theorem 3.7, page 241] and [16, Theorem 8.7, page 254], we have the following:

**Theorem 6.5.** *Let $X$ be a continuum and $n \in \mathbb{N}$ be given. Then the following are equivalent:*

(1) *$2^X$ is contractible;*

(2) *$C_n(X)$ is contractible;*
(3) *$C_\infty(X)$ is contractible;*
(4) *$C(X)$ is contractible;*
(5) *$C_K^n(X)$ is contractible for each $K \in 2^X$.*

A continuum $X$ is said to have *Kelley's property* provided that given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a, b \in X$, $d(a,b) < \delta$, and $A \in C(X)$ such that $a \in A$, then there exists $B \in C(X)$ such that $b \in B$ and $H_d(A,B) < \varepsilon$. We say that a continuum $X$ is *smooth at a point* $p \in X$ provided that for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $x \in X$, for each subcontinuum $M$ containing $p$ such that $x \in M$ and for each $y \in X$ satisfying $d(x,y) < \delta$ there is a subcontinuum $L$ containing $p$ such that $y \in L$ and $H_d(M,L) < \varepsilon$. A continuum $X$ is *smooth* if it is smooth at some point.

**Corollary 6.6.** *If $C_n(X)$ is a smooth continuum and $K \in 2^X$, then for each $n \in \mathbb{N}$, $C_K^n(X)$ is contractible.*

PROOF: Since $C_n(X)$ is smooth continuum, by [7, Corollary 4.3.1, page 253], $X$ has Kelley's property. By [16, Corollary 3.8, page 241], $C_n(X)$ is contractible. Thus, by Theorem 6.4, $C_K^n(X)$ is contractible. □

A nonempty closed proper subset (continuum) $L$ of a continuum $X$ is called;

- an $R^1$-*set (continuum)* if there exist an open set $U$ containing $L$ and two sequences $\{C_m^i\}_{m=1}^\infty$, $i = 1,2$, of components of $U$ such that $L = \limsup C_m^1 \cap \limsup C_m^2$;
- an $R^2$-*set (continuum)* if there exist an open set $U$ containing $L$ and two sequences $\{C_m^i\}_{m=1}^\infty$, $i = 1,2$, of components of $U$ such that $L = \lim C_m^1 \cap \lim C_m^2$;
- an $R^3$-*set (continuum)* if there exist an open set $U$ containing $L$ and a sequence $\{C_m\}_{m=1}^\infty$ of components of $U$ such that $L = \liminf C_m$.

**Theorem 6.7.** *Let $n \in \mathbb{N}$. If a continuum $X$ contains an $R^i$-continuum, $i \in \{1,2,3\}$, then $C_n(X)$ contains an $R^i$-set for $i \in \{1,2,3\}$, respectively.*

PROOF: Let $L$ be an $R^1$-continuum in $X$. Then there exist an open set $U$ containing $L$ and two sequences $\{C_m^i\}_{m=1}^\infty$, $i = 1,2$, of components of $U$ such that $L = \limsup C_m^1 \cap \limsup C_m^2$. By Lemma 5.9, $\langle C_m^i \rangle_n$ are components of $\langle U \rangle_n$. Let

$$\mathcal{L} = \limsup \langle C_m^1 \rangle_n \cap \limsup \langle C_m^2 \rangle_n.$$

Then, $\{\{x\} : x \in L\} \subset \mathcal{L}$ and $\mathcal{L}$ is closed. Let $A \in \mathcal{L}$ for each $i = 1,2$, let $\{A_{m_j}^i\}_{j=1}^\infty$ be sequences such that $\lim_{j\to\infty} A_{m_j}^i = A$, where $A_{m_j}^i \in \langle C_{m_j}^i \rangle_n$. Then, $A \subset L$ so that $A \in \langle U \rangle_n$. Thus, $\mathcal{L} \subset \langle U \rangle_n$ is an $R^1$-set in $C_n(X)$. The proof for $i = 2,3$ is similar. □

By [2, Corollary 3.3, page 317] and Theorem 6.7 we conclude the following result.

**Corollary 6.8.** *Let $n \in \mathbb{N}$. If a continuum $X$ contains an $R^i$-continuum for $i \in \{1,2,3\}$, then $C_n(X)$ is not contractible.*

As consequence of Theorem 6.4 and Corollary 6.8 we obtain

**Corollary 6.9.** *Let $n \in \mathbb{N}$. If a continuum $X$ contains an $R^i$-continuum for $i \in \{1,2,3\}$ and $K \in 2^X$, then $C_K^n(X)$ is not contractible.*

The following results give another proof of Corollary 6.9.

**Lemma 6.10.** *Let $X$ be a continuum, $n \in \mathbb{N}$ and $K \in 2^X$. Fix $\varepsilon > 0$, $\pi_K(N_H(C_{nK}(X),\varepsilon) \cap C_n(X))$ is an open subset of $C_K^n(X)$ containing $C_{nK}$.*

PROOF: Since $N_H(C_{nK}(X),\varepsilon) \cap C_n(X)$ is an open subset of $C_n(X)$ and $\pi_K$ is a closed map, then

$$\begin{aligned}\pi_K(C_n(X) - (N_H(C_{nK}(X),\varepsilon) \cap C_n(X)))&\\ &= \pi_K(C_n(X)) - \pi_K(N_H(C_{nK}(X),\varepsilon) \cap C_n(X))\\ &= C_K^n(X) - \pi_K(N_H(C_{nK}(X),\varepsilon) \cap C_n(X))\end{aligned}$$

is a closed subset of $C_K^n(X)$. Thus, $\pi_K(N_H(C_{nK}(X),\varepsilon) \cap C_n(X))$ is an open subset of $C_K^n(X)$ containing $C_{nK}$. $\square$

**Lemma 6.11.** *Let $n \in \mathbb{N}$ and $K \in 2^X$. If $\mathcal{U}$ is an open subset of $C_n(X)$ such that $C_{nK}(X) \subset \mathcal{U}$, then $\pi_K(\mathcal{U})$ is an open subset of $C_K^n(X)$ such that $C_{nK} \in \pi_K(\mathcal{U})$.*

PROOF: Given $\mathcal{A} \in \pi_K(\mathcal{U})$, there exists $A \in \mathcal{U}$ such that $\pi_K(A) = \mathcal{A}$. Suppose that $A \notin C_{nK}(X)$. Then, there is $\mathcal{W}$ an open subset of $C_n(X)$ such that $A \in \mathcal{W} \subset \mathcal{U} - C_{nK}(X)$. By Remark 1.2, $\pi_K(\mathcal{W})$ is an open subset of $C_K^n(X)$ containing $\mathcal{A}$ such that $\pi_K(\mathcal{W}) \subset \pi_K(\mathcal{U})$.

Now, suppose that $A \in C_{nK}(X)$. Since $C_{nK}(X) \subset \mathcal{U}$, there is $\varepsilon > 0$ such that $N_H(C_{nK}(X),\varepsilon) \cap C_n(X) \subset \mathcal{U}$. Then $A \in N_H(C_{nK}(X),\varepsilon) \cap C_n(X)$. By Lemma 6.10, $\pi_K(N_H(C_{nK}(X),\varepsilon) \cap C_n(X)) \subset \pi_K(\mathcal{U})$ is an open subset of $C_K^n(X)$. Therefore, $\pi_K(\mathcal{U})$ is an open subset of $C_K^n(X)$ such that $C_{nK} \in \pi_K(\mathcal{U})$. $\square$

**Theorem 6.12.** *Let $n \in \mathbb{N}$ and $K \in 2^X$. If a continuum $X$ contains an $R^i$-continuum, $i \in \{1,2,3\}$, then $C_K^n(X)$ contains an $R^i$-set for $i \in \{1,2,3\}$, respectively.*

PROOF: Let $L$ be an $R^1$-continuum in $X$. Then, there exist an open subset $U$ of $X$ with $L \subset U$ and two sequences $\{C_m^i\}_{m=1}^{\infty}$, $i = 1,2$, of components of $U$ such that $L = \limsup C_m^1 \cap \limsup C_m^2$. We consider two cases:

*Case I.* Assume that $K \not\subset L$. Let $k_1 \in K - L$ and $W = U \cap (X - \{k_1\})$. Note that $L \subset W$. Thus, there are subsequences $\{C^1_{m_j}\}_{j=1}^{\infty}$ and $\{C^2_{m_j}\}_{j=1}^{\infty}$ of $\{C^1_m\}_{m=1}^{\infty}$ and $\{C^2_m\}_{m=1}^{\infty}$ in $W$, respectively, such that $L = \limsup C^1_{m_j} \cap \limsup C^2_{m_j}$. Since $\langle W\rangle_n \cap C_{nK}(X) = \emptyset$, by proof of Theorem 6.7, $\mathcal{L} = \limsup\langle C^1_{m_j}\rangle_n \cap \limsup\langle C^2_{m_j}\rangle_n$ is an $R^1$-set in $\langle W\rangle_n$. From Remark 1.2,

$$\pi_K(\mathcal{L}) = \limsup \pi_K(\langle C^1_{m_j}\rangle_n) \cap \limsup \pi_K(\langle C^2_{m_j}\rangle_n)$$

is an $R^1$-set in $\pi_K(\langle W\rangle_n)$.

*Case II.* Suppose that $K \subset L$. Let $\varepsilon > 0$ such that $N_d(L, \varepsilon) \subset U$. Note that $B^H_\varepsilon(L) \subset N_H(C_{nK}(X), \varepsilon)$ and $B^H_\varepsilon(L) \cap C_n(X) \subset \langle U\rangle_n$. Set $\mathcal{W} = \langle U\rangle_n \cup (N_H(C_{nK}(X), \varepsilon) \cap C_n(X))$, which is an open subset of $C_n(X)$ such that $C_{nK}(X) \subset \mathcal{W}$. By Lemma 6.11, $\pi_K(\mathcal{W})$ is an open subset of $C^n_K(X)$ containing $C_{nK}$. Let $\mathcal{C}^i_m$ be the component of $\pi_K(W)$ containing $\pi_K(\langle C^i_m\rangle_n)$ for $i = 1, 2$. Then, for each $i = 1, 2$, $\{\mathcal{C}^i_m\}_{m=1}^{\infty}$ is a sequence in $\pi_K(\mathcal{W})$. Thus, $\mathcal{M} = \limsup \mathcal{C}^1_m \cap \limsup \mathcal{C}^2_m$ is an $R^1$-set in $\pi_K(\mathcal{W})$. The proof for $i = 2, 3$ is similar. □

**6.3 Homogeneity.** Using the induced map we have the following result.

**Proposition 6.13.** *If $X$ is a homogeneous continuum then $C^n_{\{p\}}(X)$ is homeomorphic to $C^n_{\{q\}}(X)$ for every $p, q \in X$.*

**Question 6.14.** If $X$ is a homogeneous continuum then is $C^1_A(X)$ homeomorphic to $C^1_B(X)$ for every $A, B \in C(X)$ (or $2^X$)?

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