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A note on the height of the third Stiefel–Whitney class of the canonical vector bundles over some Grassmann manifolds

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Abstract. We compute the height of the third Stiefel–Whitney characteristic class of the canonical bundles over some infinite classes of Grassmann manifolds of five dimensional vector subspaces of real vector spaces.

Keywords: Grassmann manifold; height of Stiefel–Whitney characteristic class

Classification: 57N65, 57R20

1. Introduction and results

Let X be a topological space. The height of a cohomology class $x \in \widetilde{H}^*(X;R)$ is defined as

$$\mathrm{ht}(x) = \sup\{n \in \mathbb{Z} \colon x^n \neq 0\}.$$

The knowledge of the height of the first and second Stiefel–Whitney classes of the canonical vector bundle over a real Grassmann manifold $G_k(\mathbb{R}^{n+k})$ was used to compute the cup-length of the Grassmann manifold (see R. E. Stong [4] and S. Dutta and S. S. Khare [2]), the cup-length (for a topological space X) being defined as

$$\mathrm{cup}_{\mathbb{Z}_2}(X) = \sup\{r \colon \exists\, x_1, \dots, x_r \in \widetilde{H}^*(X;\mathbb{Z}_2) \ni x_1 \cup \dots \cup x_r \neq 0\}.$$

This work expands the known results about the height of the third Stiefel–Whitney class of the canonical bundle over the Grassmann manifold $G_4(\mathbb{R}^{n+4})$, denoted w_3, which was computed in [1, Table 1].

In Section 3 we prove

Theorem 1.1. *Let $s \geq 4$. The height of w_3 in $H^*(G_5(\mathbb{R}^{n+5});\mathbb{Z}_2)$ is*

$$\mathrm{ht}(w_3) = \begin{cases} 2^s, & \text{if } n+5 = 2^s+3, \\ 2^s+3, & \text{if } n+5 = 2^s+4, \\ 2^s+6, & \text{if } n+5 = 2^s+5, \\ 2^s+7, & \text{if } n+5 = 2^s+t,\ t = 6,\ 7,\ 8. \end{cases}$$

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Although this is only a partial result, in the near future we plan to expand this result and find values of the height of the third Stiefel–Whitney class of the canonical bundle of $G_5(\mathbb{R}^{n+5})$ for all n. Using the height, we expect to find the cup-length of further classes of Grassmann manifolds.

The obtained results are computed using the method developed by R. E. Stong in [4] which we describe in the following section.

2. Method of computation

In his work [4], R. E. Stong introduced a method for convenient computation in the cohomology ring of the Grassmann manifold $G_k(\mathbb{R}^{n+k})$, the space of all k-dimensional subspaces of $\mathbb{R}^{n+k}$

$$H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_k, \overline{w}_1, \dots, \overline{w}_n]/\{w \cdot \overline{w} = 1\},$$

where w_i and $\overline{w}_i$ are Stiefel–Whitney classes of the canonical k-plane bundle γ_k over the Grassmann manifold and the dual Stiefel–Whitney classes, respectively. Similarly, w and $\overline{w}$ are total Stiefel–Whitney classes of the canonical k-plane bundle over the Grassmann manifold and its dual, respectively.

The total real flag manifold $\mathrm{Flag}(\mathbb{R}^m)$ is a space consisting of ordered m-tuples $(V_1, \dots, V_m)$ of mutually orthogonal one-dimensional vector subspaces $V_i \subset \mathbb{R}^m$. There are m canonical line bundles $l_1, \dots, l_m$ over $\mathrm{Flag}(\mathbb{R}^m)$. The total space of l_i consists of ordered pairs $((V_1, \dots, V_m), v_i)$, where $(V_1, \dots, V_m) \in \mathrm{Flag}(\mathbb{R}^m)$ and $v_i \in V_i$. The cohomology of total flag manifold is

$$H^*(\mathrm{Flag}(\mathbb{R}^m); \mathbb{Z}_2) = \mathbb{Z}_2[e_1, \dots, e_m] \Big/ \left\{ \prod_{i=1}^{m} (1 + e_i) = 1 \right\},$$

where $e_i = w_1(l_i)$ is the first Stiefel–Whitney class of the canonical line bundle l_i.

The map $\pi\colon \mathrm{Flag}(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k})$ given by $(V_1, \dots, V_{n+k}) \mapsto V_1 \oplus V_2 \oplus \dots \oplus V_k$ induces injective homomorphism on cohomology,

$$\pi^*\colon\ H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \to H^*(\mathrm{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2).$$

Moreover

$$\pi^*(w) = \prod_{i=1}^{k} (1 + e_i), \qquad \pi^*(\overline{w}) = \prod_{i=k+1}^{n+k} (1 + e_i).$$

Details can be found in [4].

The actual computation in the cohomology ring of Grassmann manifold is based on the following observations.

Proposition 2.1 ([4, Observation, page 106]). *The value of the class* $u \in H^*(G_k(\mathbb{R}^{n+k});\mathbb{Z}_2)$ *on the fundamental class of* $G_k(\mathbb{R}^{n+k})$ *is the same as the value of*

$$\pi^*(u)e_1^{k-1}e_2^{k-2}\cdots e_{k-1}e_{k+1}^{n-1}e_{k+2}^{n-2}\cdots e_{n+k-1}$$

on the fundamental class of $\mathrm{Flag}(\mathbb{R}^{n+k})$.

Proposition 2.2 ([4, Corollary, page 106]). *The nonzero monomials in* $H^{\mathrm{top}}(\mathrm{Flag}(\mathbb{R}^m);\mathbb{Z}_2)$ *are precisely those of the form*

$$e_{\sigma(1)}^{m-1}e_{\sigma(2)}^{m-2}\cdots e_{\sigma(i)}^{m-i}\cdots e_{\sigma(m)}^{0},$$

where σ *is any permutation of the elements of the set* $\{1,2,\ldots,m\}$.

3. Proof of Theorem 1.1

The image of w_3 by π^* in $H^*(\mathrm{Flag}(\mathbb{R}^{n+5});\mathbb{Z}_2)$ is the sum

$$\pi^*(w_3) = \sum_{1\le p<q<r\le 5} e_p e_q e_r.$$

Case 1: Let $n+5 = 2^s+3$, $n \ge 4$ and consider the product

$$\pi^*(w_3^{2^s})e_1^4e_2^3e_3^2e_4\cdot M = \left(\sum_{1\le p<q<r\le 5} e_p^{2^s}e_q^{2^s}e_r^{2^s}\right)e_1^4e_2^3e_3^2e_4\cdot M,$$

where $M = e_6^{2^s-3}\cdots e_{2^s+2}$. It is easy to see that all but one of the monomials in the product contain exponents greater or equal to 2^s+3 and such monomials will necessarily contribute as zero by Proposition 2.2. The only possibly nonzero monomial in the product is $e_1^4e_2^3e_3^{2^s+2}e_4^{2^s+1}e_5^{2^s}\cdot M$. To get this monomial into the form from Proposition 2.2, we need to multiply it with $e_1^{2^s-5}e_2^{2^s-5}$ or $e_1^{2^s-6}e_2^{2^s-4}$. It can be readily verified that the image $\pi^*(w_1^2w_2^{2^s-6})$ contains only the latter of the two and we have

$$\pi^*(w_3^{2^s}w_1^2w_2^{2^s-6})e_1^4e_2^3e_3^2e_4\cdot M \neq 0.$$

This proves that the height of w_3 is at least 2^s and we immediately have

$$\pi^*(w_3^{2^s+1}x)e_1^4e_2^3e_3^2e_4\cdot M = 0$$

for any class x such that $w_3^{2^s+1}x$ is in the top dimension. The Proposition 2.1 then implies that $w_3^{2^s+1}x = 0$ for any class x such that $w_3^{2^s+1}x$ is in the top dimension and by duality [3, Theorem 11.10] $w_3^{2^s+1} = 0$.

Case 2: Let $n+5 = 2^s+4$, $n \ge 4$ and $M = e_6^{2^s-2}\cdots e_{2^s+3}$. Computing

$$\pi^*(w_3^{2^s+3})e_1^4e_2^3e_3^2e_4 = \pi^*(w_3^{2^s})\pi^*(w_3^2)\pi^*(w_3)e_1^4e_2^3e_3^2e_4,$$

we are left with only the following three classes with a chance to be nonzero:

$$e_1^7e_2^6e_3^{2^s+3}e_4^{2^s+1}e_5^{2^s+2}, \qquad e_1^4e_2^3e_3^{2^s+2}e_4^{2^s+1}e_5^{2^s+3}, \qquad e_1^4e_2^3e_3^{2^s+2}e_4^{2^s+3}e_5^{2^s+1}.$$

Consider now the class $w_1^2w_2^{2^s-8}$. The image $\pi^*(w_1^2w_2^{2^s-8})$ contains the monomial $e_1^{2^s-8}e_2^{2^s-6}$ and this is the only monomial in $\pi^*(w_1^2w_2^{2^s-8})$ such that

$$\pi^*(w_3^{2^s+3}w_1^2w_2^{2^s-8})e_1^4e_2^3e_3^2e_4 \cdot M$$

is nonzero in the top dimension.

There is at least one exponent greater or equal to 2^s+4 in each monomial of the product

$$\begin{aligned}\pi^*(w_3^{2^s+4})e_1^4e_2^3e_3^2e_4 &= \pi^*(w_3^{2^s})\pi^*(w_3^4)e_1^4e_2^3e_3^2e_4\\ &= \Bigg(\sum_{1\le p<q<r\le 5} e_p^{2^s}e_q^{2^s}e_r^{2^s}\Bigg)\Bigg(\sum_{1\le p<q<r\le 5} e_p^4e_q^4e_r^4\Bigg)\pi^*(w_3^{2^s}).\end{aligned}$$

It follows that $\pi^*(w_3^{2^s+4}x)e_1^4e_2^3e_3^2e_4 \cdot M = 0$ for any x such that $w_3^{2^s+4}x$ is in the top dimension. Using Proposition 2.1 and duality [3, Theorem 11.10] as in the first case gives $w_3^{2^s+4} = 0$.

Case 3: Let $n+5 = 2^s+5$, $n \ge 4$. Multiplying

$$\pi^*(w_3^{2^s+6}) = \pi^*(w_3^{2^s})\pi^*(w_3^4)\pi^*(w_3^2),$$

we find that the only surviving monomial is

$$e_1^{10}e_2^9e_3^{2^s+2}e_4^{2^s+3}e_5^{2^s+4}.$$

In a way similar to previous cases, we find that for the class $w_1^2w_2^{2^s-10}$ the value of $\pi^*(w_3^{2^s+6}w_1^2w_2^{2^s-10})e_1^4e_2^3e_3^2e_4 \cdot M$, with $M = e_6^{2^s-1}\cdots e_{2^s+4}$, is nonzero and that $\pi^*(w_3^{2^s+7}x)e_1^4e_2^3e_3^2e_4 \cdot M = 0$ for any x with $w_3^{2^s+7}x$ in maximal dimension. Then $w_3^{2^s+7} = 0$ by Proposition 2.1 and duality [3, Theorem 11.10].

Case 4: An easy computation similar to the computations above shows that $w_3^{2^s+8} = 0$ in $G_5(\mathbb{R}^{n+5})$ for $n+5 = 2^s+8$. Let $n+5 = 2^s+6$. The possibly nonzero monomials in $\pi^*(w_3^{2^s+7})e_1^4e_2^3e_3^2e_4$ can be divided into classes according the exponents at $e_1,\dots,e_5$, see Table 1. For example, the monomial $e_1^{2^s+4}e_2^{10}e_3^9e_4^{2^s+5}e_5^{2^s+3}$ belongs to the monomial class B.

We say that a monomial p in $e_1,\dots,e_5$ is complementary to a monomial class X, if for some monomial $q \in X$ the monomial $q \cdot p$ contains all mutually different exponents from the set $\{2^s+1, 2^s+2, 2^s+3, 2^s+4, 2^s+5\}$. For the case of the monomial class B, the complementary monomials are $e_i^{2^s-8}e_j^{2^s-8}$ and $e_i^{2^s-7}e_j^{2^s-9}$.

We consider now the class $w_1^{2^s-4}w_2^{2^{s-1}-6}$. There are only even exponents in monomials of $\pi^*(w_1^{2^s-4}w_2^{2^{s-1}-6})$ so all the complementary monomials for all of the monomial classes of $\pi^*(w_3^{2^s+7})e_1^4e_2^3e_3^2e_4$ that are monomials of $\pi^*(w_1^{2^s-4}w_2^{2^{s-1}-6})$ are reduced to a set of monomials given in 'Significant complementary monomials' column in the Table 1.

Monomial class	**No.**	**Significant compl. monomials**
$A = \{8, 11, 2^s+3, 2^s+4, 2^s+5\}$	9	$e_i^{2^s-6}e_j^{2^s-10}$
$B = \{9, 10, 2^s+3, 2^s+4, 2^s+5\}$	10	$e_i^{2^s-8}e_j^{2^s-8}$
$C = \{9, 11, 2^s+2, 2^s+4, 2^s+5\}$	5	$e_i^{2^s-8}e_j^{2^s-8}$
$D = \{9, 11, 2^s+3, 2^s+3, 2^s+5\}$	4	
$E = \{9, 11, 2^s+3, 2^s+4, 2^s+4\}$	3	
$F = \{10, 11, 2^s+1, 2^s+4, 2^s+5\}$	3	$e_i^{2^s-8}e_j^{2^s-8}$, $e_i^{2^s-8}e_j^{2^s-10}e_k^2$
$G = \{10, 11, 2^s+2, 2^s+3, 2^s+5\}$	3	$e_i^{2^s-6}e_j^{2^s-10}$, $e_i^{2^s-8}e_j^{2^s-10}e_k^2$
$H = \{10, 10, 2^s+3, 2^s+3, 2^s+5\}$	2	
$I = \{10, 11, 2^s+2, 2^s+4, 2^s+4\}$	2	
$J = \{10, 10, 2^s+2, 2^s+4, 2^s+5\}$	3	
$K = \{10, 11, 2^s+3, 2^s+3, 2^s+4\}$	1	$e_i^{2^s-8}e_j^{2^s-10}e_k^2$
$L = \{10, 10, 2^s+3, 2^s+4, 2^s+4\}$	1	

Table 1. List of all monomial classes of $\pi^*(w_3^{2^s+7})e_1^4e_2^3e_3^2e_4$ with exponents from a given set with number of monomials in a class and significant complementary monomials from $\pi^*(w_1^{2^s-4}w_2^{2^{s-1}-6})$.

We now investigate the remaining significant complementary monomials. In the monomial class A we have $e_i^{2^s-6}e_j^{2^s-10} = e_i^{2^{s-1}}e_j^{2^{s-1}-4}e_i^{2^{s-1}-6}e_j^{2^{s-1}-6}$. The part $e_i^{2^{s-1}-6}e_j^{2^{s-1}-6}$ comes from $\pi^*(w_2^{2^{s-1}-6})$ with nonzero coefficient. The remaining part $e_i^{2^{s-1}}e_j^{2^{s-1}-4}$ comes from $\pi^*(w_1^{2^s-4})$ with coefficient

$$\binom{2^s-4}{2^{s-1}} = \binom{2^s-4}{2^{s-1}-4} \equiv 1 \mod 2.$$

Similar argument for $e_i^{2^s-8}e_j^{2^s-8}$ gives

$$e_i^{2^s-8}e_j^{2^s-8} = e_i^{2^{s-1}-2}e_j^{2^{s-1}-2}e_i^{2^{s-1}-6}e_j^{2^{s-1}-6}$$

with coefficient at $e_i^{2^{s-1}-2}e_j^{2^{s-1}-2}$ equal to $\binom{2^s-4}{2^{s-1}-2} = 0$ in $\pi^*(w_1^{2^s-4})$.

The last of the complementary monomial is $e_i^{2^s-8}e_j^{2^s-10}e_k^2$ and it can be written as

$$e_i^{2^{s-1}-2}e_j^{2^{s-1}-2}e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2$$

or

$$e_i^{2^{s-1}}e_j^{2^{s-1}-4}e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2.$$

Factors $e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2$ and $e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2$ come from $\pi^*(w_2^{2^{s-1}-6}) = \pi^*(w_2^{2^{s-1}-8})\pi^*(w_2^2)$ and the remaining factors in $\pi^*(w_1^{2^s-4})$ were addressed above.

From that we conclude that there are odd number of nonzero monomials in $\pi^*(w_3^{2^s+7}w_1^{2^s-4}w_2^{2^{s-1}-6})e_1^4e_2^3e_3^2e_4 \cdot M$ with $M = e_6^{2^s} \cdots e_{2^s+5}$, namely, 9 from the class A, 3 from the class F, 2 times 3 from the class G and 1 from the class K. Therefore $w_3^{2^s+7}$ is nonzero in $G_5(\mathbb{R}^{2^s+6})$ and we proved that the height of w_3 in $G_5(\mathbb{R}^{2^s+6})$ is 2^s+7. Due to the fact that the height is nondecreasing as a function of n, the height of w_3 in $G_5(\mathbb{R}^{n+5})$ for $n = 6, 7, 8$ is also $2^s + 7$.

References

[1] Balko Ľ., Lörinc J., *On the cup-length of certain classes of flag manifolds*, Acta Math. Hungar. **159** (2019), no. 2, 638–652.

[2] Dutta S., Khare S. S., *On second Stiefel–Whitney class of Grassmann manifolds and cuplength*, J. Indian Math. Soc. (N.S.) **69** (2002), no. 1–4, 237–251.

[3] Milnor J. W., Stasheff J. D., *Characteristic Classes*, Annals of Mathematics Studies, 76, Princeton University Press, Princeton, University of Tokyo Press, Tokyo, 1974.

[4] Stong R. E., *Cup products in Grassmannians*, Topology Appl. **13** (1982), no. 1, 103–113.

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