Edge-sum distinguishing labeling

Jan Bok, Nikola Jedličková

Abstract. We study edge-sum distinguishing labeling, a type of labeling recently introduced by Z. Tuza (2017) in context of labeling games.

An ESD labeling of an n-vertex graph G is an injective mapping of integers 1 to l to its vertices such that for every edge, the sum of the integers on its endpoints is unique. If l equals to n, we speak about a canonical ESD labeling.

We focus primarily on structural properties of this labeling and show for several classes of graphs if they have or do not have a canonical ESD labeling. As an application we show some implications of these results for games based on ESD labeling. We also observe that ESD labeling is closely connected to the well-known notion of magic and antimagic labelings, to the Sidon sequences and to harmonious labelings.

Keywords: graph theory; graph labeling; games on graphs

Classification: 05C78

1. Introduction and preliminaries

Graph labeling is a vivid area of combinatorics which started in the middle of 1960's. Much of the area is based on results of A. Rosa in [11] and of R. L. Graham and N. J. A. Sloane in [3]. Since then, over 200 different labelings were introduced. We refer to J. A. Gallian's survey [2], citing over 2500 papers, gathering most of the results in the area. Applications of labeling are both theoretical (A. Rosa introduced so-called *graceful* labelings to attack Ringel's conjecture on certain graph decompositions) and practical (for example the frequency assignment problem [5], [15], [6]).

We study *edge-sum distinguishing* (abbreviated as ESD) labeling, introduced by Z. Tuza in [14] in 2017. Tuza's primarily concern was to study several combinatorial games connected to this labeling. Our main objective is to study structural properties of this labeling on its own. However, as our secondary objective, we also give some results on game variants of edge-sum distinguishing labeling.

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Structure of the paper. In the rest of this section we review basic definitions and show a broader context of ESD labeling to other existing notions in combinatorics. The second section deals with structural properties of ESD labeling. For various well-known classes of graphs we show if they have a canonical ESD labeling or not. In the third section we are concerned with game variants, the original motivation of Z. Tuza. Finally, in the last section we summarize our results and propose some open problems.

1.1 Notation. We use the notation of D. B. West in [17]. All graphs in the paper are finite, undirected, connected and without multiple edges, unless we say otherwise.

1.2 Basic definitions. We need to formally define what graph labeling is. We will need vertex labelings only.

Definition 1.1. Let $G = (V, E)$ be a graph and let $L \subseteq \mathbb{N}$ be a set of labels. Then a mapping $\varphi\colon V \to L$ is called a *vertex labeling*. We further say that vertex labeling is *canonical* if $|V| = |L|$.

We will often refer to edge-weights, induced by a vertex labeling.

Definition 1.2. Let $G = (V, E)$ be a graph and φ a vertex labeling on G. The *edge-weight* of an edge xy is defined as $w_\varphi(xy) := \varphi(x) + \varphi(y)$.

Now we can finally introduce a definition of *edge-sum distinguishing labeling*.

Definition 1.3. Let $G = (V, E)$ be a graph and $L = \{1, \ldots, l\}$, $l \in \mathbb{N}$. A vertex labeling $\varphi : V \to L$ is called *edge-sum distinguishing labeling* (*ESD labeling*) if φ is injective and if

$$\forall\, e, f \in E\colon e \neq f \to w_\varphi(e) \neq w_\varphi(f).$$

We note that no ESD labeling exists in case $|L| < |V|$. We call a special case when $|L| = |V|$ a *canonical ESD labeling*.

Example 1.4. Consider a path P_n and denote its vertices consecutively $v_1, \ldots, v_n$. Choose a labeling $\varphi(v_i) = i$. Clearly, this is an ESD labeling and even a canonical ESD labeling.

1.3 Connections to existing notions.

Edge-antimagic vertex labeling. Following the usual terminology in the area of graph labelings, one could name canonical ESD labelings also as *edge-antimagic vertex labelings*. To illustrate this, let us recall that an *antimagic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. Antimagic

labelings were introduced as a natural generalization of magic labelings. We refer the reader to [1], [13] for more information on antimagic labelings and to [7], [8], [16] for a literature on magic labelings.

To our best knowledge, edge-antimagic vertex labelings were not studied yet.

Super edge-magic total labelings. A *super edge-magic total labeling* is an injection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that the weight of every edge xy defined as $w(xy) = f(x)+f(y)+f(xy)$ is equal to the same magic constant m and the vertex labels are the numbers $1, 2, \ldots, |V|$. One can observe that such labeling implies an edge-sum distinguishing labeling in a natural way. If we remove the labels of edges, the edge-weights now form an arithmetic progression. We can say about the resulting labeling even more; it is an $(a, 1)$-edge antimagic vertex labeling.

An (a, d)-*edge antimagic vertex labeling* is a one-to-one mapping f from $V(G)$ onto $\{1, 2, \ldots, |V|\}$ with the property that for every edge $xy \in E(G)$, the edge-weight set is equal to

$$\{f(x) + f(y) \colon x, y \in V(G)\} = \{a, a + d, a + 2d, \ldots, a + (|E| + 1)d\}$$

for some $a > 0$, $d \geq 0$. This definition comes from [10].

Sidon sequences. The Sidon sequences were introduced by S. Sidon in 1932 in [12]. We refer the reader to a dynamically updated survey of O'Bryant in [9]. The formulation of the following definition comes from the survey.

Definition 1.5. A *Sidon sequence* is a sequence of integers $a_1 < a_2 < \ldots$ with the property that sums $a_i + a_j$, $i \leq j$, are distinct.

ESD labeling can be reformulated in a similar fashion.

Definition 1.6. An *ESD labeling* of a graph $G = (V, E)$, where $V = \{1, \ldots, n\}$, is a sequence of integers $a_1 < a_2 < \ldots$ with the property that sums $a_i + a_j$, $i \leq j$, $(i, j) \in E$, are distinct and $a_1 = 1$.

With this new definition in hand, we see that ESD labeling is in some sense a generalization of the Sidon sequence. The difference that $a_1 = 1$ in the definition of ESD labeling could be easily dropped (but it is convenient for this paper). Also, one can observe that without this condition, the original Sidon sequences are ESD labelings of a sufficiently large complete graph with loops added to each vertex. However, again for our convenience, we consider only loopless graphs in this paper.

Harmonious labeling. *Harmonious labeling* was introduced by R. L. Graham and N. J. A. Sloane in [3]. We say that graph G with k edges is *harmonious* if its

vertices can be labeled injectively with integers modulo k so that the sum of the labels of its endpoints modulo k is unique.

The difference between harmonious and ESD labeling is that we do not take vertex labels and edge labels modulo number of edges in ESD case. In fact, ESD labelings and harmonious labelings behave differently. For example, it is conjectured that trees are harmonious and it is known that not all cycles are harmonious, see [4]. For comparison, we show that all trees and cycles have a canonical ESD labeling.

2. Structural results

2.1 Necessary condition.

Theorem 2.1. *If a graph $G = (V, E)$ such that $|V| > 1$ has a canonical ESD labeling, then the inequality $|E| \leq 2|V| - 3$ holds.*

Proof: We claim that every canonical ESD labeling of an n-vertex graph has at most $2n - 3$ different edge-weights.

To prove this, observe that the smallest possible edge-weight in such labeling is 3 and the largest possible is $2n - 1$. Also, the edge-weights of G form a subset of the set $\{3, \ldots, 2n - 1\}$ which is of the size $2n - 3$. This proves the claim.

Now if a graph G has more than $2|V| - 3$ edges we can use our claim and by the pigeonhole principle, we have two edges with the same weight, a contradiction. □

Now we will show that this bound is tight.

Theorem 2.2. *For every $n \in \mathbb{N}$, $n > 1$, there exists an n-vertex graph G_n with $|E(G_n)| = 2n - 3$ which has a canonical ESD labeling.*

Proof: For G_2 take K_2 and for G_3 take K_3. These cases are trivial.

For $n > 3$, take a complete bipartite graph $K_{2,n-2}$ and add an edge between the two vertices of the part of size 2. See Figure 1 for an example.

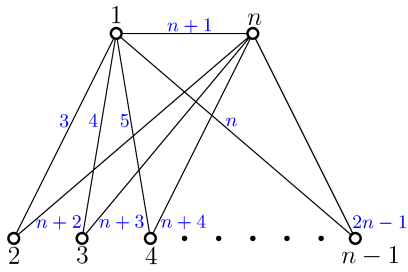

Figure 1. An example of an ESD graph with $2n - 3$ edges.

We will show that this graph has a canonical ESD labeling. We will denote x_1, x_2 the vertices of the part of size 2 and $y_1, \ldots, y_{n-2}$ the vertices of the other part.

Now we define a labeling φ in the following way.

- Let $\varphi(x_1) = 1$ and $\varphi(x_2) = n$.
- Let $\varphi(y_i) = i + 1$ for $1 \leq i \leq n - 2$.

Observe that the edges incident with x_1 have edge-weights from 3 to $n + 1$. Furthermore, the edges incident with x_2, except for the edge x_1x_2, have edge-weights ranging from $n + 2$ to $2n - 1$. All these weights appear exactly once and thus we are done. □

2.2 Fan graphs. In the previous part we showed a necessary condition for graph to have a canonical ESD labeling. The point of this part is to show that this condition is not sufficient in general by proving that *fan graphs*, which have $2n-3$ edges, do not have a canonical ESD labeling if their order is bigger than 8.

Definition 2.3. A *fan graph* F_n is a path P_{n-1} and one other vertex v (we call it the *central vertex*) joined by an edge with every vertex of the path. See Figure 2 for an example.

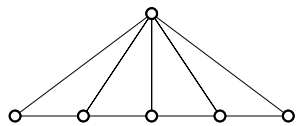

FIGURE 2. A fan graph F_6.

Theorem 2.4. *A fan graph F_n does not have a canonical ESD labeling if and only if $n \geq 8$.*

PROOF: Note that F_n for n up to 7 has a canonical ESD labeling, as we can see on Figure 4. It is obvious that F_2 and F_3 have canonical ESD labelings.

From Theorem 2.1 we know that we have at most $2n-3$ different edge-weights. Since a fan graph of order n has exactly $2n-3$ edges we need to use every possible edge-weight from the set $\{3, \ldots, 2n - 1\}$ exactly once.

The edge-weights 3 and 4 can be obtained in exactly one possible way. In the first case on an edge with endpoints labeled 1 and 2, in the second case on an edge with endpoints 1 and 3. The edge-weight 5 can be obtained in two ways. Either as the weight of an edge with endpoints 2 and 3 or as the weight on an edge with endpoints 1 and 4. We get two possible subgraphs S_1 and S_2.

By a similar analysis, one can get the labeled subgraphs S_3 and S_4.

Hence, exactly one of the labeled subgraphs S_1 or S_2 has to be in F_n and, analogously, one of the S_3 and S_4 as well. However, in all graphs S_i, $i \in \{1, \ldots, 4\}$, one of its vertices has to be the central vertex. Since $n \geq 8$, we see that the minimum possible label in S_3 and S_4 is 5. Also, the maximum label on S_1 and S_2 is 4. Therefore, we cannot properly label the central vertex and the theorem follows. □

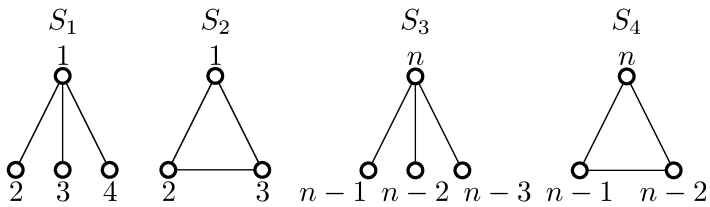

FIGURE 3. The subgraphs from the proof of Theorem 2.4.

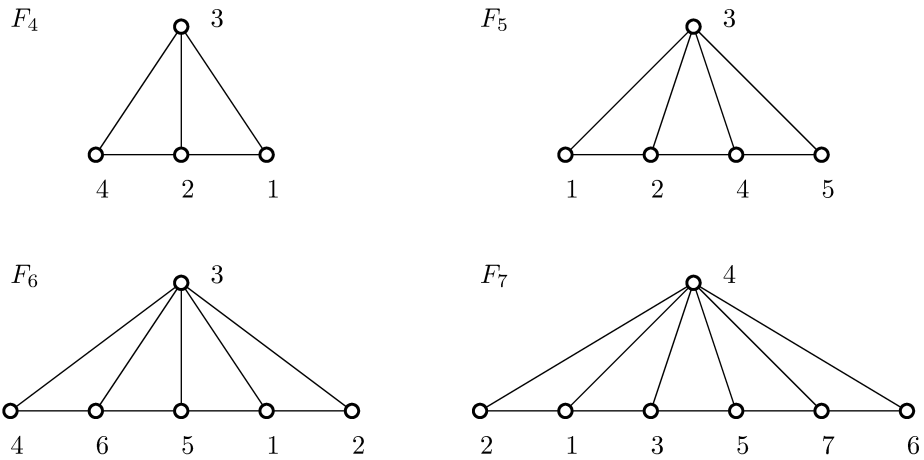

FIGURE 4. Canonical ESD labelings for F_4, F_5, F_6, and F_7.

2.3 Complete bipartite graphs. We need to introduce a notion of isomorphism for vertex labelings.

Definition 2.5. Vertex labelings φ_1 and φ_2 on G are isomorphic if there exists an automorphism f of G such that $\varphi_1(v) = \varphi_2(f(v))$ for every $v \in V(G)$.

We will prove the following theorem, covering all cases for complete bipartite graphs.

Theorem 2.6. *Let $K_{p,q}$ be a complete bipartite graph on $n = p + q$ vertices, $p \leq q$, then the following holds.*

(1) *For $p, q > 2$ there is no ESD labeling on $K_{p,q}$.*
(2) *If $p = 2$, then there exists exactly one possible ESD labeling up to isomorphism.*
(3) *If $p = 1$, then every canonical labeling is an ESD labeling.*

Proof: (1) Suppose for a contradiction that we have some canonical ESD labeling φ. Denote the parts of $K_{p,q}$ by P and Q. We will divide the proof into two cases.

- There exist two vertices v_1, v_2, in P and two vertices u_1, u_2 in Q such that $\varphi(v_2) = \varphi(v_1) + 1$ and $\varphi(u_2) = \varphi(u_1) + 1$. Then $w_\varphi(v_1u_2) = w_\varphi(v_2u_1)$, and we get a contradiction.
- There exists a part (without loss of generality P) such that $\varphi(v_1) \neq \varphi(v_2) + 1$ for every $v_1, v_2 \in P$. Since P is of size at least 3, there exist two vertices $v_1', v_2' \in P$ with labels smaller than n. Thus there exists a vertex $u_1 \in Q$ with label $\varphi(v_1') + 1$ and $u_2 \in Q$ with label $\varphi(v_2') + 1$. Then $w_\varphi(v_1'u_2) = w_\varphi(v_2'u_1)$, a contradiction.

(2) We denote the vertices of the part of the size 2 as v_1, v_2. The vertices of the other part will be $u_1, \ldots, u_q$. Let ψ be a vertex labeling of $K_{2,q}$ defined as follows:

- $\psi(v_1) = 1$;
- $\psi(v_2) = n$;
- $\psi(u_i) = i + 1$ for $i \in \{1, \ldots, q\}$.

Observe that ψ is indeed a canonical ESD labeling. For $q = 2$, one can easily check that this is the only canonical ESD labeling up to isomorphism.

Now, for a contradiction, assume that a canonical ESD labeling ψ', nonisomorphic to ψ, exists. Furthermore, $n > 4$, and we can assume that $\psi'(v_1) < \psi'(v_2)$. Either $\psi'(v_1) \neq 1$ or $\psi'(v_2) \neq n$. We distinguish two cases.

(a) It holds that $\psi'(v_2) = \psi'(v_1) + 1$. Since $n > 4$, we can find two vertices a_1, a_2 in the other part such that $\psi'(a_2) = \psi'(a_1) + 1$. Similarly as in case (1) of this theorem, $w_{\psi'}(v_1a_2) = w_{\psi'}(v_2a_1)$ and we get a contradiction.
(b) It holds that $\psi'(v_2) \neq \psi'(v_1) + 1$. Then there exist two distinct vertices $u_j, u_k \in \{u_1, \ldots, u_q\}$ such that one of the following holds. Either $\psi'(u_j) = \psi'(v_1) + 1$ and $\psi'(u_k) = \psi'(v_2) + 1$, or $\psi'(u_j) = \psi'(v_1) - 1$ and $\psi'(u_k) = \psi'(v_2) - 1$. In both cases $w_{\psi'}(v_1u_k) = w_{\psi'}(v_2u_j)$ and we are done.

We conclude that no such ψ' exists.

(3) Every edge in a canonical labeling of $K_{1,q}$ has a unique sum since every edge is incident to the central vertex of degree q. □

We note that the first part of Theorem 2.6 can be proved by using Theorem 2.1 but we think that our proof is more clear.

2.4 Trees. We already showed that paths and stars are ESD graphs. The following theorem solves the general case of trees.

Theorem 2.7. *Every tree has a canonical ESD labeling.*

Proof: Let T be an n-vertex tree with root in $v_1 \in V(T)$. We will denote by $v_1, \ldots, v_n$ an ordering of vertices visited in a breadth-first search on T, starting in v_1. We define a labeling φ as $\varphi(v_k) := k$ for all $v_k \in V(T)$. We want to show that φ is a canonical ESD labeling.

Consider some vertex v_i, $i > 1$, and its parent v_j. Denote by T' the tree induced by vertices $v_1, \ldots, v_{i-1}$. See Figure 5 for an illustration. We claim that the following holds:

$$w_\varphi(v_i v_j) > w_\varphi(v_a v_b), \qquad \forall\, v_a v_b \in E(T').$$

By the level of a vertex we mean its distance to root vertex v_1. Without loss of generality, assume that $a < b$. We distinguish these cases.

- The edge $v_a v_b$ has both endpoints on a level lower or equal to the level of v_j. Then $a < j$ and $b < i$ and from that $a + b < i + j$.
- If $v_a = v_j$, then v_j is the common parent of v_b and v_i. Thus $b < i$ and from that $b + j < i + j$.
- The vertex v_a is on the same level as v_j and $v_a \neq v_j$. Then $a < j$ and $b < i$, implying that $a + b < i + j$.

We proved the claim and the theorem follows. □

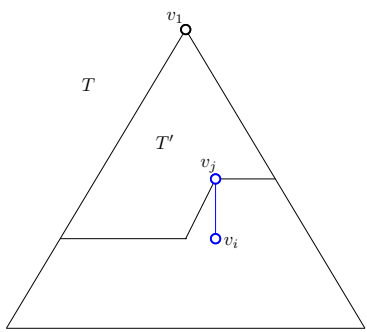

Figure 5. An illustration of situation in Theorem 2.7.

2.5 Cycles.

Theorem 2.8. *Every cycle graph C_n is an ESD graph.*

PROOF: Let us denote the vertices of C_n as $v_1, \ldots, v_n$ in a circular order. We distinguish two cases:

(1) If n is even, then we assign labels as follows:

- $\varphi(v_i) = i$ for all $i \in \{1, \ldots, n-2\}$;
- $\varphi(v_{n-1}) = n$;
- $\varphi(v_n) = n-1$.

Weights of the edges v_iv_{i+1} for $i \in \{1, \ldots, n-3\}$ are odd integers $3, 5, \ldots, 2n-5$. The weight of the edge $v_{n-1}v_n$ is $2n-1$ and therefore is odd as well. The remaining edges will be even; $w_\varphi(v_nv_1) = n$ and $w_\varphi(v_{n-2}v_{n-1}) = 2n-2$. We conclude that the edge-weights are unique.

(2) If n is odd we assign labels as follows:

- $\varphi(v_i) = i$ for all $i \in \{1, \ldots, n\}$.

The weights of the edges between v_iv_{i+1} for $i \in \{1, \ldots, n-1\}$ will be odd integers $3, 5, \ldots, 2n-1$. The weight of the edge v_1v_n is equal to $n+1$ and therefore it is even. Again, all edge-weights are unique and we get a canonical ESD labeling. □

2.6 Generalized sunlet graphs. We recall that a graph is *unicyclic* if it contains exactly one cycle.

Definition 2.9. A *generalized sunlet graph* S_k^p is a unicyclic graph obtained by taking a cycle graph C_k, with $V(C_k) = \{c_1, \ldots, c_k\}$, and joining path graphs R_i, $i \in \{1, \ldots, k\}$ of order p to this cycle so that one of the endpoints of R_i is identified with c_i.

Theorem 2.10. *Let S_k^p be a generalized sunlet graph. If k is odd and p is even, then S_k^p has a canonical ESD labeling.*

PROOF: We denote the vertices of S_k^p in the following way.

- Vertices on the cycle are $v_1, v_{p+1}, v_{2p+1}, \ldots, v_{(k-1)p+1}$.
- Vertices on the path joined to the vertex v_{ip+1} are consecutively $v_{ip+1}, \ldots, v_{(i+1)p}$ for $1 \le i \le k$.

We define a vertex labeling φ as $\varphi(v_i) := i$. We claim that φ is a canonical ESD labeling. All edge-weights on attached paths are odd, because we get them as a sum of two consecutive numbers. Furthermore, all edge-weights on a path joined to vertex v_{ip+1} are smaller than edge-weights on a path joined to vertex $v_{(i+1)p+1}$. Thus all edge-weights on paths are distinct. All edge-weights on the cycle expect for the edge $v_1v_{(k-1)p+1}$ are in the form $k'p+2$ for some $k' \in \mathbb{N}$. Thus they are all even and distinct.

It remains to show that the edge $v_1v_{(k-1)p+1}$ has an edge-weight different from all others. For a contradiction we assume that the edge-weight $(k-1)p+2$ was already used. It is even, so it can be only used on the cycle. Thus, $k-1$ must be a sum of two distinct consecutive natural numbers. That gives a contradiction, because $k-1$ is even. □

For the other parity conditions we were not able to prove that there is always an ESD labeling. Thus we leave as an open problem to determine if all generalized sunlet graphs have a canonical ESD labeling. Small examples suggest that it might be true.

Theorem 2.11. *Let S_k^p be a generalized sunlet graph. If k and p are odd or k is even and p is odd or even, then S_k^p has an ESD labeling with label set L of size $(p+1)k-2$.*

Proof: In both cases of parity of k, the unique cycle in S_k^p will be labeled in the same way as in Theorem 2.8. Observe that the greatest edge-weight on the edges of cycle is $2k-1$.

The rest of the vertices is labeled by the following procedure. Start with label $i := 2k-1$ and label by i an unlabeled vertex which is adjacent to the vertex with the minimum label. Increment i by one and repeat the step. We see that in every step we get one new edge-weight. Furthermore, this edge-weight is always greater than any previous edge-weight created during this procedure and all these edge-weights are greater than any edge-weight on an edge in the cycle. Thus, the resulting labeling is ESD. Furthermore, we labeled the cycle with k labels with $1,\dots,k$ and then the remaining $p(k-1)$ vertices with labels $2k-1,\dots,(p-1)k-2$. This implies that the set of labels L is of size $(p-1)k-2$. □

2.7 Grids.

Definition 2.12. A $k\times l$ grid graph $G_{k,l}$ is the Cartesian product of path graphs P_k and P_l.

Theorem 2.13. *Let $G_{k,l}$ be a grid graph. If k or l is even then $G_{k,l}$ has a canonical ESD labeling.*

Proof: Without loss of generality assume that k, the number of columns, is even. Let us denote the vertices in the ith row by $v_{(i-1)k+1},\dots,v_{ik}$ for every $i\in\{1,\dots,l\}$. We define a canonical vertex labeling φ as $\varphi(v_i) := i$. We want to show that φ is an ESD labeling on $G_{k,l}$.

The graph $G_{k,l}$ with labeling φ has the following edge-weights:

- $2(i-1)k+3,\dots,2ik-1$ in the ith row for every $1\le i\le l$;
- $2ik+3,\dots,2(i+1)k-1$ in the $(i+1)$th row for every $0\le i\le l-1$;

- $(2i-1)k+2, \ldots, (2i+1)k$ on edges between the ith and the $(i+1)$th row $1 \leq i \leq l-1$.

All edge-weights on rows are odd and all edge-weights in the ith row are smaller than all edge-weights in the $(i+1)$th row. A similar argument holds for all edge-weights in columns. This concludes the proof. □

2.8 Complete graphs. From Theorem 2.1 it is clear that complete graphs K_n for $n > 3$ do not have a canonical ESD labeling. However, the following theorem provides a simple way how to find an ESD labeling. We recall that *Fibonacci sequence* is defined as $F_0 := 0$, $F_1 := 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. We note that the following theorem implies that for any n-vertex graph, F_{n+1} labels suffice to construct an ESD labeling.

Theorem 2.14. *There exists an ESD labeling with F_{n+1} labels for every complete graph K_n.*

Proof: For given K_n we label its vertices $\{v_1, \ldots, v_n\}$ by function φ_n, defined as $\varphi_n(v_i) := F_{i+1}$.

We show that this is an ESD labeling by induction. We see that for K_1 and K_2, φ_1 and φ_2 are clearly ESD labelings. Now we want to prove that φ_n is an ESD labeling. We see that $v_1, \ldots, v_{n-1}$ are labeled as in φ_{n-1}. The largest possible sum on an edge in φ_{n-1} is $F_n + F_{n-1}$. The only new label in φ_n is F_{n+1} and the minimum possible sum on an edge incident with v_n is $F_{n+1} + F_2 = F_n + F_{n-1} + 1$. Thus, assuming that φ_{n-1} is an ESD labeling, φ_n is an ESD labeling as well. □

3. Games with ESD labeling

Z. Tuza in his paper [14] emphasized that only few papers on graph labeling games exist. He defined a new game from ESD labeling.

Definition 3.1. We call a vertex of graph *free* if it is not labeled yet.

Definition 3.2. Let $G = (V, E)$ be a graph and $L = \{1, \ldots, l\}$ its set of labels. Alice and Bob are two players who alternate after every move. Alice starts. In each move, player chooses a free vertex of G and assigns an unused label to it. The move is *legal* if the resulting edge-weights are unique.

The game ends if there is no legal move possible or an ESD labeling is created. Alice wins if an ESD labeling is created, otherwise Bob wins.

We say that an ESD labeling game is *canonical* on G if $|L| = |V(G)|$.

One can also define other variants of this game. For example, Bob can be the starting one. Also, our definition of game is a Maker–Breaker type of game, but it is possible to define Achievement and Avoidance type of this game as well.

Proposition 3.3. *If a graph G does not have a canonical ESD labeling then Bob has a winning strategy in the canonical game on G.*

Proof: If a graph G does not have a canonical ESD labeling then Alice cannot make any canonical ESD labeling and Bob eventually wins. □

Theorem 3.4. *Alice wins every canonical game on a star S_n.*

Proof: We already proved in Theorem 2.6 that every canonical vertex labeling on a star graph is edge-sum distinguishing. Thus Alice wins every game regardless on the course of the game. □

Theorem 3.5. *Bob wins every canonical game on a complete bipartite graph $K_{p,q}$, $p \leq q$, where $p = 2$.*

Proof: We recall Theorem 2.6. The graph $K_{p,q}$, $p \leq q$, where $p = 2$, needs to have labels 1 and $p + q$ on the smaller part. Thus a winning strategy for Bob is to assign a label w, such that $1 < w < p + q$, on a free vertex of the smaller part in his first move. Now it is not possible to build a canonical ESD labeling and Bob wins. □

Z. Tuza also asked [14, Problem 3.1] the following question: Given $G = (V, E)$, for which values of l can Alice win the edge-sum distinguishing labeling game? We partially answer this question by the following theorem.

Theorem 3.6. *Let G be a graph, Δ its maximum degree, and L its set of labels. If $|L| \geq (\Delta^2 + 1)n + \Delta\binom{n-1}{2}$, then Alice has a winning strategy.*

Proof: For each vertex v of G, define a set S_v as the set of labels available for v. In the beginning of every game, $S_v = L$ for every $v \in V(G)$.

Our goal is to build a winning strategy for Alice. In kth move, a player assigns to a free vertex v some label $\varphi(v) \in S_v$. We update the set of labels in the following three steps right after the player's choice.

(1) We delete $\varphi(v)$ from S_u for every $u \in V(G)$. This label cannot be used twice, since ESD labeling is an injective mapping.

(2) For every free vertex y incident to v we delete all labels $l_{y,e}$ such that $l_{y,e} + \varphi(v) = w_\varphi(e)$ for some edge e with both endpoint vertices labeled and incident with v. In this process, we delete at most $\binom{k-1}{2}$ labels from S_y.

(3) For every free vertex z and for every vertex $z' \in N(z)$, such that z' is already labeled, we delete from S_z all labels l' such that

$$l' + \varphi(z') = w_\varphi(vv'), \qquad \forall\, v' \in N(v).$$

Within these steps, we delete at most Δ^2 labels from label set of every free vertex.

If the label set for every free vertex is nonempty before every move, Alice wins. Let us count how many labels are deleted in course of the game for every free vertex.

- We delete at most $n-1$ labels through all first steps.
- We delete at most $\Delta\binom{n-1}{2}$ labels through all second steps.
- We delete at most $\Delta^2 n$ labels in third steps.

Summarized, we delete at most $(\Delta^2+1)n + \Delta\binom{n-1}{2} - 1$ labels. If we have one extra label available, we can always find a label for a free vertex and our bound is proved. Note an important fact that it does not matter how Bob plays and the resulting labeling is ESD. □

Observe that this theorem also gives us a bound on the size of label set for general graphs. This follows by taking Proposition 3.3 into account.

Also, by a similar analysis, one can obtain the following theorem for path graphs.

Theorem 3.7. *Let P_n be a path graph on n vertices. If $|L| \geq 5n$, then Alice wins every game on P_n.*

4. Concluding remarks

We studied a new type of graph labeling, introduced by Z. Tuza, which is similar to magic (and antimagic) labelings, harmonious labelings and has a relation to the Sidon sequences. We would like to highlight our main results.

- We proved that trees, cycles and complete bipartite graphs with one part of size 2 have a canonical ESD labeling.
- We proved that in some cases grid graphs and generalized sunlet graphs do have a canonical ESD labeling.
- We showed that fan graphs and complete bipartite graphs with both parts of size at least 3 do not have a canonical ESD labeling.
- We studied a Maker–Breaker type of game, applied our previous results and derived a general bound on number of labels such that Maker wins the game.

Open problems. Aside from Tuza's original game-oriented problems proposed in [14], we emphasize the following question, arising from the results in this paper.

Problem 4.1. What is the maximum possible number of edges for n-vertex connected graphs so that every graph with such number of edges has a canonical ESD labeling?

From Theorem 2.7 we see that to answer this question one needs to resolve the case of unicyclic graphs which is now only partially solved.

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