On the continuity of the elements of the Ellis semigroup and other properties

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Abstract. We consider discrete dynamical systems whose phase spaces are compact metrizable countable spaces. In the first part of the article, we study some properties that guarantee the continuity of all functions of the corresponding Ellis semigroup. For instance, if every accumulation point of X is fixed, we give a necessary and sufficient condition on a point $a \in X'$ in order that all functions of the Ellis semigroup $E(X, f)$ be continuous at the given point a. In the second part, we consider transitive dynamical systems. We show that if (X, f) is a transitive dynamical system and either every function of $E(X, f)$ is continuous or $|\omega_f(x)| = 1$ for each accumulation point x of X, then $E(X, f)$ is homeomorphic to X. Several examples are given to illustrate our results.

Keywords: discrete dynamical system; Ellis semigroup; p-iterate; p-limit point; ultrafilter; compact metric countable space

Classification: 54G20, 54D80

1. Introduction and preliminaries

A dynamical system (X, f) will consist of a compact metric infinite space X and a continuous function $f\colon X \to X$ (usually, this kind of dynamical systems are called discrete dynamical systems and we short the name for convenience). The *orbit* of a point $x \in X$ is the set $\mathcal{O}_f(x) := \{x, f(x), f^2(x), f^3(x), \dots\}$. The dynamical system (X, f) is *transitive* if there is a point with dense orbit. The dynamical system (X, f) is *weakly almost periodic* (WAP) if $E(X, f)$ is relatively compact in the weak topology on $C(X)$. A point $x \in X$ is called *periodic* if there is $n \geq 1$ such that $f^n(x) = x$, and its *period* is $\min\{n \in \mathbb{N}\colon f^n(x) = x\}$. The symbol P_f stands for the set of all periods of the periodic points of a dynamical systems (X, f). A point x is called *eventually periodic* if its orbit is finite. The *ω-limit set* of $x \in X$, denoted by $\omega_f(x)$, is the set of points $y \in X$ for which there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $f^{n_k}(x) \longrightarrow y$. Observe that

DOI 10.14712/1213-7243.2021.014
Research of the first-named author was supported by the PAPIIT grant no. IN-105318. The third author was supported by the VIE-UIS grant #2422.

for each $y \in \mathcal{O}_f(x)$, we have that $\omega_f(y) = \omega_f(x)$. If $\mathcal{O}_f(y)$ contains a periodic point x, then $\omega_f(y) = \mathcal{O}_f(x)$. For a space X we denote by $\mathcal{N}(x)$ the set of all neighborhoods of $x \in X$, and the set of all accumulation points of X will be simply denoted by X'. The Stone–Čech compactification $\beta(\mathbb{N})$ of $\mathbb{N}$ with the discrete topology will be identified with the set of ultrafilters over $\mathbb{N}$. Its remainder denoted by $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$, is the set of all free ultrafilters on $\mathbb{N}$, where, as usual, each natural number n is identified with the fixed ultrafilter consisting of all subsets of $\mathbb{N}$ containing n.

Since our phase spaces are compact metric countable spaces, we remind the reader the classical result from [13] which asserts that every compact metric countable space is homeomorphic to a countable successor ordinal with the order topology. In this context, some of the most attractive phase spaces have the form $\omega^\alpha+1$, where $\alpha \geq 1$ is a countable ordinal.

The *Ellis semigroup* of a dynamical system (X, f), denoted $E(X, f)$, is defined as the pointwise closure of $\{f^n \colon n \in \mathbb{N}\}$ in the compact space X^X with composition of functions as its algebraic operation. The Ellis semigroup is equipped with the topology inhered from the product space X^X. This semigroup was introduced by R. Ellis in [4], and it is a very important tool in the study of the topological behavior of dynamical systems. Subsequently, in [5] R. Ellis and M. Nerurkar showed that a dynamical system (X, f) is WAP if and only if every element of $E(X, f)$ is continuous. In addition, the article [11] offers an excellent survey concerning applications of the Ellis semigroup. In the paper [10], the authors initiated the study of the continuity and discontinuity of the elements of $E(X, f) \setminus \{f^n \colon n \in \mathbb{N}\}$. For instance, they point out that if X is a convergent sequence with its limit point, then either all elements are continuous or all are discontinuous (this result was later improved in [8]). In a different context, P. Szuca in [14] showed that if $X = [0, 1]$, the function $f \colon [0, 1] \to [0, 1]$ is onto and f^p is continuous for some $p \in \mathbb{N}^*$, then all the elements of $E([0, 1], f)$ are continuous. Using the Cantor set as a phase space and generalized shift maps, the continuity and discontinuity of the elements of the Ellis semigroup were studied in [7]. The main tool that have been used in all these investigations is the combinatorial properties of the ultrafilters on $\mathbb{N}$. Certainly, the Ellis semigroup can be described in terms of the notion of convergence with respect to an ultrafilter. Indeed, given $p \in \mathbb{N}^*$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in a space X, we say that a point $x \in X$ is the *p-limit point* of the sequence, in symbols $x = p - \lim_{n\to\infty} x_n$, if for every neighborhood V of x, $\{n \in \mathbb{N} \colon f^n(x) \in V\} \in p$. Observe that a point $x \in X$ is an accumulation point of a countable set $\{x_n \colon n \in \mathbb{N}\}$ of X if and only if there is $p \in \mathbb{N}^*$ such that $x = p - \lim_{n\to\infty} x_n$.

The notion of a p-limit point has been used in several branches of mathematics (see for instance [3] and [6, page 179]). A. Blass in [2] and N. Hindman in [12] formally established the connection between "the iteration in topological dynamics" and "the convergence with respect to an ultrafilter" by considering a more general iteration of the function f as follows: Let X be a compact space and $f\colon X \to X$ a continuous function. For $p \in \mathbb{N}^*$, the *p-iterate* of f is the function $f^p\colon X \to X$ defined by

$$f^p(x) = p - \lim_{n\to\infty} f^n(x)$$

for each $x \in X$. The description of the Ellis semigroup and its operation in terms of the p-iterates are the following, see [2], [12]:

$$E(X, f) = \{f^p\colon p \in \beta(\mathbb{N})\}$$
$$f^p \circ f^q = f^{q+p} \qquad \text{for each } \ p, q \in \beta(\mathbb{N}).$$

This result is a consequence of the fact that $\beta(\mathbb{N})$ is the Stone–Čech compactification of $\mathbb{N}$ with the discrete topology. More explicitly, the continuous map $\mathbb{N} \to X^X$ defined by $n \mapsto f^n$, extends to a continuous map $\beta(\mathbb{N}) \to X^X$ defined by $p \mapsto f^p$. More generally, when we have some sequence $(x_n)_{n\in\mathbb{N}}$ in X the continuous map $n \mapsto x^n$ extends to a continuous map $p \mapsto p - \lim_{n\to\infty} x_n$. Additionally, when f^p is a continuous function for any $p \in \beta(\mathbb{N})$, $f^p \circ f^q = f^q \circ f^p$ for all $q \in \beta(\mathbb{N})$. In particular, $f^n \circ f^q = f^q \circ f^n$ for each $n \in \mathbb{N}$ and $q \in \beta(\mathbb{N})$.

We define $E(X, f)^*$ as the limit point set of the sequence $(f_n)_{n\in\mathbb{N}}$. From the above, the set $E(X, f)^*$ is the continue image of $\mathbb{N}^*$ under the map $p \mapsto f^p$. Besides, we have that $\omega_f(x) = \{f^p(x)\colon p \in \mathbb{N}^*\}$ for each $x \in X$. When the orbit of w is dense in X, then $X = \{f^p(w)\colon p \in \beta(\mathbb{N})\}$ and thus X is a continuous image of $\beta(\mathbb{N})$.

Recently (for instance see [8] and [9]), we have investigated the structure of the Ellis semigroup of a dynamical system and the topological properties of some of its elements. Our main purpose moves in two directions: The first one concerns with the continuity and discontinuity of the p-iterates which is dealt in the third section, and the second one concerns about a general question stated in [9]:

Question 1.1. Given two compact metric countable spaces X and Y, is there a continuous function $f\colon X \to X$ such that $E(X, f)$ is homeomorphic to Y?

We provide a partial answer to this question in the fourth section.

2. Basic properties

In this section, we state several useful results that were proved in the articles [8] and [9]. Our first lemma is precisely Lemma 2.1 from [9].

Lemma 2.1. *Let (X, f) be a dynamical system and $x \in X$.*

(i) *Assume that x is periodic with period n and let $l < n$. Then, $p \in \big(n\mathbb{N}+l\big)^*$ if and only if $f^p(x) = f^l(x)$.*

(ii) *Suppose that x is eventually periodic and that $m \in \mathbb{N}$ is the smallest positive integer such that $f^m(x)$ is a periodic point. If n is the period of $f^m(x)$ and $p \in \big(n\mathbb{N}+l\big)^*$ for some $l < n$, then $f^p(x) = f^l(f^{nj}(x))$ where $j = \min\{i \colon m \le ni + l\}$.*

(iii) *Suppose that the orbit of x is infinite and $\omega_f(x) = \mathcal{O}_f(y)$ for some periodic point $y \in X$ with period n. If $p, q \in (n\mathbb{N}+l)^*$ for some $l < n$, then $f^p(x) = f^q(x)$.*

The next statement is well known in the literature, see, e.g., Lemma 2.4 of [8].

Lemma 2.2. *Let (X, f) be a dynamical system. If $\omega_f(x)$ is finite, then every point of $\omega_f(x)$ is periodic. In particular, if $\omega_f(x)$ has an isolated point in $\overline{\mathcal{O}_f(x)}$, then every point of $\omega_f(x)$ is periodic.*

Lemma 2.3. *Let (X, f) be a dynamical system, $x \in X$ and $m \in \mathbb{N}$. There is an integer $M > 0$ such that $|\mathcal{O}_f(x)| < M$ for each $x \in X$ if and only if $E(X, f)$ is finite.*

We omit the proof of the following well-known result.

Lemma 2.4. *Let (X, f) be a dynamical system such that X has a dense subset consisting of isolated points. If X has a point with infinite orbit, then $\{f^n \colon n \in \mathbb{N}\}$ is infinite and discrete in $E(X, f)$ and $f^n \neq f^p$ for every $(n, p) \in \mathbb{N} \times \mathbb{N}^*$.*

3. Continuity of the p-iterates

In the context of compact metric countable spaces, we showed in [8, Theorem 3.11] that if all points of $a \in X'$ are periodic, then for $b \in X'$ either each f^q is discontinuous at b for every $q \in \mathbb{N}^*$ or each f^q is continuous at b for all $q \in \mathbb{N}^*$. An example where this assertion fails assuming that all points of X' are eventually periodic is also given in [8]. Here, our main task is to give a necessary and sufficient condition on the space X in order that all p-iterates be discontinuous at a given point. Hence, we have a necessary and sufficient condition on the space X to make (X, f) weakly almost periodic. To have this done we shall need some preliminary lemmas.

The next result corresponds to Lemma 3.7 and Theorem 3.8 of [8].

Lemma 3.1. *Let (X, f) be a dynamical system such that X is a compact metric countable space and every point of X' is periodic. For every $x \in X$, there exists a periodic point $y \in X$ such that $\omega_f(x) = \mathcal{O}_f(y)$. If x has an infinite orbit and $y \in \omega_f(x)$ is fixed, then $f^n(x) \longrightarrow y$.*

The following two additional results are needed to establish our theorems.

Lemma 3.2. *Let $f\colon X \to X$ be a continuous function such that every accumulation point is fixed. The set $\{x \in X\colon d(x, f(x)) \geq \varepsilon\}$ is finite for each $\varepsilon > 0$.*

Proof: Assume, towards a contradiction, that there is $\varepsilon_0 > 0$ such that $H = \{x \in X\colon d(x, f(x)) \geq \varepsilon_0\}$ is infinite. Since X is compact and metric, there is a non constant sequence $(x_n)_{n\in\mathbb{N}}$ in H and $x \in X'$ such that $x_n \longrightarrow x$. As f is continuous and x is fixed, then $f(x_n) \longrightarrow f(x) = x$ which implies that there is $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, f(x_{n_0})) < \varepsilon_0$, but this is a contradiction. This shows that H is finite. □

Lemma 3.3. *Let (X, f) be a dynamical system such that X is a compact metric countable space and every accumulation point is periodic. Let $p \in \mathbb{N}^*$ and $b \in X$ be an isolated point. If there is a sequence $(a_n)_{n\in\mathbb{N}}$ in X such that $f^p(a_n) \longrightarrow b$, then b is periodic and $\mathcal{O}_f(b) = \omega_f(b) = \omega_f(a_n)$ for every positive integer except finitely many.*

Proof: It follows directly from the assumption that $B = \{n \in \mathbb{N}\colon f^p(a_n) = b\}$ is cofinite and hence $b \in \omega_f(a_n) \subseteq \overline{\mathcal{O}_f(a_n)}$ for every $n \in B$. By Lemma 2.2, we have that b is a periodic point. So, $\mathcal{O}_f(b) = \omega_f(b)$. By Lemma 3.1, for every $n \in B$ there is a periodic point $y_n \in X$ such that $\omega_f(a_n) = \mathcal{O}_f(y_n)$ and since $b \in \mathcal{O}_f(y_n)$, we conclude that $\omega_f(b) = \mathcal{O}_f(b) = \mathcal{O}_f(y_n) = \omega_f(a_n)$ for all $n \in B$. □

Theorem 3.4. *Let (X, f) be a dynamical system such that X is a compact metric countable space and every accumulation point of X is fixed. For every $a \in X'$, the following statements are equivalent:*

(1) *There is $p \in \mathbb{N}^*$ such that f^p is discontinuous at $a \in X'$.*
(2) *There are a periodic point $b \in X \setminus \{a\}$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in X such that $a_n \longrightarrow a$ and $b \in \overline{\mathcal{O}_f(a_n)}$ for all $n \in \mathbb{N}$.*
(3) *The function f^p is discontinuous at a for each $p \in \mathbb{N}^*$.*

Proof: (1) $\Rightarrow$ (2). Suppose that f^p is discontinuous at $a \in X'$. Then, there is a nontrivial sequence $(a_n)_{n\in\mathbb{N}}$ in X such that $a_n \longrightarrow a$ and $f^p(a_n)$ does not converge to a. Since X is compact and metric, there are a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ and $b \in X \setminus \{a\}$ such that $f^p(a_{n_k}) \longrightarrow b$. Assume, without loss of generality, that this subsequence is $(a_n)_{n\in\mathbb{N}}$. In virtue of Lemma 3.3, we only consider the case when $b \in X'$. Since $a \neq b$, $f^p(a_n) \longrightarrow b$ and $a_n \longrightarrow a$, there are a clopen set $V \in \mathcal{N}(a)$ and $N \in \mathbb{N}$ such that $b \notin V$, $a_n \in V$ and $f^p(a_n) \in X \setminus V$ for each $n \geq N$. For every $n \geq N$ there is $d_n \in \mathcal{O}_f(a_n) \cap V$ such that $f(d_n) \in X \setminus V$, this is possible since the set $A_n = \{m \in \mathbb{N}\colon f^m(a_n) \notin V\} \in p$ and hence it is infinite. Then, by Lemma 3.2, we have that the set $B = \{d_n\colon n \geq N\}$ is finite. Hence, there exists $d \in B$ for which the set $H = \{n \in \mathbb{N}\colon d = d_n\}$ is infinite. To finish the proof it suffices to show that $b \in \omega_f(a_n)$ for all $n \in H$. Indeed, we consider two cases. Suppose first that $\mathcal{O}_f(d)$ is infinite, then there is $e \in X'$ such that $e \in \omega_f(d)$. As e is fixed, by Lemma 3.1, $f^m(d) \longrightarrow e$. Analogously it is shown that $f^m(a_n) \xrightarrow[m\to\infty]{} e$ because of $e \in \omega_f(a_n)$ for each $n \in H$. Consequently, we obtain that $f^q(a_n) = e$ for each $q \in \mathbb{N}^*$ and for each all $n \in H$. This implies that $b = e$ since $f^p(a_n) \xrightarrow[n\in H]{} b$. Thus $b \in \omega_f(a_n)$ for all $n \in H$. For the second case, we assume that $\mathcal{O}_f(d)$ is finite. As $d \in \mathcal{O}_f(a_n)$ for each $n \in H$, we also have that the orbit of a_n is finite for all $n \in H$. By Lemma 3.1, we may choose a periodic point $e \in X$ such that $\mathcal{O}_f(e) = \omega_f(d) = \{f^q(d)\colon q \in \mathbb{N}^*\} = \{f^q(a_n)\colon q \in \mathbb{N}^*\} = \omega_f(a_n)$ for $n \in H$. Since $\mathcal{O}_f(e)$ is finite and $f^p(a_n) \longrightarrow b$, we obtain that $b \in \omega_f(a_n)$ for each $n \in H$.

(2) $\Rightarrow$ (3). We have to analyze two possible cases.

Case I. Suppose b is isolated. By assumption, b is periodic. Hence, if $b \in \mathcal{O}_f(y)$, then $\mathcal{O}_f(y)$ is finite and so $\mathcal{O}_f(b) = \omega_f(y) = \{f^p(y)\colon p \in \mathbb{N}^*\}$. Thus, from the hypothesis we obtain that $f^p(a_n) \in \omega_f(a_n) = \mathcal{O}_f(b)$ for every $n \in \mathbb{N}$ and every $p \in \mathbb{N}^*$. As a is fixed and $b \neq a$, then $a \notin \mathcal{O}_f(b)$ and the sequence $(f^p(a_n))_{n\in\mathbb{N}}$ cannot converge to a for any $p \in \mathbb{N}^*$. Thus f is discontinuous at a.

Case II. Suppose $b \in X'$. Observe that if $b \in \overline{\mathcal{O}_f(y)}$ and $\mathcal{O}_f(y)$ is finite, then $\omega_f(y) = \{b\}$. On the other hand, if $b \in \overline{\mathcal{O}_f(y)}$ and $\mathcal{O}_f(y)$ is infinite, then it follows from Lemma 3.1 that $\omega_f(y) = \{b\}$. Thus, we have that $f^p(a_n) = b$ for each $p \in \mathbb{N}^*$ and for each $n \in \mathbb{N}$. Thus f is discontinuous at a.

(3) $\Rightarrow$ (1). It is evident. $\square$

Corollary 3.5. *Let (X, f) be a dynamical system such that X is a compact metric countable space and every accumulation point of X is fixed. If the orbit of every isolated point of X is finite, then every function of $E(X, f)$ is continuous.*

Proof: Assume that there is $p \in \mathbb{N}^*$ such that f^p is discontinuous at $a \in X'$. It follows from Theorem 3.4 (2) that there are a periodic point $b \in X \setminus \{a\}$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in X such that $a_n \longrightarrow a$ and $b \in \overline{\mathcal{O}_f(a_n)}$ for all $n \in \mathbb{N}$. Since each accumulation point of X is fixed, we may assume that a_n is isolated for every $n \in \mathbb{N}^*$, hence $b \in \mathcal{O}_f(a_n)$ for all $n \in \mathbb{N}$. Now, let V and W be disjoint open sets such that $\overline{V} \cap \overline{W} = \emptyset$, $a \in V$ and $b \in W$. Without loss of generality, we may assume that $a_n \in V$ for all $n \in \mathbb{N}$. Then we can find a sequence $(k_n)_{n\in\mathbb{N}}$ such that $f^{k_n}(a_n) \in V$ and $f^{k_n+1}(a_n) \in W$ for every $n \in \mathbb{N}$. We may assume that

$f^{k_n}(a_n) \longrightarrow c \in \overline{V}$, but this is impossible since $f^{k_n+1}(a_n) \longrightarrow f(c) = c \in \overline{W}$. Therefore, f^p is continuous. $\square$

By the previous results, we have a necessary and sufficient condition to make (X, f) weakly almost periodic, when X is a compact metric countable space and every accumulation point is fixed.

The conclusion of Theorem 3.4 is not true if we replace the hypothesis "every accumulation point is fixed" by the hypothesis "every accumulation point is periodic". Indeed, the next two examples witness that the condition (2) of Theorem 3.4 holds together with either the assumption "f^p is continuous for every $p \in \mathbb{N}^*$" or the assumption "f^p is discontinuous for all $p \in \mathbb{N}^*$".

The phase space of the following dynamical systems is the ordinal space $2\omega + 1$ (two disjoint convergent sequences) which will be identified with the following subspace of $\mathbb{R}$:

$$X = \{a_n \colon n \in \mathbb{N}\} \cup \{b_n \colon n \in \mathbb{N}\} \cup \{a, b\},$$

where $a_n < a_{n+1} < a < b_n < b_{n+1} < b$ for every $n \in \mathbb{N}$, $a_n \longrightarrow a$ and $b_n \longrightarrow b$.

Example 3.6. Define a function $f \colon X \to X$ as follows:

a) $f(a) = b$ and $f(b) = a$,
b) $f(a_n) = b_n$ for each $n \in \mathbb{N}$, and
c) $f(b_n) = a_{n+1}$ for each $n \in \mathbb{N}$.

That is,

$$a_0 \to b_0 \to a_1 \to b_1 \to a_2 \to b_2 \to a_3 \to b_3 \cdots a_n \to b_n$$
$$\to a_{n+1} \to b_{n+1} \to a_{n+2} \to b_{n+2} \to \cdots$$

It is evident that f is continuous. From the definition of f we have, for all $n, m \in \mathbb{N}$, the following:

i) $f^{2m}(a_n) = a_{n+m}$,
ii) $f^{2m+1}(a_n) = b_{n+m}$,
iii) $f^{2m}(b_n) = b_{n+m}$, and
iv) $f^{2m+1}(b_n) = a_{n+m+1}$.

Conditions i)–iv) imply the following:

(1) $f^p(a_n) = f^p(a) = a$ for every $p \in (2\mathbb{N})^*$,
(2) $f^p(a_n) = f^p(b) = b$ for every $p \in (2\mathbb{N}+1)^*$,
(3) $f^p(b_n) = f^p(b) = b$ for every $p \in (2\mathbb{N})^*$, and
(4) $f^p(b_n) = f^p(a) = a$ for every $p \in (2\mathbb{N}+1)^*$.

Note that,

(1) $\omega_f(x) = \{a, b\}$ for every $x \in X$.
(2) $E(X, f)$ is infinite.
(3) The point a_1 has a dense orbit.

Then we have that the accumulation points a and b have period equal to 2 and both satisfy the second condition of Theorem 3.4. However, the function f^p is continuous for every $p \in \mathbb{N}^*$. In adition, we have a transitive dynamical system with $|\omega_f(x)| \leq 2$ for every $x \in X$, and $E(X, f)$ is infinite and countable.

Example 3.7. We define a function $f\colon X \to X$ as follows:

a) $f(a) = b$ and $f(b) = a$,
b) $f(a_0) = a_0$,
c) $f(a_n) = b_{n-1}$ for every $0 < n \in \mathbb{N}$, and
d) $f(b_n) = a_n$ for every $n \in \mathbb{N}$.

That is,

$$\begin{aligned} b_n \to a_n \to b_{n-1} \to a_{n-1} \to b_{n-2} \to a_{n-2} \to b_{n-3} \\ \to a_{n-3} \to \cdots b_2 \to a_2 \to b_1 \to a_1 \to b_0 \to a_0. \end{aligned}$$

It is not hard to prove that f is continuous. From the definition of f we easily have that for each $x \in X \setminus \{a, b\}$ there exists $n \in \mathbb{N}$ such that $f^n(x) = a_0$. Hence, $f^p(x) = a_0$ for every $x \in X \setminus \{a, b\}$ and every $p \in \mathbb{N}^*$. Since $f^p(a) = a$ and $f^p(b) = b$ for all $p \in (2\mathbb{N})^*$; and $f^p(a) = b$ and $f^p(b) = a$ for all $p \in (2\mathbb{N}+1)^*$, we conclude that f^p is discontinuous at a and b for each $p \in \mathbb{N}^*$ and $|\omega_f(x)| \leq 2$ for each $x \in X$. Finally, observe that $a_0 \in \overline{\mathcal{O}_f(a_n)}$ for all $n \in \mathbb{N}$, so (2) of Theorem 3.4 is satisfied.

In the next theorem, we show a generalization of (2) $\Rightarrow$ (3) in Theorem 3.4.

Theorem 3.8. *Let (X, f) be a dynamical system such that X is a compact metric countable space and every accumulation point of X is periodic. Let $a \in X'$. If there are a sequence $(a_n)_{n\in\mathbb{N}}$ in X and a periodic point $b \in X \setminus \mathcal{O}_f(a)$ such that $a_n \longrightarrow a$ and $b \in \overline{\mathcal{O}_f(a_n)}$ for every $n \in \mathbb{N}$, then f^p is discontinuous at a for each $p \in \mathbb{N}^*$.*

PROOF: First we show a particular case. Suppose that $b \in \mathcal{O}_f(a_n)$ for infinitely many $n \in \mathbb{N}$. Hence, from the periodicity of b, we have that $\mathcal{O}_f(b) = \omega_f(b) = \omega_f(a_n) = \{f^p(a_n) : p \in \mathbb{N}^*\}$ is finite for infinitely many $n \in \mathbb{N}$. Since $\mathcal{O}_f(a) \cap \mathcal{O}_f(b) = \emptyset$, the sequence $(f^p(a_n))_{n\in\mathbb{N}}$ cannot converge to $f^p(a) \in \mathcal{O}_f(a)$ for any $p \in \mathbb{N}^*$. Thus f^p is not continuous at a for any $p \in \mathbb{N}^*$.

For the proof of the theorem we consider two cases: (i) Suppose b is isolated. Then $b \in \mathcal{O}_f(a_n)$ for every $n \in \mathbb{N}$ and we are done from the result we proved above.

(ii) Assume that $b \in X' \setminus \left(\bigcup_{n\in\mathbb{N}} \mathcal{O}_f(a_n)\right)$. From the result proved above, we can assume that $\mathcal{O}_f(a_n)$ is infinite for every $n \in \mathbb{N}$. Then, we must have that $b \in \omega_f(a_n)$ for every $n \in \mathbb{N}$. Now, in virtue of Lemma 3.1, for every $n \in \mathbb{N}$ there is a periodic point $y_n \in X$ such that $\omega_f(a_n) = \mathcal{O}_f(y_n)$. Since b is periodic and

$b \in \mathcal{O}_f(y_n)$, then $\mathcal{O}_f(b) = \mathcal{O}_f(y_n)$ for all $n \in \mathbb{N}$. Thus $f^p(a_n) \in \mathcal{O}_f(b)$ for all n and all $p \in \mathbb{N}^*$. Since $\mathcal{O}_f(a) \cap \mathcal{O}_f(b) = \emptyset$, we conclude as before that f^p is not continuous at a for any $p \in \mathbb{N}^*$. $\square$

In the next result we slightly modify the proof of the previous theorem to get an interesting statement.

Theorem 3.9. *Let (X, f) be a dynamical system such that X is a compact metric countable space and every accumulation point of X is periodic. Let $a \in X'$ and $(a_n)_{n\in\mathbb{N}}$ be a sequence in X such that $f^p(a_n) \longrightarrow f^p(a)$ for some $p \in \mathbb{N}^*$. Suppose $b \in X$ is a periodic point and $b \in \bigcap_{n\in\mathbb{N}} \overline{\mathcal{O}_f(a_n)}$, then $b \in \mathcal{O}_f(a)$.*

Proof: Let $a \in X'$, $(a_n)_n$ in X and $p \in \mathbb{N}^*$ as in the hypothesis. First, suppose that b is isolated. Then, $b \in \bigcap_{n\in\mathbb{N}} \mathcal{O}_f(a_n)$ and so $\mathcal{O}_f(b) = \omega_f(b) = \omega_f(a_n)$ for every $n \in \mathbb{N}$. Since $f^p(a_n) \longrightarrow f^p(a) \in \mathcal{O}_f(a)$ and $f^p(a_n) \in \omega_f(a_n) = \mathcal{O}_f(b)$ for each $n \in \mathbb{N}$, we must have that $\mathcal{O}_f(a) \cap \mathcal{O}_f(b) \neq \emptyset$ and so $b \in \mathcal{O}_f(a)$. Now, suppose that $b \in X'$. Notice that if $b \in \mathcal{O}_f(a_n)$ for some $n \in \mathbb{N}$, then $\mathcal{O}_f(b) = \omega_f(b) = \omega_f(a_n)$. Thus we have that $b \in \omega_f(a_n)$ for all $n \in \mathbb{N}$. By Lemma 3.1, there exists a periodic point $y_n \in X$ such that $\omega_f(a_n) = \mathcal{O}_f(y_n)$. Since $b \in \omega_f(a_n)$, then $\omega_f(a_n) = \mathcal{O}_f(b)$ for all $n \in \mathbb{N}$. Thus, we have shown that $f^p(a_n) \in \mathcal{O}_f(b)$ for every $n \in \mathbb{N}$. Then, as before, we have $\mathcal{O}_f(a) \cap \mathcal{O}_f(b) \neq \emptyset$ and thus $b \in \mathcal{O}_f(a)$. $\square$

Concerning the last theorem we have the following example.

Example 3.10. We consider again the countable ordinal space $2\omega + 1$ identified with the subspace X of $\mathbb{R}$ from above. Define the function $f\colon X \to X$ as follows:

a) $f(a) = b$ and $f(b) = a$,
b) $f(a_0) = b$,
c) $f(a_n) = b_{n-1}$ for each $n > 0$, and
d) $f(b_n) = a_n$ for each $n \in \mathbb{N}$.

The function f is evidently continuous. From the definition we can see that

$$\begin{aligned} b_n \to a_n \to b_{n-1} \to a_{n-1} \to b_{n-2} \to a_{n-2} \to \cdots \to b_2 \\ \to a_2 \to b_1 \to a_1 \to b_0 \to a_0 \to b. \end{aligned}$$

Hence, all points are eventually periodic. Besides, we have the following properties:

i) For every $x \in X$ there is $n \in \mathbb{N}$ such that $f^n(x) = b$.
ii) $f^p(x) = b$ for each $x \in X \setminus \{a, b\}$ and for each $p \in \mathbb{N}^*$.
iii) $f^p(a) = a$ for all $p \in (2\mathbb{N})^*$.
iv) $f^p(a) = b$ for all $p \in (2\mathbb{N}+1)^*$.
v) $f^p(b) = a$ for all $p \in (2\mathbb{N}+1)^*$.

Thus, we have that f^p is discontinuous at a for all $p \in (2\mathbb{N})^*$ and we also have that $b \in \mathcal{O}_f(x)$ for every $x \in X$. Moreover, f^p is discontinuous at b for all $p \in (2\mathbb{N}+1)^*$.

Theorem 3.4 suggests the following problem.

Problem 3.11. Let (X, f) be a dynamical system such that X is a compact metric countable space such that every accumulation point of X is periodic. Find a necessary and sufficient condition on a point $a \in X$, like in Theorem 3.4, in order that f^p is discontinuous at a for each $p \in \mathbb{N}^*$.

4. Transitive dynamical systems

Continuing the work presented in [9], in this section we focus our attention on transitive dynamical systems. We recall some questions from that paper. The first one is whether $E(X, f)$ is countable for every transitive system (X, f), see Questions 4.6 in [9]. For instance, this happens when every function in $E(X, f)$ is continuous, see [9, Theorem 2.9, Theorem 3.3]. Below we extend this result. The second question is whether or not there is a continuous function $f\colon X \to X$ such that $E(X, f)$ is homeomorphic to Y, where X and Y are arbitrary compact metric countable spaces. See Questions 4.8 in [9]. In this section, X will be always a compact metric countable space.

We need the following result from [9], see Lemma 3.1 and Theorem 2.3.

Lemma 4.1. *Let (X, f) be a dynamical system.*

(i) *Then $E(X, f) \setminus \{f^n\colon n \in \mathbb{N}\}$ is finite if and only if there is $m \in \mathbb{N}$ such that $|\omega_f(x)| \leq m$ for each $x \in X$.*
(ii) *If (X, f) is transitive and $y \in X'$, then $f(y) \in X'$.*
(iii) *If w has a dense orbit, then w is isolated.*

Theorem 4.2. *Let (X, f) be a transitive dynamical system where X is a compact metrizable countable space. If either f^p is continuous for each $p \in \mathbb{N}^*$, or $|\omega_f(x)| = 1$ for each $x \in X'$, then $E(X, f)$ is homeomorphic to X.*

Proof: In [1] E. Akin and E. Glasner proved that if (X, f) is transitive, X is compact metric space and f^p is continuous for each $p \in \mathbb{N}^*$, then $E(X, f)$ is homeomorphic to X. Here, we present another proof of this fact which also holds under the assumption $|\omega_f(x)| = 1$ for each $x \in X'$. Let w be a point of X whose orbit is dense in X. According to Lemma 4.1 the point w must be isolated. Now, consider the function $h\colon E(X, f) \to X$ defined by $h(f^p) = f^p(w)$ for every $p \in \beta(\mathbb{N})$. This function is continuous since it is the restriction of the projection map $\pi_w\colon X^X \to X$ to $E(X, f)$. It follows from Lemma 2.4 that h

is surjective since $X = \{f^p(w) \colon p \in \beta(\mathbb{N})\}$. To prove the theorem it suffices to show that the function h is injective. To have this done, first observe that $f^p(f^n(w)) = f^n(f^p(w))$ for all $n \in \mathbb{N}$ and for all $p \in \beta(\mathbb{N})$, and the orbit of w is the collection of isolated points of X. Hence, we obtain that $f^p(x) = f^q(x)$ for every isolated point $x \in X$ whenever $f^p(w) = f^q(w)$ for some $p, q \in \beta(\mathbb{N})$.

Suppose first that f^p is continuous for each $p \in \mathbb{N}^*$. Then, if $p, q \in \mathbb{N}^*$ and f^p and f^q agree on all the points of a dense orbit, then we obtain that $f^p = f^q$. This shows h is injective.

Now, assume that $|\omega_f(x)| = 1$ for each $x \in X'$. Then for every $x \in X'$ there is $z_x \in X'$ such that $\omega_f(x) = \{z_x\}$ and hence we obtain that $f^p(x) = z_x$ for all $p \in \mathbb{N}^*$. Thus we have that $f^p = f^q$ if and only if $f^p(w) = f^q(w)$ for all $p, q \in \beta(\mathbb{N})$. So, h is injective.

Therefore, in both cases, h is a homeomorphism between $E(X, f)$ and X. □

Question 4.8 is related to the second condition of the previous theorem.

In Example 3.6, we have a transitive dynamical system such that $E(X, f)$ is infinite and countable with the restriction $|\omega_f(x)| = 2$ for every $x \in X$. In this direction, we proof that the cardinality of the Ellis semigroup is countable under some restriction on ω-sets.

Theorem 4.3. *Let (X, f) be a transitive dynamical system such that X is a compact metric countable space. If there is $m \in \mathbb{N}$ such that $|\omega_f(x)| \leq m$ for every $x \in X'$, then $E(X, f)$ is countable infinite.*

Proof: Let $E^* = E(X, f) \setminus \{f^n \colon n \in \mathbb{N}\}$. It suffices to show that E^* is countable. First notice that E^* is equal to $\{f^p \colon p \in \mathbb{N}^*\}$ by Lemma 2.4. Since X' is f-invariant (by Lemma 4.1 (ii)), then $(X', f \restriction X')$ is a well defined dynamical system. By Lemma 4.1 (i), $E(X', f \restriction X')$ is finite. Let $w \in X$ with a dense orbit. Consider the function $\varphi \colon E^* \to E(X', f \restriction X') \times X'$ given by $\varphi(g) = (g \restriction X', g(w))$. It suffices to show that φ is injective. In fact, let $g \in E^*$, then $g = f^p$ for some $p \in \mathbb{N}^*$. Notice that every isolated point is of the form $f^l(w)$ for some $l \in \mathbb{N}$. Thus $g(f^l(w)) = f^p(f^l(w)) = f^l(f^p(w)) = f^l(g(w))$. Therefore g is completely determined by $g \restriction X'$ and $g(w)$. Hence φ is injective. □

As consequence of Theorem 4.3 and Theorem 2.7 of [9], we obtain the following characterization.

Corollary 4.4. *Let (X, f) be a transitive dynamical system where X is a compact metrizable countable space. Then the set P_f is finite if and only if $E(X, f)$ is countable.*

Proof: First we suppose that the set P_f is finite. Hence, there is $m \in \mathbb{N}$ such that $|\omega_f(x)| \leq m$ for every $x \in X'$. By Theorem 4.3, we conclude that $E(X, f)$

is countable. Now, we assume that $E(X,f)$ is countable. If P_f were infinite, by Theorem 2.7 of [9], then $E(X,f)$ would be homeomorphic to the Cantor set $2^{\mathbb{N}}$, which is impossible. So, P_f is finite. $\square$

In the next example, we show that there exists a continuous function f such that the dynamical system (ω^3+1, f) is transitive and has a sequence of accumulation points with arbitrarily large period. By Theorem 2.7 of [9], we have that $E(\omega^3+1, f)$ is homeomorphic to $2^{\mathbb{N}}$. This example differs from Example 4.1 of [8] since this dynamical system is transitive and in the other one all points have finite orbit. Notice also that, in the example below, all p-iterates are discontinuous.

The phase space of the next two examples is going to be ω^3+1 which for our convenience will be identified with the following subspace of $\mathbb{R}$:

$$X = \{d_{i,j,k} \colon i,j,k \in \mathbb{N}\} \cup \{d_{j,k} \colon j,k \in \mathbb{N}\} \cup \{d_k \colon k \in \mathbb{N}\} \cup \{d\},$$

where $(d_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence such that $d_k \xrightarrow[k\to\infty]{} d$; $(d_{j,0})_{j\in\mathbb{N}}$ is a strictly increasing sequence contained in $(-\infty, d_0)$ such that $d_{j,0} \xrightarrow[j\to\infty]{} d_0$; for each positive $k \in \mathbb{N}$, the sequence $(d_{j,k})_{j\in\mathbb{N}}$ is strictly increasing, it is contained in (d_{k-1}, d_k) and $d_{j,k} \xrightarrow[j\to\infty]{} d_k$; $D_{0,0} := \{d_{i,0,0} \colon i \in \mathbb{N}\}$ is a strictly increasing sequence such that $d_{i,0,0} \xrightarrow[i\to\infty]{} d_{0,0}$ and it is contained in $(-\infty, d_{0,0})$; $D_{0,k} = \{d_{i,0,k} \colon i \in \mathbb{N}\}$ is a strictly increasing sequence with $d_{i,0,k} \xrightarrow[i\to\infty]{} d_{0,k}$ and contained in $(d_{k-1}, d_{0,k})$ for each $k \in \mathbb{N}\setminus\{0\}$ $D_{j,k} := \{d_{i,j,k} \colon i \in \mathbb{N}\}$ is a strictly increasing sequence with $d_{i,j,k} \xrightarrow[i\to\infty]{} d_{j,k}$ and contained in $(d_{j-1,k}, d_{j,k})$ for every $j \in \mathbb{N}\setminus\{0\}$ and for every $k \in \mathbb{N}$.

We are ready to describe our first example.

Example 4.5. There is a function $f\colon \omega^3+1 \to \omega^3+1$ such that

(1) $\mathcal{O}_f(d_{0,0,0})$ is dense.
(2) All points of $(\omega^3+1)'$ have finite orbit.
(3) The accumulation point d_{a_n} has period $n+2$, where $a_0 = 0$ and $a_n = a_{n-1}+n+1$ for every $n \in \mathbb{N}$.
(4) The Ellis semigroup $E(\omega^3+1, f)$ is homeomorphic to $2^{\mathbb{N}}$.
(5) The function f^p is discontinuous for all $p \in \mathbb{N}^*$.

The function f is defined as follows:

a) $f(d) = d$.
b) $f(d_{a_n}) = d_{a_n+n+1} = d_{a_{n+1}-1}$ for each $n \in \mathbb{N}$.
c) $f(d_{a_n+k}) = d_{a_n+k-1}$ for each $n \in \mathbb{N}$ and $0 < k \le n+1$.
d) $f(d_{0,0}) = d$ and $f(d_{0,a_n}) = d_{a_{n-1}}$ for each $n > 0$.
e) $f(d_{i,a_n}) = d_{i-1,a_n+n+1} = d_{i-1,a_{n+1}-1}$ for each $n \in \mathbb{N}$ and $i > 0$.
f) $f(d_{i,a_n+k}) = d_{i,a_n+k-1}$ for each $i \in \mathbb{N}$, $n \in \mathbb{N}$ and $0 < k \le n+1$.

g) $f(d_{0,0,0}) = d_{0,0,1}$ and $f(d_{n,0,0}) = d_{0,0,a_n}$ for each $n > 0$.
h) $f(d_{i,j,0}) = d_{i+1,j-1,1}$ for each $i \in \mathbb{N}$ and $j > 0$.
i) $f(d_{i,0,1}) = d_{i+1,0,1}$ for each $i \in \mathbb{N}$.
j) $f(d_{i,j,1}) = d_{i,j,0}$ for each $i \in \mathbb{N}$ and $j > 0$.
k) $f(d_{i,j,a_n}) = d_{i,j-1,a_n+n+1}$ for each $i \in \mathbb{N}$ and $j, n > 0$.
l) $f(d_{i,0,a_n}) = d_{0,i+1,a_n-1}$ for each $i \in \mathbb{N}$ and $n > 0$.
m) $f(d_{i,j,k}) = d_{i,j,k-1}$ for each $i, j \in \mathbb{N}$ and $k \notin H$ and $k \notin \{a_n + 1 \colon n \in \mathbb{N}\}$.
n) $f(d_{i,j,k}) = d_{i,j,k-1}$ for each $i \in \mathbb{N}$, $j > 0$ and $k \in \{a_n + 1 \colon n > 0\}$.
o) $f(d_{i,0,k}) = d_{i+1,0,k-1}$ for each $i \in \mathbb{N}$ and $k \in \{a_n + 1 \colon n > 0\}$.

To have some idea about the orbits we describe some of them in the next diagram:

$$
\begin{aligned}
\underline{d_{0,0,0}} &\to d_{0,0,1} \to \underline{d_{1,0,0}} \to d_{0,0,2} \to d_{0,1,1} \to d_{0,1,0} \to d_{1,0,1} \to \underline{d_{2,0,0}} \\
&\to d_{0,0,5} \to d_{0,1,4} \to d_{0,1,3} \to d_{0,1,2} \to d_{0,0,4} \to d_{0,0,3} \to d_{0,0,2} \\
&\to d_{1,0,2} \to d_{0,2,1} \to d_{0,2,0} \to d_{1,1,1} \to d_{1,1,0} \to d_{2,0,1} \to \underline{d_{3,0,0}} \\
&\to d_{0,0,9} \to d_{0,1,8} \to d_{0,1,7} \to d_{0,1,6} \to d_{0,1,5} \to d_{0,0,8} \to d_{0,1,7} \\
&\to d_{0,1,6} \to d_{1,0,5} \to d_{0,2,4} \to d_{0,2,3} \to d_{0,2,2} \to d_{1,1,4} \to d_{2,1,3} \\
&\to d_{1,1,2} \to d_{1,0,4} \to d_{2,0,3} \to d_{2,0,2} \to d_{0,3,1} \to d_{0,3,0} \to d_{1,2,1} \\
&\to d_{1,1,0} \to d_{3,1,1} \to d_{3,1,0} \to d_{3,0,1} \to d_{4,0,0} \to \underline{d_{0,0,14}} \to \dots
\end{aligned}
$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$
\begin{aligned}
&\to \underline{d_{n,0,0}} \to d_{0,0,a_n} \to d_{0,1,a_{n-1}} \to \cdots \to d_{0,1,a_{n-1}} \\
&\to d_{0,0,a_n-1} \to \cdots \to d_{1,0,a_{n-1}} \to d_{0,2,a_{n-1}-1} \to \cdots \to d_{0,2,a_{n-2}} \\
&\to d_{n,0,1} \to \underline{d_{n+1,0,0}} \to \cdots
\end{aligned}
$$

$$\vdots \qquad \vdots \qquad \vdots$$

Notice that:

i) $f(D_{0,0}) = \{d_{0,0,a_n} \colon n \in \mathbb{N}\}$.
ii) $f(D_{j,0}) = D_{j-1,1} \setminus \{d_{0,j-1,1}\}$ for each $j > 0$.
ii) $f(D_{0,1}) = D_{0,0} \setminus \{d_{0,0,0}\}$ and $f(D_{ij}) = D_{(i-1)j}$ for each $i, j > 0$.
iii) $f(D_{j,1}) = D_{j,0}$ for each $j > 0$.
iv) $f(D_{0,a_n}) = \{d_{0,j+1,a_n-1} \colon j \in \mathbb{N}\}$ for each $n > 0$.
v) $f(D_{j,a_n}) = D_{j-1,a_n+n+1} \setminus \{d_{0,j-1,a_n+n+1}\}$ for each $j, n > 0$.
vi) $f(D_{j,k}) = D_{j,k-1}$ for each $j \in \mathbb{N}$ and $k \notin H \cup \{a_n + 1 \colon n \in \mathbb{N}\}$.
vii) $f(D_{j,k}) = D_{j,k-1}$ for each $j > 0$ and $k \in \{a_n + 1 \colon n \in \mathbb{N}\}$.
viii) $f(D_{0,k}) = D_{0,k-1} \setminus \{d_{0,0,k-1}\}$ for each $k \in \{a_n + 1 \colon n \in \mathbb{N}\}$.

By conditions a), b), c), iv), v), vi), vii) and viii), we have that f is continuous at d, d_k for every $k > 0$; f is continuous at d_0 by conditions b) and ii); and f is continuous at d_{ij}, by conditions d), e), f) and identities from i) to viii) for each $i, j \in \mathbb{N}$. Consider the sequence $(d_{i0})_{i\in\mathbb{N}}$ that converges to d_0. By conditions d), e) and f) for every $i \in \mathbb{N}$ there exist $l_i \in \mathbb{N}$ such that $f^{l_i}(d_{i0}) = d$. Then, for each $i \in \mathbb{N}$, $p \in \mathbb{N}^*$ $f^p(d_{i0}) = d$ and $f^p(d_0) = d_1$. So, we conclude that f^p is discontinuous at d_0 for every $p \in \mathbb{N}^*$. Since there are periodic points of arbitrarily large period, the Ellis semigroup $E(\omega^3 + 1, f)$ is homeomorphic to $2^{\mathbb{N}}$, by Theorem 2.7 of [9].

Our next example satisfies the second conditions of Theorem 4.2 and all the p-iterates are discontinuous for $p \in \mathbb{N}^*$.

Example 4.6. There is a continuous function $f\colon \omega^3 + 1 \to \omega^3 + 1$ such that:

(1) $\mathcal{O}_f(d_{0,0,0})$ is dense.
(2) The points $d_{c_n-1,0}$ and $d_{c_n+1,0}$ have infinite orbits, where $c_n = 2+3n$ for every $n \in \mathbb{N}$.
(3) $|\omega_f(x)| = 1$ for every $x \in (\omega^3 + 1)'$.
(4) $E(\omega^3 + 1, f)$ is homeomorphic to $\omega^3 + 1$.
(5) f^p is discontinuous for all $p \in \mathbb{N}^*$.

Our function f is defined as follows:

a) $f(d) = d$ and $f(d_n) = d_n$ for each $n \in \mathbb{N}$.
b) $f(d_{0,0}) = d$, $f(d_{1,0}) = d_{3,0}$, $f(d_{2,0}) = d_{0,0}$ and $f(d_{0,1}) = d_0$.
c) $f(d_{c_n-1,0}) = d_{c_{n-1}-1,0}$ for each $n > 0$.
d) $f(d_{c_n,0}) = d_{c_{n-1},0}$ for each $n > 0$.
e) $f(d_{c_n+1,0}) = d_{c_{n+1}+1,0}$ for each $n \in \mathbb{N}$.
f) $f(d_{j,k}) = d_{j-1,k}$ for each $j > 0$ and $k > 0$.
g) $f(d_{0,k}) = d_{k-1}$ for each $k > 0$.
h) $f(d_{i,j,k}) = d_{i+1,j-1,k}$ for each $i \in \mathbb{N}$ and $j, k > 0$.
i) $f(d_{i,0,2}) = d_{0,i,1}$ for each $i \in \mathbb{N}$.
j) $f(d_{i,0,k}) = d_{0,i+1,k-1}$ for each $i \in \mathbb{N}$ and $k > 2$.
k) $f(d_{i,0,1}) = d_{0,c_{i+1}-1,0}$ for each $i \in \mathbb{N}$.
l) $f(d_{i,c_0,0}) = d_{i+1,0,0}$ for each $i \in \mathbb{N}$.
m) $f(d_{i,c_n,0}) = d_{i+1,c_{n-1},0}$ for each $i \in \mathbb{N}$ and $n > 0$.
n) $f(d_{i,c_n-1,0}) = d_{i+1,c_{n-1}-1,0}$ for each $i \in \mathbb{N}$ and $n > 0$.
o) $f(d_0, c_{n+1}, 0) = d_{0,c_n,0}$ for each $n \in \mathbb{N}$.
p) $f(d_{i,c_n+1,0}) = d_{i-1,c_{n+1}+1,0}$ for each $i > 0$ and $n \in \mathbb{N}$.
q) $f(d_{i,4,0}) = d_{i,1,0}$ for each $i \in \mathbb{N}$.
r) $f(d_{i,1,0}) = d_{i,3,0}$ for each $i \in \mathbb{N}$.
s) $f(d_{i,0,0}) = d_{0,0,i+2}$ for each $i \in \mathbb{N}$.

t) $f(d_{i,2,0}) = d_{i+1,0,0}$ for each $i \in \mathbb{N}$.

In the next diagram, we can see the behavior of the orbits on the isolated points.

$$d_{0,0,0} \to d_{0,0,2} \to d_{0,0,1} \to d_{0,4,0} \to d_{0,1,0} \to d_{0,3,0} \to d_{0,2,0} \to d_{1,0,0}$$

$$\begin{aligned} d_{1,0,0} &\to d_{0,0,3} \to d_{0,1,2} \to d_{1,0,2} \to d_{0,1,1} \to d_{1,0,1} \to d_{0,7,0} \to d_{1,4,0} \to d_{1,1,0} \\ &\to d_{1,3,0} \to d_{0,6,0} \to d_{0,5,0} \to d_{1,2,0} \to d_{2,0,0} \end{aligned}$$

$$\begin{aligned} d_{2,0,0} &\to d_{0,0,4} \to d_{0,1,3} \to d_{1,0,3} \to d_{0,2,2} \to d_{1,1,2} \to d_{2,0,2} \to d_{0,2,1} \to d_{1,1,1} \\ &\to d_{2,0,1} \to d_{0,10,0} \to d_{1,7,0} \to d_{2,4,0} \to d_{2,1,0} \to d_{2,3,0} \to d_{1,6,0} \to d_{0,9,0} \\ &\to d_{0,8,0} \to d_{1,5,0} \to d_{2,2,0} \to d_{3,0,0} \end{aligned}$$

$$\begin{aligned} d_{3,0,0} &\to d_{0,0,5} \to d_{0,1,4} \to d_{1,0,4} \to d_{0,2,3} \to d_{1,1,3} \to d_{2,0,3} \to d_{0,3,2} \to d_{1,2,2} \\ &\to d_{2,1,2} \to d_{3,0,2} \to d_{0,3,1} \to d_{0,2,2} \to d_{1,1,4} \to d_{2,1,3} \to d_{1,1,2} \to d_{1,0,4} \\ &\to d_{2,0,3} \to d_{2,0,2} \to d_{0,3,1} \to d_{1,2,1} \to d_{2,1,1} \to d_{3,0,1} \to d_{4,0,0} \end{aligned}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\begin{aligned} d_{n,0,0} &\to d_{0,0,n+2} \to d_{0,1,n+1} \to \cdots \to d_{1,0,n+1} \cdots \to d_{n,0,1} \\ &\to d_{0,c_{n+1}-1,0} \to \cdots \to d_{n,2,0} \to d_{n+1,0,0} \cdots \end{aligned}$$

$$\vdots \qquad \vdots \qquad \vdots$$

The following properties are satisfied.

i) $f(D_{j,k}) = D_{j-1,k}$ for each $j > 0$ and $k > 0$.

ii) $f(D_{c_0,0}) = D_{0,0} \setminus \{d_{0,0,0}\}$ and $f(D_{c_n,0}) = D_{c_{n-1},0} \setminus \{d_{0,c_{n-1},0}\}$ for each $n > 0$.

iii) $f(D_{c_n+1,0}) = D_{c_{n+1}+1,0} \cup \{d_{0,c_n,0}\}$ for each $n \in \mathbb{N}$.

iv) $f(D_{c_n-1,0}) = D_{c_{n-1}-1,0} \setminus \{d_{0,c_{n-1}-1,0}\}$ for each $n \neq 0$.

v) $f(D_{0,1}) = \{d_{0,c_{n+1}-1,0} \colon n \in \mathbb{N}\}$.

vi) $f(D_{0,0}) = \{d_{0,0,k+2} \colon k \in \mathbb{N}\}$.

vii) $f(D_{1,0}) = D_{3,0}$ and $f(D_{2,0}) = D_{0,0}$.

viii) $f(D_{0,2}) = \{d_{0,j,1} \colon j \in \mathbb{N}\}$ and $f(D_{0,k}) = \{d_{0,j+1,k-1} \colon j \in \mathbb{N}\}$ for each $k > 2$.

Observe that clauses a) and i) imply that f is continuous at d, at d_i for each $i > 0$ and at d_{ij} for every $i, j > 0$. For $j > 1$, f is continuous at d_{0j} by clauses g) and viii). By d), e), ii), iii) and iv), we have that f is continuous at d_0 and at $d_{i,0}$ for every $i > 2$. By b), v), vi) and vii), f is continuous at the points $d_{0,0}$, d_{10}, d_{20} and $d_{0,1}$. Therefore, f is continuous.

It is evident that the orbit $\mathcal{O}_f(d_{0,0,0})$ is dense. Also it is evident that the points $d_{c_n-1,0}$ and $d_{c_n+1,0}$ have infinite orbits for every $n \in \mathbb{N}$. The follow relationships follows directly from the definition:

I) $f^p(d) = d$ and $f^p(d_n) = d_n$ for each $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$.

II) $f^p(d_{0,0}) = d$, $f^p(d_{2,0}) = d$ and $f^p(d_{0,1}) = d_0$.

III) $f^p(d_{c_n,0}) = d$ for each $n > 0$.

IV) $f^p(d_{c_n+1,0}) = d_0 = f^p(d_{c_n-1,0})$ for each $n \in \mathbb{N}$.

V) $f^p(d_{j,k}) = d_{k-1}$ for each $j > 0$ and $k > 0$.

All these properties imply clause (3), condition (4) follows from Theorem 4.2 and the last condition (5) follows from clauses III) and IV).

Concerning the previous example we formulate the following question.

Question 4.7. Is there a continuous function $f\colon \omega^3+1 \to \omega^3+1$ such (ω^3+1, f) is transitive and $E(\omega^3+1, f)^*$ contains both continuous and discontinuous functions?

With respect to Theorem 4.2, it is natural to ask the following.

Question 4.8. Let $(\omega^\alpha + 1, f)$ be a transitive dynamical system where $\alpha \geq 1$ is a countable ordinal. If $1 < \sup\{|\omega_f(x)|\colon x \in (\omega^\alpha + 1)'\} < \omega$, is $E(\omega^\alpha + 1, f)$ homeomorphic to $\omega^\beta + 1$ for some countable ordinal $\beta \geq 1$?

Acknowledgement. We thank the referee for providing constructive comments that help improving the contents of this paper.

References

[1] Akin E., Glasner E., *WAP systems and labeled subshifts*, Mem. Amer. Math. Soc. **262** (2019), no. 1265, v+116 pages.

[2] Blass A., *Ultrafilters: where topological dynamics = algebra = combinatorics*, Topology Proc. **18** (1993), 33–56.

[3] Bernstein A. R., *A new kind of compactness for topological spaces*, Fund. Math. **66** (1969/70), 185–193.

[4] Ellis R., *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272–281.

[5] Ellis R., Nerurkar M., *Weakly almost periodic flows*, Trans. Amer. Math. Soc. **313** (1989), no. 1, 103–119.

[6] Furstenberg H., *Recurrence in Ergodic Theory and Combinatorial Number Theory*, M. B. Porter Lectures, Princeton University Press, Princeton, 1981.

[7] García-Ferreira S., *Dynamical properties of certain continuous self maps of the Cantor set*, Topology Appl. **159** (2012), no. 7, 1719–1733.

[8] García-Ferreira S., Rodriguez-López Y., Uzcátegui C., *Iterates of dynamical systems on compact metrizable countable spaces*, Topology Appl. **180** (2015), 100–110.

[9] García-Ferreira S., Rodríguez-López Y., Uzcátegui C., *Cardinality of the Ellis semigroup on compact metric countable spaces*, Semigroup Forum **97** (2018), no. 1, 162–176.
[10] García-Ferreira S., Sanchis M., *Ultrafilter-limit points in metric dynamical systems*, Comment. Math. Univ. Carolin. **48** (2007), no. 3, 465–485.
[11] Glasner E., *Enveloping semigroups in topological dynamics*, Topology Appl. **154** (2007), no. 11, 2344–2363.
[12] Hindman N., *Ultrafilters and Ramsey theory-an update*, Set Theory and Its Applications, Lecture Notes in Mathematics, 1401, Springer, Berlin, 1989, pages 97–118.
[13] Mazurkiewicz S., Sierpiński W., *Contribution à la topologie des ensembles dénombrables*, Fundamenta Mathematicae **1** (1920), 17–27 (Spanish).
[14] Szuca P., *$\mathcal{F}$-limit points in dynamical systems defined on the interval*, Cent. Eur. J. Math. **11** (2013), no. 1, 170–176.

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(*Received* August 20, 2019, *revised* May 9, 2020)