# Boundedness and compactness of some operators on discrete Morrey spaces

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*Abstract.* We consider discrete versions of Morrey spaces introduced by Gunawan et al. in papers published in 2018 and 2019. We prove continuity and compactness of multiplication operators and commutators acting on them.

*Keywords:* discrete Morrey space; multiplication operator; compactness

*Classification:* 42B35, 46B45, 46B50

## 1. Introduction

Morrey spaces have been studied extensively by many authors on several contexts and for different purposes since the seminal paper [6] by Morrey. These spaces can be considered as generalizations of the classical Lebesgue spaces, BMO and Lipschitz spaces, and turn out to be very suitable for certain kind of classical operators in analysis and partial differential equations.

Recently, H. Gunawan et al. in [2], [3] have introduced discrete versions of these spaces, and they have studied inclusion relations, and the action of the Hardy–Littlewood maximal operator and Riesz potentials on them.

The analysis of discrete versions of several function spaces, or classical operators in harmonic analysis, has been a subject of great interest along the years. Its study goes back to several authors, for example, [5], [7], [8], [9], just to mention a few.

Our goal is to study properties of continuity and compactness of classical operators such as convolutions, multiplication operators and commutators acting on these sequence spaces. In order to do that, the first step will be to obtain a sufficient condition to determine if a family in this sequence space is totally bounded. Later, we will apply this result to obtain compactness of multiplication operators and commutators. All of these will be done in the following sections of this work.

We will employ standard notation along this note, and as usual, we shall denote by $C$ a constant, probably different at different occurrences.

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## 2. Preliminaries

To define the discrete Morrey spaces $l_q^p$ we will be always assuming along this note that $1 \le p \le q < \infty$.

The discrete Morrey spaces for dimension $n = 1$ have been defined by H. Gunawan et al., see [2], in the following way:

Given $m \in \mathbb{Z}$, $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set

$$l_q^p = \{x = (x_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \colon \|x\|_{l_q^p} < \infty\},$$

where

$$\|x\|_{l_q^p} = \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{1/q-1/p} \left( \sum_{k=m-N}^{m+N} |x_k|^p \right)^{1/p},$$

is called a discrete Morrey space.

It turns out that $\left(l_q^p, \|\cdot\|_{l_q^p}\right)$ is a Banach space such that $l_p^p = l^p$ is the classical space of $p$-summable sequences indexed by $\mathbb{Z}$. Also, it can be viewed as a subspace of the continuous Morrey space $M_q^p(\mathbb{R})$ equipped with the norm

$$\|f\|_{M_q^p} = \sup_{a\in\mathbb{R}, r>0} r^{1/q-1/p} \left( \int_{a-r}^{a+r} |f(y)|^p \, \mathrm{d}y \right)^{1/p},$$

if we consider a sequence $(x_k)_{k\in\mathbb{Z}} \in l_q^p$ as a step function

$$\sum_{k\in\mathbb{Z}} x_k \chi_{[k-1/2,k+1/2)}.$$

However, as in [2], we will consider here $l_q^p$ as a space on its own.

In the paper [3], the authors also consider the Hardy–Littlewood maximal operator $M$ on the space $l_q^p$, in fact, they proved that this operator is bounded from $l_q^p$ into itself for $1 < p \le q < \infty$. The operator $M$ is defined for each $m \in \mathbb{Z}$ by

$$Mx(m) = \sup_{N\in\mathbb{N}_0} \frac{1}{2N+1} \sum_{k=m-N}^{m+N} |x_k|.$$

As we mentioned in the introduction of this paper, we first state a result about precompactness for families in $l_q^p$ that will be very useful to prove compactness of some operators acting on these spaces. In order to state it, we will make use of the following lemma that was proved in [4].

**Lemma 1** ([4])**.** *Let $X$ be a metric space. Assume that for every $\varepsilon > 0$, there exists $\delta > 0$, a metric space $W$, and a mapping $\Phi \colon X \to W$ so that $\Phi(X)$ is totally bounded, and whenever $x, y \in X$ are such that $d_W(\Phi(x), \Phi(y)) < \delta$, then $d_X(x, y) < \varepsilon$. Then $X$ is totally bounded.*

Our corresponding result on precompactness is as follows.

**Lemma 2.** *Let $\mathcal{F} = \{x_\alpha : \alpha \in \mathcal{A}\}$ a subset of $l^p_q$. Assume that the following conditions hold:*

(1) *There exists $C > 0$ such that $\|x_\alpha\|_{l^p_q} \le C$ for every $\alpha \in \mathcal{A}$.*
(2) *Given $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that*

$$\frac{1}{(2N+1)^{1/p-1/q}} \Bigg( \sum_{|k-m| \le N,\, |k|>K} |x_\alpha(k)|^p \Bigg)^{1/p} < \varepsilon$$

*for each $\alpha \in \mathcal{A}$, each $N \in \mathbb{N}_0$ and every $m \in \mathbb{Z}$.*

*Then, the set $\mathcal{F}$ is totally bounded.*

Proof: Define $\Phi : \mathcal{F} \to \mathbb{R}^{2K+3}$ by

$$\Phi(x_\alpha) = (x_\alpha(-K-1), \dots, x_\alpha(K+1)).$$

Assumption 1 implies the existence of a constant $C_1 = C_1(K,p,q)$ such that

$$\|\Phi(x_\alpha)\|_{(\mathbb{R}^{2K+3}, \|\cdot\|_2)} \sim \Bigg( \sum_{k=-K-1}^{K+1} |x_\alpha(k)|^p \Bigg)^{1/p} \le C_1$$

for every $\alpha \in \mathcal{A}$.

Since in finite dimensions boundedness and total boundedness are equivalent, we conclude that $\Phi(\mathcal{F})$ is a totally bounded subset of $\mathbb{R}^{2K+3}$.

Now, given $\varepsilon > 0$, if $\alpha, \beta \in \mathcal{A}$ and

$$\|\Phi(x_\alpha) - \Phi(x_\beta)\|_{(\mathbb{R}^{2K+3}, \|\cdot\|_p)} < \varepsilon, \tag{1}$$

we will have that for any $N \in \mathbb{N}_0$ and any $m \in \mathbb{Z}$

$$\begin{aligned} \frac{1}{(2N+1)^{1/p-1/q}} & \Bigg( \sum_{k=m-N}^{m+N} |x_\alpha(k) - x_\beta(k)|^p \Bigg)^{1/p} \\ & \le \frac{1}{(2N+1)^{1/p-1/q}} \Bigg( \sum_{|k-m| \le N,\, |k|>K} |x_\alpha(k) - x_\beta(k)|^p \Bigg)^{1/p} \\ & \quad + \frac{1}{(2N+1)^{1/p-1/q}} \Bigg( \sum_{|k-m| \le N,\, |k| \le K} |x_\alpha(k) - x_\beta(k)|^p \Bigg)^{1/p} \\ & < 2\varepsilon + \varepsilon \end{aligned}$$

because of assumption 2 and estimate (1). Thus

$$\|x_\alpha - x_\beta\|_{l^p_q} < 3\varepsilon.$$

Finally, we invoke Lemma 1 to conclude that $\mathcal{F}$ is totally bounded. □

The next result proves the continuity of an appropriate convolution operator on the sequence space $l^p_q$.

**Proposition 3.** *Let $y \in l^1$ and define for $1 \leq p \leq q < \infty$, $x \in l^p_q$, the sequence $C_y$, where*

$$C_y(k) = x * y(k) = \sum_{n \in \mathbb{Z}} x(n)y(k-n).$$

*Then, the operator $C_y$ is bounded from $l^p_q$ into itself.*

Proof: Given any $m \in \mathbb{Z}$ and any $N \in \mathbb{N}_0$, we can apply Minkowski's inequality for integrals to obtain

$$\begin{aligned}\left( \sum_{k=m-N}^{m+N} |C_y(k)|^p \right)^{1/p} &= \left( \sum_{k=m-N}^{m+N} \left| \sum_{j \in \mathbb{Z}} x(j)y(k-j) \right|^p \right)^{1/p} \\ &= \left( \sum_{k=m-N}^{m+N} \left| \sum_{l \in \mathbb{Z}} x(k-l)y(l) \right|^p \right)^{1/p} \\ &\leq \sum_{l \in \mathbb{Z}} \left( \sum_{s=m-N-l}^{m+N-l} |x(s)|^p \right)^{1/p} |y(l)| \\ &= \sum_{l \in \mathbb{Z}} \left( \sum_{s=(m-l)-N}^{(m-l)+N} |x(s)|^p \right)^{1/p} |y(l)| \\ &\leq \|y\|_{l^1} \|x\|_{l^p_q} (2N+1)^{1/p-1/q}\end{aligned}$$

and this implies that

$$\|C_y(x)\|_{l^p_q} \leq \|y\|_{l^1} \|x\|_{l^p_q},$$

as we wanted to show. □

## 3. Main results

In this section we study multiplication operators and commutators. We will see that under appropriate conditions, both are compact operators. We adapt to our discrete case some of the ideas employed in [1].

For a sequence $z$, let us denote by $M_z$ the multiplication operator $M_z(x) = xz = (x_k z_k)_{k \in \mathbb{Z}}$.

**Theorem 4.** *Let $y \in l^1$, $z \in l^\infty$.*

a) *For $1 \leq p \leq q < \infty$ the operator $M_z$ is bounded from $l^p_q$ into itself.*
b) *If additionally $z \in c_0$, the space of sequences converging to zero, then, the operator $M_z C_y$ is compact on $l^p_q$.*

PROOF: Let us consider $x \in l^p_q$.

Then for each $m \in \mathbb{Z}$ and each $N \in \mathbb{N}_0$ we have

$$\Big(\sum_{k=m-N}^{m+N} |x_k|^p\Big)^{1/p} \le \|x\|_{l^p_q}(2N+1)^{1/p-1/q}$$

and this clearly implies

$$\Big(\sum_{k=m-N}^{m+N} |x_k z_k|^p\Big)^{1/p} \le \|z\|_{l^\infty}\|x\|_{l^p_q}(2N+1)^{1/p-1/q},$$

in other words, $M_z(x) \in l^p_q$ and

$$\|M_z(x)\|_{l^p_q} \le \|z\|_{l^\infty}\|x\|_{l^p_q},$$

which shows part a).

Now, assume that $z \in c_0$ and let $B$ be any bounded set of $l^p_q$.

Thus, there exists $M > 0$ such that for every $x \in B$

$$\|x\|_{l^p_q} \le M.$$

We need to check that $\{M_z C_y(x)\colon x \in B\}$ is a relatively compact subset of $l^p_q$, and for this, it will suffice to prove that both conditions of Lemma 2 are satisfied.

Let $x \in B$, $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$.

By Minkowski's inequality for integrals we have

$$\begin{aligned}\Big(\sum_{k=m-N}^{m+N} |M_z C_y(x)(k)|^p\Big)^{1/p} &= \Big(\sum_{k=m-N}^{m+N} \Big|\sum_{j\in\mathbb{Z}} x(j)y(k-j)z_k\Big|^p\Big)^{1/p}\\ &\le \|y\|_{l^1}\|x\|_{l^p_q}(2N+1)^{1/p-1/q}\|z\|_{l^\infty},\end{aligned}$$

and this implies

$$\|M_z C_y(x)\|_{l^p_q} \le M\|z\|_{l^\infty}\|y\|_{l^1}$$

uniformly on $x \in B$, therefore, condition (1) in Lemma 2 is satisfied.

Now, let $\varepsilon > 0$. Let $K$ be a fixed natural number whose size will be chosen appropriately later.

Then, for any $m \in \mathbb{Z}$ and any $N \in \mathbb{N}_0$, we will get for every $x \in B$

$$\begin{aligned}\Big(\sum_{|k-m|\le N,\, |k|>K} |M_z C_y(x)(k)|^p\Big)^{1/p} &\le \|y\|_{l^1}\|x\|_{l^p_q}(2N+1)^{1/p-1/q}\sup_{|k|>K}|z_k|\\ &< M\varepsilon\|y\|_{l^1}(2N+1)^{1/p-1/q},\end{aligned}$$

as long as $K$ is selected large enough. This shows that condition (2) in Lemma 2 holds. $\square$

The following lemma although very simple, will be useful to prove compactness of the commutator $[M_z, C_y]$.

**Lemma 5.** *For $z \in c_0$ and $y \in l^1$ define the sequence*

$$\xi(n) := \sum_{k\in\mathbb{Z}} |z_n - z_{n-k}||y_k|, \qquad n \in \mathbb{Z}.$$

*Then, $\xi \in c_0$.*

Proof: Clearly $\xi \in l^\infty$. Given $\varepsilon > 0$ let us select $K_1$ large enough so that

$$\sum_{|k|>K_1} |y_k| < \varepsilon.$$

Now, for $n \in \mathbb{Z}$ we have

$$\begin{aligned}\xi(n) &= \sum_{|k|\le K_1} |z_n - z_{n-k}||y_k| + \sum_{|k|>K_1} |z_n - z_{n-k}||y_k| \\ &\le \sum_{|k|\le K_1} |z_n - z_{n-k}||y_k| + 2\|z\|_{l^\infty}\varepsilon.\end{aligned}$$

Since $z \in c_0$, we can choose $K \gg K_1$ such that for $|n| > K$

$$\sum_{|k|\le K_1} |z_n - z_{n-k}||y_k| \le \varepsilon\|y\|_{l^1}.$$

Finally, collecting the previous estimates we get

$$0 \le \xi(n) < \varepsilon\|y\|_{l^1} + 2\|z\|_{l^\infty}\varepsilon$$

for $|n| > K$, that is, $\xi \in c_0$. □

**Remark 6.** Notice that the previous proof works even for $z \in c$, the space of convergent sequences.

**Theorem 7.** *Let $z \in c_0$, $y \in l^1$, $1 < p \le q < \infty$. Then, the commutator $[M_z, C_y]$ is a compact operator on $l_q^p$.*

Proof: By Lemma 5, given $\varepsilon > 0$ there exists $K$ large enough such that

$$\sup_{|n|>K} \sum_{k\in\mathbb{Z}} |z_n - z_{n-k}||y_k| < \varepsilon.$$

Let us remind that the commutator of the operators $M_z$ and $C_y$ is defined as

$$[M_z, C_y] := M_z C_y - C_y M_z.$$

Hence, given a sequence $x = (x_n)_{n\in\mathbb{Z}}$, we can proceed formally as follows:

For $n \in \mathbb{Z}$

$$
\begin{aligned}
|[M_z, C_y](x)(n)| &\leq \sum_{k\in\mathbb{Z}} |z_n - z_{n-k}||x_{n-k}||y_k| \\
&= \sum_{k\in\mathbb{Z}} |z_n - z_{n-k}|^{1/p'} |y_k|^{1/p'} |z_n - z_{n-k}|^{1/p} |y_k|^{1/p} |x_{n-k}| \\
&\leq 2^{1/p} \|z\|_{l^\infty}^{1/p} \Bigg( \sum_{k\in\mathbb{Z}} |z_n - z_{n-k}||y_k| \Bigg)^{1/p'} \Bigg( \sum_{k\in\mathbb{Z}} |x_{n-k}|^p |y_k| \Bigg)^{1/p}. \qquad (2)
\end{aligned}
$$

If $T_K$ denotes the operator of truncation, that is

$$x \longmapsto (\ldots, 0, x_{-K}, \ldots, x_K, 0, \ldots),$$

and $I$ denotes the identity operator, we will have for $|n| > K$ using (2)

$$
\begin{aligned}
&|(I - T_K)[M_z, C_y](x)(n)| \\
&\qquad \leq (2\|z\|_{l^\infty})^{1/p} \sup_{|n|>K} \Bigg( \sum_{k\in\mathbb{Z}} |z_n - z_{n-k}||y_k| \Bigg)^{1/p'} \Bigg( \sum_{k\in\mathbb{Z}} |x_{n-k}|^p |y_k| \Bigg)^{1/p} \\
&\qquad \leq (2\varepsilon^{p/p'} \|z\|_{l^\infty})^{1/p} \Bigg( \sum_{k\in\mathbb{Z}} |x_{n-k}|^p |y_k| \Bigg)^{1/p}.
\end{aligned}
$$

Now, taking into consideration that $(I - T_K)(\xi)(n) = 0$ for $|n| \leq K$, for any sequence $\xi$, we get for $x \in l^p_q$, for every $m \in \mathbb{Z}$ and for every $N \in \mathbb{N}_0$

$$
\begin{aligned}
&\Bigg[ \sum_{n=m-N}^{m+N} |(I - T_K)[M_z, C_y](x)(n)|^p \Bigg]^{1/p} \\
&\qquad \leq (2\varepsilon^{p/p'} \|z\|_{l^\infty})^{1/p} \Bigg( \sum_{n=m-N}^{m+N} \sum_{k\in\mathbb{Z}} |x_{n-k}|^p |y_k| \Bigg)^{1/p} \\
&\qquad = (2\varepsilon^{p/p'} \|z\|_{l^\infty})^{1/p} \Bigg( \sum_{k\in\mathbb{Z}} |y_k| \Bigg( \sum_{j=m-N-k}^{m+N-k} |x_j|^p \Bigg) \Bigg)^{1/p} \\
&\qquad \leq (2\varepsilon^{p/p'} \|z\|_{l^\infty})^{1/p} \|y\|_{l^1}^{1/p} \|x\|_{l^p_q} (2N+1)^{1/p-1/q},
\end{aligned}
$$

which implies that

$$\|(I - T_K)[M_z, C_y]\|_{l^p_q \to l^p_q} \leq (2\varepsilon^{p/p'} \|z\|_{l^\infty})^{1/p} \|y\|_{l^1}^{1/p},$$

or equivalently

$$\|[M_z, C_y] - T_K[M_z, C_y]\|_{l^p_q \to l^p_q} \leq (2\varepsilon^{p/p'} \|z\|_{l^\infty})^{1/p} \|y\|_{l^1}^{1/p}. \qquad (3)$$

Since the operator $T_K[M_z, C_y]$ is compact because it is of finite rank, the relation (3) implies that $[M_z, C_y]$ is compact.

The proof of Theorem 7 is completed. $\square$

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