Comment.Math.Univ.Carolin. 62,2 (2021) 243–257

Coarse homotopy on metric spaces and their corona

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Abstract. This paper discusses properties of the Higson corona by means of a quotient on coarse ultrafilters on a proper metric space. We use this description to show that the corona functor is faithful and reflects isomorphisms.

Keywords: Higson corona; coarse geometry

Classification: 51F99, 54H99

1. Introduction

The corona $\nu'(X)$ of a metric space X has been introduced in [12] and studied in [13], [14], [3], [15], [7], [8].

The Stone–Čech compactification is a functor β from the category of completely regular spaces to the category of compact Hausdorff spaces. Note that by [1, Theorem 2.1] if X is a completely regular space and G a group then

$$\check{H}^n_F(X;G) = \check{H}^n(\beta X, G).$$

The left side denotes n-dimensional Čech type functional cohomology based on finite open covers and the right side denote n-dimensional Čech cohomology.

This resembles [8, Corollary 35] where sheaf cohomology based on finite coarse covers of a metric space X is related to sheaf cohomology on the corona $\nu'(X)$. This property and other properties which we are going to discuss in this paper suggest that the corona functor is the Stone–Čech boundary version of a space in the coarse category.

We start with the first quite elementary property:

Theorem A. *If* mCoarse *denotes the category of metric spaces and coarse maps modulo close and* Top *the category of topological spaces and continuous maps then the functor*

$$\nu' \colon \mathrm{mCoarse} \to \mathrm{Top}$$

is faithful.

DOI 10.14712/1213-7243.2021.011

A direct consequence of this result is that ν' reflects isomorphisms.

We examine in which way the corona functor ν' is related to the Higson corona ν of [17]. Originally the Higson corona has been defined on a proper metric space X as the boundary of the compactification determined by an algebra of bounded functions called the Higson functions. Already [13] showed that there exists a homeomorphism $\nu(X) = \nu'(X)$. We provide an explicit homeomorphism and show ν, ν' agree on morphisms too.

Theorem B. *If X is a proper metric space then there is a homeomorphism*

$$\nu'(X) \to \nu(X).$$

Here the right side denotes the Higson corona of [17]. *If $f\colon X \to Y$ is a coarse map between proper metric spaces then $\nu'(f)$, $\nu(f)$ are homeomorphic (the same map pre-and postcomposed by a homeomorphism).*

The asymptotic product of two metric spaces has been introduced in [9] as the limit of a pullback diagram in the coarse category. Note [6, Theorem 1] shows the following: If X, Y are hyperbolic proper geodesic metric spaces then their asymptotic product $X * Y$ is hyperbolic proper geodesic and therefore its Gromov boundary $\partial(X*Y)$ is defined. There is a homeomorphism $\partial(X*Y) = \partial(X)\times\partial(Y)$ which is the main result of [6].

If the asymptotic product $X * Y$ of two metric spaces X, Y is well defined then $\nu'(X * Y)$ is the pullback of

$$\begin{tikzcd} & \nu'(Y) \arrow[d, "\nu'(d(\cdot,q))"] \\ \nu'(X) \arrow[r, "\nu'(d(\cdot,p))"'] & \nu'(\mathbb{Z}_+) \end{tikzcd}$$

2. Metric spaces

Definition 1. Let (X, d) be a metric space. Then the *coarse structure associated to d* on X consists of those subsets $E \subseteq X^2$ for which

$$\sup_{(x,y)\in E} d(x,y) < \infty.$$

We call an element of the coarse structure *entourage*. In what follows we assume the metric d to be finite for every $(x, y) \in X^2$.

Definition 2. A map $f\colon X \to Y$ between metric spaces is called *coarse* if

- $E \subseteq X^2$ being an entourage implies that $f^{\times 2}(E)$ is an entourage (*coarsely uniform*);
- and if $A \subseteq Y$ is bounded then $f^{-1}(A)$ is bounded (*coarsely proper*).

Two maps $f, g\colon X \to Y$ between metric spaces are called *close* if

$$f \times g(\Delta_X)$$

is an entourage in Y. Here Δ_X denotes the diagonal in X^2.

Notation 3. A map $f\colon X \to Y$ between metric spaces is called

- *coarsely surjective* if there is an entourage $E \subseteq Y^2$ such that
$$E[\operatorname{im} f] = Y;$$
- *coarsely injective* if for every entourage $F \subseteq Y^2$ the set $(f^{\times 2})^{-1}(F)$ is an entourage in X.

Two subsets $A, B \subseteq X$ are called *not coarsely disjoint* if there is an entourage $E \subseteq X^2$ such that the set

$$E[A] \cap E[B]$$

is not bounded. We write $A \curlywedge B$ in this case.

Two subsets $A, B \subseteq X$ are called *asymptotically alike* if there is an entourage $E \subseteq X^2$ such that

$$E[A] = B.$$

We write $A \lambda B$ in this case.

Remark 4. We study metric spaces up to coarse equivalence. A coarse map $f\colon X \to Y$ between metric spaces is a *coarse equivalence* if:

- there is a coarse map $g\colon Y \to X$ such that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X;
- or equivalently if f is both coarsely injective and coarsely surjective.

Notation 5. If X is a metric space and $U_1, \ldots, U_n \subseteq X$ are subsets, then $(U_i)_i$ are said to *coarsely cover* X if for every entourage $E \subseteq X^2$ the set

$$E[U_1^c] \cap \cdots \cap E[U_n^c]$$

is bounded.

3. The corona functor

Definition 6. If X is a metric space a system $\mathcal{F}$ of subsets of X is called a *coarse ultrafilter* if

(1) $A, B \in \mathcal{F}$ then $A \curlywedge B$;
(2) $A, B \subseteq X$ are subsets with $A \cup B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$;
(3) $X \in \mathcal{F}$.

Lemma 7. *If $f\colon X \to Y$ is a coarse map between metric spaces and $\mathcal{F}$ is a coarse ultrafilter on X then*

$$f_*\mathcal{F} := \{A \subseteq Y\colon f^{-1}(A) \in \mathcal{F}\}$$

is a coarse ultrafilter on Y.

PROOF: See [8]. □

Definition 8. We define a relation on coarse ultrafilters on X: two coarse ultrafilters $\mathcal{F}, \mathcal{G}$ are *asymptotically alike*, written $A\lambda B$ if for every $A \in \mathcal{F}$, $B \in \mathcal{G}$:

$$A \curlywedge B.$$

Remark 9. By [8] the relation λ is an equivalence relation on coarse ultrafilters on X. If two coarse ultrafilters $\mathcal{F}, \mathcal{G}$ on X are asymptotically alike and $f\colon X \to Y$ is a coarse map to a metric space Y then $f_*\mathcal{F}\lambda f_*\mathcal{G}$ on Y.

Definition 10. Let X be a metric space. Denote by $\nu'(X)$ the set of coarse ultrafilters modulo asymptotically alike on X. The relation "$\curlywedge$" on subsets of $\nu'(X)$ is defined as follow. Define, for a subset $A \subseteq X$,

$$\mathsf{cl}(A) = \{[\mathcal{F}] \in \nu'(X)\colon A \in \mathcal{F}\}.$$

Then $\pi_1 \not\curlywedge \pi_2$ if and only if there exist subsets $A, B \subseteq X$ such that $A \not\curlywedge B$ and $\pi_1 \subseteq \mathsf{cl}(A), \pi_2 \subseteq \mathsf{cl}(B)$.

Remark 11. The relation "$\curlywedge$" on subsets of $\nu'(X)$ defines a proximity relation on $\nu'(X)$ which induces a compact topology. By [8] the mapping f_* between coarse ultrafilters induces a continuous map $\nu'(f)$ between the quotients. Thus ν' is a functor mapping coarse metric spaces to compact topological spaces.

The topology on $\nu'(X)$ is generated by $(\mathsf{cl}(A))^c_{A\subseteq X}$. Coarse covers determine finite open covers, see [8].

4. On morphisms

Lemma 12. *Let $f\colon X \to Y$ be a map between metric spaces. Then*

(1) *a map f is a coarse map if*
   - *for any bounded subset $B \subseteq X$, $f(B)$ is bounded;*
   - *for every subsets $A, B \subseteq X$, the relation $A \curlywedge B$ implies $f(A) \curlywedge f(B)$;*

(2) *a coarse map f is coarsely injective if $A \not\curlywedge B$ implies $f(A) \not\curlywedge f(B)$;*

(3) *a map f is coarsely surjective if the relation $f(X) \not\curlywedge B$ in Y implies B is bounded.*

Proof: (1) First we show f is coarsely proper. If $B \subseteq Y$ is bounded then $B \not\curlywedge Y$. This implies $f^{-1}(B) \not\curlywedge X$. Thus $f^{-1}(B)$ is bounded.

Now we show f is coarsely uniform: Suppose $A, B \subseteq X$ are two subsets with $f(A)\bar{\lambda}f(B)$. Then there is an unbounded subset $S \subseteq f(A)$ with $S \not\curlywedge f(B)$ or there is an unbounded subset $T \subseteq f(B)$ with $T \not\curlywedge f(A)$. Assume the former. Then $f^{-1}(S) \not\curlywedge B$. Since f maps bounded sets to bounded sets the set $f^{-1}(S) \cap A$ is unbounded. Thus $A\bar{\lambda}B$. Thus we have shown $A\lambda B$ implies $f(A)\lambda f(B)$. By [11, Theorem 2.3] we can conclude that f is coarsely uniform.

(2) This is [8, Lemma 41].

(3) This is easy. □

Theorem 13. *If $f, g\colon X \to Y$ are two coarse maps between metric spaces and $\nu'(f) = \nu'(g)$ then f, g are close.*

Proof: Suppose f, g are not close. By [11, Proposition 2.15] there is a subset $A \subseteq X$ with $f(A)\bar{\lambda}g(A)$. This implies there is a subset $S \subseteq A$ with $f(S) \not\curlywedge g(S)$. Now by [7, Proposition 4.7] there is a coarse ultrafilter $\mathcal{F}$ on X with $S \in \mathcal{F}$. Then $f(S) \in f_*\mathcal{F}$ and $g(S) \in g_*\mathcal{F}$. Since $f(S) \not\curlywedge g(S)$ this implies $f_*\mathcal{F} \neq g_*\mathcal{F}$. Thus $\nu'(f)$, $\nu'(g)$ are not the same map. □

Corollary 14. *If* mCoarse *denotes the category of metric spaces and coarse maps modulo close and* Top *the category of topological spaces and continuous maps then the functor*

$$\nu'\colon \mathrm{mCoarse} \to \mathrm{Top}$$

is faithful.

Corollary 15. *The functor* $\nu'\colon \mathrm{mCoarse} \to \mathrm{Top}$ *reflects epimorphisms and monomorphisms.*

Proof: It is general theory that a faithful functor reflects epimorphisms and monomorphisms. This fact can also be found in [16, Exercise 1.6. vii]. Since by Corollary 14 the functor ν' is faithful the result follows. □

Corollary 16. *The functor* $\nu'\colon \mathrm{mCoarse} \to \mathrm{Top}$ *reflects isomorphisms.*

Proof: Suppose $f\colon X \to Y$ is a coarse map between metric spaces such that $\nu'(f)$ is an isomorphism in Top. Then $\nu'(f)$ is both a monomorphism and an epimorphism. The proof of [8, Theorem 40] can be generalized to hold for metric spaces. Then the map f is coarsely surjective. By Corollary 15 the map f is a monomorphism in mCoarse. By a proof similar to the one of [4, Proposition 3.A.16] every monomorphism is coarsely injective. Since f is coarsely injective and coarsely surjective it is a coarse equivalence. □

Definition 17. Let Y be a locally compact topological space. A bounded continuous map $\varphi\colon Y \to \mathbb{R}$ is said to *vanish at infinity* if for every $\varepsilon > 0$ there is a compact set $K \subseteq Y$ such that $y \in K^c$ implies $|f(y)| < \varepsilon$.

Definition 18. Let $R > 0$ be a number. A metric space X is said to be R-*discrete* if for every $x, y \in X$ the relation $x \neq y$ implies $d(x,y) \geq R$.

Theorem 19. *If X is a proper metric space then there is a homeomorphism*

$$\nu'(X) \to \nu(X).$$

Here the right side denotes the Higson corona of [17]. *If $f\colon X \to Y$ is a coarse map between proper metric spaces then $\nu'(f)$, $\nu(f)$ are homeomorphic (the same map pre-and postcomposed by a homeomorphism).*

PROOF: Let X be a proper metric space. First we show that $h'(X) := X \sqcup \nu'(X)$ is a compactification of X. Closed sets on $h'(X)$ are generated by $(\bar{A} \cup \mathsf{cl}(A))_{A \subseteq X}$. We show this topology is compact. If $(\bar{A}_i \cup \mathsf{cl}(A_i))_i^c$ is an open cover of $h'(X)$ then there is a subcover

$$(\bar{A}_1 \cup \mathsf{cl}(A_1))_1^c, \ldots, (\bar{A}_n \cup \mathsf{cl}(A_n))^c$$

such that $\mathsf{cl}(A_1)^c, \ldots, \mathsf{cl}(A_n)^c$ is a cover of $\nu'(X)$. Now this implies $A_1^c, \ldots, A_n^c$ are a coarse cover of X. Thus $\bar{A}_1 \cap \cdots \cap \bar{A}_n$ is both bounded and closed. Then there is a subcover

$$(\bar{A}_{n+1} \cup \mathsf{cl}(A_{n+1}))^c, \ldots, (\bar{A}_{n+m} \cup \mathsf{cl}(A_{n+m}))^c$$

of $(\bar{A}_i \cup \mathsf{cl}(A_i))_i^c$ such that $\bar{A}_{n+1}^c, \ldots, \bar{A}_{n+m}^c$ covers $\bar{A}_1 \cap \cdots \cap \bar{A}_n$. Then

$$(\bar{A}_1 \cup \mathsf{cl}(A_1))^c, \ldots, (\bar{A}_{n+m} \cup \mathsf{cl}(A_{n+m}))^c$$

are a subcover of $(\bar{A}_i \cup \mathsf{cl}(A_i))_i^c$ that cover $h'(X)$.

Now $X, \nu'(X)$ both appear as subspaces of $h'(X)$. We show the inclusion $X \to h'(X)$ is dense:

$$\overline{X}^{h'} = \bigcap_{\bar{A} \cup \mathsf{cl}(A) \supseteq X} (\bar{A} \cup \mathsf{cl}(A)) = X \cup \mathsf{cl}(X) = h'(X).$$

The Higson compactification $h(X)$ is determined by the C^*-algebra of Higson functions whose definition we now recall from [17]: A bounded continuous function $\varphi\colon X \to \mathbb{R}$ is called *Higson* if the function

$$\begin{aligned} d\varphi\colon\quad & X^2 \to \mathbb{R} \\ & (x,y) \mapsto \varphi(y) - \varphi(x) \end{aligned}$$

when restricted to E vanishes at infinity for every entourage $E \subseteq X^2$.

Note [13, Proposition 1] shows Higson functions on X can be extended to $h'(X)$. For the convenience of the reader we recall it.

Without loss of generality assume that X is R-discrete for some $R > 0$. Then every coarse ultrafilter $\mathcal{F}$ on X is determined by an ultrafilter σ on X by the proof of [8, Theorem 17]. If σ is an ultrafilter on X then a bounded continuous function $\varphi\colon X \to \mathbb{R}$ determines an ultrafilter $\varphi_*\sigma := \{A\colon \varphi^{-1}(A) \in \sigma\}$ on $\mathbb{R}$. Since the image of φ is bounded and therefore relatively compact the ultrafilter $\varphi_*\sigma$ converges to a point $\sigma\text{-}\lim\varphi \in \mathbb{R}$.

If two ultrafilters σ, τ induce asymptotically alike coarse ultrafilters and φ is a Higson function then $\sigma\text{-}\lim\varphi = \tau\text{-}\lim\varphi$: Suppose $\sigma\text{-}\lim\varphi \neq \tau\text{-}\lim\varphi$. Then there exist neighborhoods $U \ni \sigma\text{-}\lim\varphi$ and $V \ni \tau\text{-}\lim\varphi$ such that $d(U,V) > 0$. Let $E \subseteq X^2$ be an entourage. Then

$$\begin{aligned} d\varphi\colon \varphi^{-1}(U) \times \varphi^{-1}(V) \cap E &\to \mathbb{R} \\ (x,y) &\to \varphi(y) - \varphi(x) \end{aligned}$$

vanishes at infinity. Since $d(U,V) > 0$ this implies that $\varphi^{-1}(U) \times \varphi^{-1}(V) \cap E$ is bounded. Now E was an arbitrary entourage thus $\varphi^{-1}(U)$, $\varphi^{-1}(V)$ are coarsely disjoint. Since $\varphi^{-1}(U) \in \sigma$, $\varphi^{-1}(V) \in \tau$ the ultrafilters σ, τ induce coarse ultrafilters which are not asymptotically alike.

If $\mathcal{F}$ is a coarse ultrafilter on X induced by an ultrafilter σ and φ a Higson function then denote by $\mathcal{F}\text{-}\lim\varphi$ the point $\sigma\text{-}\lim\varphi$ in $\mathbb{R}$. By the above $\mathcal{F}\text{-}\lim\varphi$ is well defined modulo asymptotically alike of $\mathcal{F}$.

If $\varphi\colon X \to \mathbb{R}$ is a Higson function then there is an extension

$$\begin{aligned} \widehat{\varphi}\colon h'(X) &\to \mathbb{R} \\ x &\mapsto \begin{cases} \varphi(x), & x \in X, \\ \mathcal{F}\text{-}\lim\varphi, & x = \mathcal{F} \in \nu'(X). \end{cases} \end{aligned}$$

We have shown $\widehat{\varphi}$ is well defined. Now we show $\widehat{\varphi}$ is continuous: Let $A \subseteq \mathbb{R}$ be a closed set. If $\mathcal{F}\text{-}\lim\varphi \in A$ fix an ultrafilter σ on X that induces $\mathcal{F}$. Then $\varphi^{-1}(A) \in \sigma$. This implies $\mathcal{F} \in \mathsf{cl}(\varphi^{-1}(A))$. On the other hand if $\mathcal{F} \in \mathsf{cl}(\varphi^{-1}(A))$ then there is an ultrafilter σ on X with $\varphi^{-1}(A) \in \sigma$ that induces $\mathcal{F}$. This implies

σ- $\lim \varphi \in A$, thus $\mathcal{F}$- $\lim \varphi \in A$. Now

$$\widehat{\varphi}^{-1}(A) = \varphi^{-1}(A) \cup \{\mathcal{F}\colon \mathcal{F}\text{-}\lim \varphi \in A\} = \varphi^{-1}(A) \cup \mathsf{cl}(\varphi^{-1}(A))$$

is closed.

Denote by $(C_h(X))^{h'}$ the set of extensions of Higson functions on X to $h'(X)$. By [2] the C^*-algebra of Higson functions $C_h(X)$ determines the compactification $h'(X)$ if and only if $(C_h(X))^{h'}$ separates points of $\nu'(X)$.

We show $(C_h(X))^{h'}$ separates points of $\nu'(X)$: Let $\mathcal{F}, \mathcal{G} \in \nu'(X)$ be two coarse ultrafilters with $\mathcal{F} \bar{\curlywedge} \mathcal{G}$. Then there exist elements $U \in \mathcal{F}$, $V \in \mathcal{G}$ with $U \not\curlywedge V$. Without loss of generality assume that U, V are disjoint such that $d(x,U) + d(x,V) \neq 0$ for every $x \in X$. Then define a function

$$\begin{aligned} \varphi\colon X &\to \mathbb{R} \\ x &\mapsto \frac{d(x,U)}{d(x,U) + d(x,V)} \,. \end{aligned}$$

By [5, Lemma 2.2] the function $d\varphi|_E$ vanishes at infinity for every entourage $E \subseteq X^2$. Now $\varphi|_U \equiv 0$ and $\varphi|_V \equiv 1$. This implies $\mathcal{F}$- $\lim \varphi = 0$ and $\mathcal{G}$- $\lim \varphi = 1$.

If $f\colon X \to Y$ is a coarse map between R-discrete for some $R > 0$ proper metric spaces and $\varphi\colon Y \to \mathbb{R}$ a Higson function then $\varphi \circ f\colon X \to \mathbb{R}$ is a Higson function: Since X is R-discrete the map f is continuous, therefore $\varphi \circ f$ is continuous. The map $\varphi \circ f$ is bounded since φ is bounded. Let $E \subseteq X^2$ be an entourage and $\varepsilon > 0$ a number. Then $f^{\times 2}(E) \subseteq Y^2$ is an entourage. This implies $(d\varphi)|_{f^{\times 2}(E)}$ vanishes at infinity. Thus there is a compact set $K \subseteq Y$ such that

$$|d(\varphi(x,y)| < \varepsilon$$

whenever $(x,y) \in f^{\times 2}(E) \cap (K^2)^c$. Since K is bounded the set $f^{-1}(K) \subseteq X$ is bounded. The set $f^{-1}(K)$ is finite since X is R-discrete and therefore $f^{-1}(K)$ is compact. Then

$$|d(\varphi \circ f)(x,y)| < \varepsilon$$

whenever $(x,y) \in E \cap (f^{-1}(K))^2$.

Now we provide an explicit homeomorphism $\nu(X) \to \nu'(X)$. Denote by

$$\begin{aligned} e_{C_h(X)}\colon Z &\to \mathbb{R}^{C_h(X)} \\ x &\mapsto (\varphi(x))_\varphi \end{aligned}$$

the evaluation map for X.

Note $e_{C_h(X)}$ is a topological embedding and $\nu(X) := \overline{e_{C_h(X)}(X)} \setminus e_{C_h(X)}(X)$ by [2]. A point $p \in \nu(X)$ is represented by a net $(x_i)_i$ such that for every Higson function $\varphi \in C_h(X)$ the net $\varphi(x_i)_i$ converges in $\mathbb{R}$. Define $F_i := \{x_j : j \geq i\}$

for every i. Then $\sigma := \{F_i : i\}$ is a filter on X such that $\varphi_*\sigma$ converges to $\lim_i \varphi(x_i)$ for every Higson function φ on X. An ultrafilter σ' which is finer than σ determines a coarse ultrafilter $\mathcal{F}$. We have shown above that the association $\Phi_X : p \mapsto \mathcal{F}$ is well defined modulo asymptotically alike.

Now we show the map Φ_X is injective: Let $p, q \in \nu(X)$ be two points. If $\Phi_X(p) = \Phi_X(q)$ then $\Phi_X(p) - \lim\varphi = \Phi_X(q) - \lim\varphi$ for every Higson function φ. This implies $p = q$ in $\mathbb{R}^{C_h(X)}$.

We show Φ_X is surjective: If σ is an ultrafilter on X that determines a coarse ultrafilter on X then there is a net $(x_i)_i$ on X which constitutes a section of σ. Since $\varphi(x_i)_i$ is a section of $\varphi_*\sigma$ for every Higson function φ the net $\varphi(x_i)_i$ converges to σ- $\lim\varphi$ in $\mathbb{R}$. Thus $(x_i)_i$ converges to a point in $\nu(X)$.

Now we show Φ_X is continuous: If $A \subseteq X$ is a subset then $\Phi_X^{-1}(\mathsf{cl}(A))$ is a subset of $\nu(X)$. We show it is closed. If $p \in \Phi_X^{-1}(\mathsf{cl}(A))$ then there is a net $(x_i)_i \subseteq X$ that converges to p. The net $(x_i)_i$ is a section of an ultrafilter σ with $A \in \sigma$. Thus there exists i with $x_j \in A$ for every $j \geq i$. If on the other hand $(x_i)_i$ is a net in X and there exists i with $x_j \in A$ for every $j \geq i$ then $(x_i)_i$ is a section of an ultrafilter σ on X with $A \in \sigma$. This implies if $(x_i)_i$ converges to $p \in \nu(X)$ then $p \in \Phi_X^{-1}(\mathsf{cl}(A))$. Thus we have shown

$$\Phi_X^{-1}(\mathsf{cl}(A)) = \overline{e_{C_h(X)}(A)} \setminus e_{C_h(X)}(A)$$

is closed. This way we have obtained that Φ_X is a homeomorphism.

Now we define a map

$$f_* : \mathbb{R}^{C_h(X)} \to \mathbb{R}^{C_h(Y)}$$
$$(x_\varphi)_{\varphi \in C_h(X)} \mapsto (x_{\varphi\circ f})_{\varphi \in C_h(Y)}.$$

We show $f_*(\overline{e_{C_h(X)}(X)}) \subseteq \overline{e_{C_h(Y)}(Y)}$: If $(x_\varphi)_\varphi \in \overline{e_{C_h(X)}(X)}$ then there is a net $(x_i)_i \subseteq X$ such that $\lim_i \varphi(x_i) = x_\varphi$ for every $\varphi \in C_h(X)$. Then $f(x_i)_i \subseteq Y$ is a net such that $\lim_i \varphi(f(x_i)) = x_{\varphi\circ f}$ for every $\varphi \in C_h(Y)$.

Now $\nu(f) := f_*|_{\overline{e_{C_h(X)}(X)} \setminus e_{C_h(X)}(X)}$. Then

$$\nu(f) = \Phi_Y^{-1} \circ \nu'(f) \circ \Phi_X.$$

$\square$

5. A Künneth formula

This is [9, Definition 25]:

Definition 20 (asymptotic product)**.** If X, Y are metric spaces fix points $p \in X$ and $q \in Y$ and a constant $R \geq 0$ large enough. Then the *asymptotic product*

$X * Y$ of X and Y is defined by

$$X * Y := \{(x,y) \in X \times Y \colon |d(p,x) - d(q,y)| \leq R\}$$

as a subspace of $X \times Y$. We define the projection $p_1 \colon X * Y \to X$ by $(x,y) \mapsto x$ and the projection $p_2 \colon X * Y \to Y$ by $(x,y) \mapsto y$. Note that the projections are coarse maps. In what follows we denote by $d(p,\cdot)$, $d(q,\cdot)$ coarse maps $X \to \mathbb{R}_+$, $Y \to \mathbb{R}_+$ defined by $x \in X \mapsto d(p,x)$, $y \in Y \mapsto d(q,y)$.

Remark 21. Let X, Y be metric spaces. Now $X*Y$ of Definition 20 is determined by points $p \in X$, $q \in Y$ and constant $R \geq 0$. If X or Y has nice properties then $X * Y$ does not depend on the choice of p, q, R up to coarse equivalence. If that is the case we say the asymptotic product is well defined. Then by [9, Lemma 27] the diagram

$$\begin{tikzcd} X * Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "d(q,\cdot)"] \\ X \arrow[r, "d(p,\cdot)"'] & \mathbb{R}_+ \end{tikzcd}$$

is a pullback diagram in Coarse.

Lemma 22. *Let X, Y be metric spaces such that the asymptotic product is well defined. The following statements hold:*

(1) *If $A \subseteq X$, $B \subseteq Y$ are subsets then $(A \times B) \cap (X * Y)$ is bounded if A is bounded or B is bounded.*
(2) *If $(U_i)_i$ is a coarse cover of X and $(V_j)_j$ a coarse cover of Y then $((U_i \times V_j) \cap (X * Y))_{ij}$ is a coarse cover of $X * Y$.*
(3) *Let $\mathcal{F}, \mathcal{G}$ be coarse ultrafilters on X, Y, respectively, with $d(p,\cdot)_*\mathcal{F}\lambda \times d(q,\cdot)_*\mathcal{G}$. Choose the constant of $X * Y$ large enough. Then*

$$\mathcal{F} * \mathcal{G} := \{(A \times B) \cap (X * Y) \colon A \in \mathcal{F},\ B \in \mathcal{G}\}$$

is a coarse ultrafilter on $X * Y$.

Proof: (1) Suppose A is bounded. Then $(x,y) \in A * Y$ implies $x \in A$ and $|d(x,p) - d(y,q)| \leq R$. Let $S \geq 0$ be such that $A \subseteq B(p,S)$. Then $y \in B(q, R+S)$. Thus $A * Y$ is bounded. Similarly if B is bounded then $X * B$ is bounded.

(2) Let $E \subseteq (X * Y)^2$ be an entourage. Then

$$\begin{aligned}\bigcap_{ij} E[(U_i \times V_j)^c \cap (X * Y)] &\subseteq \bigcap_{ij} E[(U_i \times V_j)^c] \cap (X * Y)\\ &= \bigcap_{ij} (E[U_i^c \times Y] \cup E[X \times V_j^c]) \cap (X * Y)\\ &= \Big(\bigcap_i E[U_i^c \times Y] \cap (X * Y)\Big) \cup \Big(\bigcap_j E[X \times V_j^c] \cap (X * Y)\Big)\end{aligned}$$

is bounded. Thus $((U_i \times V_j) \cap (X * Y))_{ij}$ is a coarse cover of $X * Y$.

Alternative proof: $(p_1^{-1}(U_i) \cap p_2^{-1}(V_j))_{ij}$.

(3) Let $i\colon X * Y \to X \times Y$ be the inclusion. At first we prove

$$i_*(\mathcal{F} * \mathcal{G}) = \{A \times B\colon A \in \mathcal{F},\ B \in \mathcal{G}\}$$

is a coarse ultrafilter on $X \times Y$. We check the axioms of a coarse ultrafilter on $i_*(\mathcal{F} * \mathcal{G})$:

(1) If $A_1 \times B_1, A_2 \times B_2 \in i_*(\mathcal{F} * \mathcal{G})$ then $A_1, A_2 \in \mathcal{F}$, $B_1, B_2 \in \mathcal{G}$. This implies $A_1 \curlywedge A_2$ in X and $B_1 \curlywedge B_2$ in Y. Then $A_1 \times B_1 \curlywedge A_2 \times B_2$ in $X \times Y$.

(2) Let $A_1 \times B_1, A_2 \times B_2 \subseteq X \times Y$ be two subsets with $(A_1 \times B_1) \cup (A_2 \times B_2) \in i_*(\mathcal{F} * \mathcal{G})$. Since $(A_1 \cup A_2) \times (B_1 \cup B_2) \supseteq (A_1 \times B_1) \cup (A_2 \times B_2)$ this implies $(A_1 \cup A_2) \times (B_1 \cup B_2) \in i_*(\mathcal{F} * \mathcal{G})$. Thus $(A_1 \cup A_2) \in \mathcal{F}, (B_1 \cup B_2) \in \mathcal{G}$. This implies $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$. Then $A_1 \times (B_1 \cup B_2) \in i_*(\mathcal{F} * \mathcal{G})$ or $A_2 \times (B_1 \cup B_2) \in i_*(\mathcal{F} * \mathcal{G})$. Suppose $A_1 \times (B_1 \cup B_2) \in i_*(\mathcal{F} * \mathcal{G})$. Since $A_1 \times B_1$ is maximal among factors of two subsets of X, Y contained in $A_1 \times (B_1 \cup B_2), (A_1 \times B_1) \cup (A_2 \times B_2) \in i_*(\mathcal{F} * \mathcal{G})$ we obtain $A_1 \times B_1 \in i_*(\mathcal{F} * \mathcal{G})$.

(3) $X \times Y \in i_*(\mathcal{F} * \mathcal{G})$ since $X \in \mathcal{F}, Y \in \mathcal{G}$.

Let $A \times B \in i_*(\mathcal{F} * \mathcal{G})$ be an element. Since $d(p,\cdot)_*\mathcal{F} \curlywedge d(q,\cdot)_*\mathcal{G}$ the sets $d(p,\cdot)(A)$, $d(q,\cdot)(B)$ are close in $\mathbb{R}_+$. Thus there exists an $R \geq 0$ and unbounded subsets $A' \subseteq A$, $B' \subseteq B$ with

$$|d(p,a) - d(q,b)| \leq R$$

for $a \in A'$, $b \in B'$. Thus we have shown $A \times B \curlywedge X * Y$. Choose the constant of $X * Y$ large enough then $X * Y \in i_*(\mathcal{F} * \mathcal{G})$. We can thus restrict $i_*(\mathcal{F} * \mathcal{G})$ to $X * Y$ and obtain $\mathcal{F} * \mathcal{G} = (i_*(\mathcal{F} * \mathcal{G}))|_{X*Y}$. This way we have shown $\mathcal{F} * \mathcal{G}$ is a coarse ultrafilter. □

Theorem 23. *Let X, Y be metric spaces such that their asymptotic product is well defined. Define*

$$\nu'(X) * \nu'(Y) := \{(\mathcal{F}, \mathcal{G}) \in \nu'(X) \times \nu'(Y)\colon \nu'(d(p,\cdot))(\mathcal{F}) = \nu'(d(q,\cdot))(\mathcal{G})\}.$$

Then the map

$$\langle \nu'(p_1), \nu'(p_2)\rangle \colon \nu'(X * Y) \to \nu'(X) * \nu'(Y)$$

is a homeomorphism.

Proof: We prove $\langle \nu'(p_1), \nu'(p_2)\rangle$ is well defined: Let $\mathcal{F}$ be a coarse ultrafilter on $X * Y$ then $p_{1*}\mathcal{F}$, $p_{2*}\mathcal{F}$ are coarse ultrafilters on X, Y, respectively. Since $d(p,\cdot) \circ p_1$, $d(q,\cdot) \circ p_2$ are close the coarse ultrafilters $d(p,\cdot)_* p_{1*}\mathcal{F}$, $d(q,\cdot)_* p_{2*}\mathcal{F}$ are asymptotically alike. Thus we have shown $(p_{1*}\mathcal{F}, p_{2*}\mathcal{F}) \in \nu'(X) * \nu'(Y)$.

Now we prove $\langle \nu'(p_1), \nu'(p_2)\rangle$ is surjective: Let $(\mathcal{F},\mathcal{G}) \in \nu'(X) * \nu'(Y)$ be a point. By Lemma 22 the system of subsets $\mathcal{F} * \mathcal{G}$ is a coarse ultrafilter on $X * Y$. Denote by $p_1' \colon X \times Y \to X$, $p_2' \colon X \times Y \to Y$ the projection to the first, second factor, respectively, and by $i \colon X * Y \to X \times Y$ the inclusion. Then $p_1 = p_1' \circ i$, $p_2 = p_2' \circ i$. Since $i_*(\mathcal{F} * \mathcal{G}) = \{A \times B \colon A \in \mathcal{F}, B \in \mathcal{G}\}$ we obtain the relations $p_{1*}' i_*(\mathcal{F} * \mathcal{G})\lambda\mathcal{F}$, $p_{2*}' i_*(\mathcal{F} * \mathcal{G})\lambda\mathcal{G}$. Thus we have proved $\langle \nu'(p_1), \nu'(p_2)\rangle \times (\mathcal{F} * \mathcal{G}) = (\mathcal{F},\mathcal{G})$.

Now we prove $(\nu'(p_1)(\mathcal{F})) * (\nu'(p_2)(\mathcal{F})) = \mathcal{F}$ for every point $\mathcal{F} \in \nu'(X * Y)$: Let $A \in \mathcal{F}$ be an element. Then $(p_1(A) \times p_2(A)) \cap (X * Y) \in (p_{1*}\mathcal{F}) * (p_{2*}\mathcal{F})$. Since $A \subseteq (p_1(A) \times p_2(A)) \cap (X * Y)$ we obtain $(p_{1*}\mathcal{F}) * (p_{2*}\mathcal{F}) \subseteq \mathcal{F}$. Thus $(p_{1*}\mathcal{F}) * (p_{2*}\mathcal{F})\lambda\mathcal{F}$. This way we have shown $\langle \nu'(p_1), \nu'(p_2)\rangle$ is bijective.

Since $\nu'(X * Y)$ is compact and $\nu'(X) * \nu'(Y)$ is Hausdorff we obtain that $\langle \nu'(p_1), \nu'(p_2)\rangle$ is a homeomorphism. $\square$

6. Space of rays

Definition 24 (space of rays)**.** Let Y be a compact topological space. As a set the *space of rays* $\digamma(Y)$ of Y is $Y \times \mathbb{Z}_+$. A subset $E \subseteq Y^2$ is an entourage if for every countable subset $((x_k, i_k), (y_k, j_k))_k \subseteq E$ the following properties hold:

(1) The set $(i_k, j_k)_k$ is an entourage in $\mathbb{Z}_+$.
(2) If $(i_k)_k \to \infty$ then $(x_k)_k$ and $(y_k)_k$ have the same limit points.

This makes $\digamma(Y)$ a coarse space.

Theorem 25. *If $f \colon X \to Y$ is a continuous map between compact topological spaces*

- *then it induces a coarse map by*

$$\begin{aligned} \digamma(f) \colon \digamma(X) &\to \digamma(Y) \\ (x,i) &\mapsto (f(x), i). \end{aligned}$$

- *If f is a homeomorphism then $\digamma(f)$ is a coarse equivalence.*

Proof: $\circ$ We show $\digamma(f)$ is coarsely uniform and coarsely proper. First we show $\digamma(f)$ is coarsely uniform: Suppose $((x_i, n_i), (y_i, m_i))_i$ is a countable entourage in $\digamma(X)$ such that $(n_i)_i$ is a strictly monotone sequence in $\mathbb{Z}_+$ and $(x_i)_i$ converges to x. Then $(n_i, m_i)_i$ is an entourage in $\mathbb{Z}_+$ and $(y_i)_i$ converges to x. Since f is a continuous map $f(x_i)_i$ and $f(y_i)_i$ both converge to $f(x)$. Thus we can conclude that

$$((f(x_i), n_i), (f(y_i), m_i))_i$$

is an entourage in $\digamma(Y)$.

Now we show $\digamma(f)$ is coarsely proper: If $B \subseteq \digamma(Y)$ is bounded we can write $B = \bigcup_i B_i \times i$ with $B_i \subseteq Y$, $i \in \mathbb{Z}_+$, where the number of i that appear is finite. Then

$$f^{-1}(B) = \bigcup_i f^{-1}(B_i) \times i$$

is bounded.

$\circ$ If f is a homeomorphism then there is a topological inverse $g\colon Y \to X$ of f. Now $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Then

$$\digamma(f) \circ \digamma(g) = \digamma(f \circ g) = \digamma(\mathrm{id}_Y) = \mathrm{id}_{\digamma(Y)}$$

and

$$\digamma(g) \circ \digamma(f) = \digamma(g \circ f) = \digamma(\mathrm{id}_X) = \mathrm{id}_{\digamma(X)}.$$

$\square$

Corollary 26. *Denote by* kTop *the category of compact topological spaces and continuous maps and by* Coarse *denote the category of coarse spaces and coarse maps modulo close. Then $\digamma$ is a functor*

$$\digamma\colon \mathrm{kTop} \to \mathrm{Coarse}.$$

Proposition 27. *Denote by $\mathcal{F}_0$ a coarse ultrafilter on $\mathbb{Z}_+$, the choice is not important. For every $y \in Y$ denote by i_y the inclusion $y \times \mathbb{Z}_+ \to \digamma(Y)$. The map*

$$\begin{aligned} \eta_Y\colon Y &\to \nu' \circ \digamma(Y) \\ y &\mapsto \nu'(i_y)(\mathcal{F}_0) \end{aligned}$$

for every metric space Y defines a natural transformation $\eta\colon \mathbb{1}_{\mathrm{kTop}} \to \nu' \circ \digamma$.

Proof: If $f\colon Y \to Z$ is a continuous map between compact spaces we show the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\eta_Y"'] & Z \arrow[d, "\eta_Z"] \\ \nu' \circ \digamma(Y) \arrow[r, "\nu' \circ \digamma(f)"'] & \nu' \circ \digamma(Z) \end{tikzcd}$$

commutes. Down and then right: a point $y \in Y$ is mapped by η_Y to $\nu'(i_y)(\mathcal{F}_0)$. Then

$$\nu' \circ F(f)(\nu'(i_y)(\mathcal{F}_0)) = F(f)_* \circ i_{y*}(\mathcal{F}_0) = (F(f) \circ i_y)_*(\mathcal{F}_0) = i_{f(y)*}(\mathcal{F}_0).$$

Right and then down: a point $y \in Y$ is mapped by f to $f(y)$. Then

$$\eta_Z(f(y)) = \nu'(i_{f(y)})(\mathcal{F}_0).$$

The map η_Y is continuous for every compact space Y: Let $(y_i)_i$ be a net in Y that converges to y. Then $(\nu'(i_{y_i})(\mathcal{F}_0))_i$ converges in $\eta_Y(Y)$ to $\nu'(i_y)(\mathcal{F}_0)$: Let $A \subseteq \nu' \circ F(Y)$ be a set such that $\nu'(i_y)(\mathcal{F}_0) \in \mathsf{cl}(A)^c$. Thus there is some $B \in \mathcal{F}_0$ such that $y \times B \not\curlywedge A$. Now for almost all i the relation $(y_i \times B) \not\curlywedge A$ holds, thus $\nu'(i_{y_i})(\mathcal{F}_0) \in \mathsf{cl}(A)^c$ for almost all i. $\square$

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(*Received* June 9, 2019, *revised* February 7, 2020)