Notes on generalizations of Bézout rings

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Abstract. In this paper, we give new characterizations of the P-2-Bézout property of trivial ring extensions. Also, we investigate the transfer of this property to homomorphic images and to finite direct products. Our results generate original examples which enrich the current literature with new examples of non-2-Bézout P-2-Bézout rings and examples of non-P-Bézout P-2-Bézout rings.

Keywords: P-Bézout ring; 2-Bézout ring; P-2-Bézout ring; trivial rings extension; homomorphic image; finite direct product

Classification: 13A15, 13B10, 13F05, 13D02

1. Introduction

All rings considered below are commutative with unit and all modules are unital. A ring R is a coherent ring if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R [8, Theorem 2.3.2, page 45].

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. Ring R is called the trivial ring extension of A by E. Recall that a prime ideal of R always has the form $Q \propto E$, where Q is a prime ideal of A [9, Theorem 25.1 (3)]. Considerable work, part of it summarized in Glaz's book [8] and Huckaba's book [9] where R is called the idealization of E by A, has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in commutative and non-commutative ring theory, see for instance [8], [9], [15].

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence of R-modules:

$$F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

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where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are finitely generated and finitely presented R-modules, respectively.

Given nonnegative integers n and d, a ring R is called an (n,d)-ring if every n-presented R-module has projective dimension less than or equal to d; and a weak (n,d)-ring if every n-presented cyclic R-module has projective dimension less than or equal to d, equivalently, if every $(n-1)$-presented ideal of R has projective dimension less than or equal to $d-1$, see for instance [5], [10].

A ring R said to be Bézout ring if every finitely generated ideal of R is principal. Examples of Bézout rings are valuation rings, elementary divisor rings and Hermite rings. For instance see [7], [11], [12]. In [2] and [4], the authors generalize that notion and introduce the notions of P-Bézout and 2-Bézout rings as follows: A ring R is called P-Bézout ring if every finitely generated prime ideal of R is principal, and R is 2-Bézout ring if every finitely presented ideal of R is principal. In [6] we generalized the notions of P-Bézout rings and 2-Bézout rings to the notion P-2-Bézout rings. A ring R is said to be P-2-Bézout ring if every finitely presented prime ideal of R is principal.

In this paper, we give new characterizations of the P-2 Bézout property of various trivial ring extensions. Also, we investigate the transfer of this property to homomorphic images and to finite direct products. Then, we construct new examples of non-2-Bézout P-2-Bézout rings and examples of non-P-Bézout P-2-Bézout rings.

2. Main results

In this section, we study a new possible transfer of the P-2 Bézout property of various trivial ring extensions. First, we examine the trivial ring extension of a local ring (A,M) by an A-module E such that $ME=0$.

Proposition 2.1. *Let A be a local ring with finitely generated maximal ideal M, E an A/M-vector space of finite rank and let $R := A \propto E$ be the trivial ring extension of A by E. If R is a P-2-Bézout ring, then so is A.*

Proof: Assume that R is a P-2-Bézout ring and let $Q := \sum_{i=1}^{i=n} Aa_i$ be a finitely presented prime ideal of A and set $P := Q \propto E = \sum_{i=1}^{i=n} R(a_i,0) + \sum_{i=1}^{i=m} R(0,x_i)$ where $(x_i)_{i=1}^{i=m}$ is a basis of the (A/M)-vector space E. Then, P is a finitely presented prime ideal of R. Indeed, consider the exact sequence of A-modules

$$0 \longrightarrow \mathrm{Ker}(u) \longrightarrow A^n \xrightarrow{u} Q \longrightarrow 0$$

where $u\big((\alpha_i)_{i=1}^{i=n}\big) = \sum_{i=1}^{i=n} \alpha_i a_i$ and the exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(v) \longrightarrow R^{n+m} \xrightarrow{v} P \longrightarrow 0$$

where $v\big((\alpha_i, e_i)_{i=1}^{i=n+m}\big) = \sum_{i=1}^{i=n}(\alpha_i, e_i)(a_i, 0) + \sum_{i=n+1}^{i=n+m}(\alpha_i, e_i)(0, x_{n+m+1-i})$. Clearly, $\mathrm{Ker}(v) = (\mathrm{Ker}(u) \oplus M^m) \propto E^{n+m}$ since $(x_i)_{i=1}^{i=m}$ is a basis of the (A/M)-vector space E. But, $\mathrm{Ker}(u)$ is a finitely generated A-module since Q is finitely presented. Therefore, $\mathrm{Ker}(v)$ is a finitely generated R-module and so P is a finitely presented prime ideal of R. Hence, $P := Q \propto E = R(a, e)$ for some element (a, e) of R because R is a P-2-Bézout ring. Thus, $Q = Aa$, as desired. $\square$

Proposition 2.2. *Let A be a domain, M a finitely generated maximal ideal of A, E a nonzero A-module such that $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. If R is a P-2-Bézout ring and E is an (A/M)-vector space of finite rank, then E is an (A/M)-vector space of rank one.*

Proof: Assume that R is a P-2-Bézout ring and E is an (A/M)-vector space of finite rank. Let $(x_i)_{i=1}^{i=m}$ be its basis, then $P := 0 \propto E = \sum_{i=1}^{i=m} R(0, x_i)$ is a finitely presented prime ideal of R. Indeed, consider the exact sequence of R-modules:

$$0 \longrightarrow \mathrm{Ker}(u) \longrightarrow R^m \xrightarrow{u} 0 \propto E \longrightarrow 0$$

where $u\big((c_i, g_i)_{i=1}^{i=m}\big) = \sum_{i=1}^{i=m}(c_i, g_i)(0, x_i) = \big(0, \sum_{i=1}^{i=m} c_i x_i\big) = \big(0, \sum_{i=1}^{i=m} \bar{c}_i x_i\big)$. Clearly, $\mathrm{Ker}(u) = M^m \propto E^m$ since $(x_i)_{i=1}^{i=m}$ is a basis of the (A/M)-vector space E. Hence, $\mathrm{Ker}(u)$ is a finitely generated R-module since M is a finitely generated ideal of A. Then $0 \propto E$ is a finitely presented prime ideal of R by the above exact sequence and so $P = R(0, x)$ for some $x \in E$ since R is a P-2-Bézout ring. Hence, $E = Ax$. $\square$

Now, we give an example of non P-2-Bézout ring.

Example 2.1. Let $A := K[[X]]$ where K is a commutative field, $M := (X)$ and $E := (A/(X))^2$. Then, $R := A \propto E$ is not a P-2-Bézout ring since $0 \propto E$ is a finitely presented prime ideal of R by [10, Theorem 2.6 (2)] which is not principal by Proposition 2.2 since E is an (A/M)-vector space of rank two.

Proposition 2.3. *Let A be a local ring with maximal ideal M, E an A-module such that $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. Then, R is a P-2-Bézout ring provided E is an (A/M)-vector space of infinite rank.*

Proof: It suffices to show that there is no finitely presented prime ideal I of R. Suppose on the contrary I be a finitely presented prime ideal of R. Then, I is projective since R is a weak $(2, 0)$-ring by [14, Theorem 2.1 (1)] and so it is free since R is local by [1, Theorem 3.2 (1)], that is $I = Ra$ for some regular element $a \in R$. This is a contradiction since $I \subseteq M \propto E$ and $(M \propto E)(0, e) = (0, 0)$ for each $e \in E - \{0\}$. Hence R is a P-2-Bézout ring. $\square$

Theorem 2.1. *Let A be a local domain with maximal ideal M, $E \neq 0$ an A-module such that $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E.*

(1) *Assume that A does not contain any nonzero finitely presented prime ideal and E is an (A/M)-vector space of rank one. Then, R is a P-2-Bézout ring.*
(2) *Assume that A contains a nonzero principal prime ideal and M is a finitely generated ideal of A. Then, R is a P-2-Bézout ring if and only if E is an (A/M)-vector space of infinite rank.*

Proof: (1) Assume that A does not contain any nonzero finitely presented prime ideal and E is an (A/M)-vector space of rank one. Let $Q := P \propto E$ be a finitely presented prime ideal of R. Then, P is a finitely presented prime ideal of A. Indeed, let $(b_i, f_i)_{i=1}^{i=n}$ be a minimal generated set of $P \propto E$. Clearly, $P \propto E \subseteq M \propto E$ since R is a local ring with maximal ideal $M \propto E$. Consider the exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(v) \longrightarrow R^n \xrightarrow{v} P \propto E \longrightarrow 0$$

where $v\big((a_i, e_i)_{i=1}^{i=n}\big) = \sum_{i=1}^{i=n}(a_i, e_i)(b_i, f_i) = \big(\sum_{i=1}^{i=n} a_i b_i, \sum_{i=1}^{i=n} a_i f_i\big)$ (since $b_i \in M$ for each $i = 1, \dots, n$). But, $\mathrm{Ker}(v) \subseteq (M \propto E)^n$ by [16, Lemma 4.43, page 134] since $(b_i, f_i)_{i=1}^{i=n}$ is a minimal generated set of $P \propto E$. Hence, $\mathrm{Ker}(v) = X \propto E^n$ where $X := \{(a_i)_{i=1}^n \in A^n \colon \sum_{i=1}^{i=n} a_i b_i = 0\}$, is a finitely generated R-module (since $P \propto E$ is a finitely presented ideal of R). On the other hand, consider the exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(u) \longrightarrow A^n \xrightarrow{u} P \longrightarrow 0$$

where u is defined by $u\big((a_i)_{i=1}^{i=n}\big) = \sum_{i=1}^{i=n} a_i b_i$. Clearly, $\mathrm{Ker}(u) = X$ which is a finitely generated A-module since $\mathrm{Ker}(v)$ is a finitely generated R-module. Then, P is a finitely presented prime ideal of A and so $P = 0$ by hypothesis. Therefore, $Q := 0 \propto E = 0 \propto Ae = R(0, e)$ where (e) is a basis of the (A/M)-vector space E. Hence, R is a P-2-Bézout ring.

(2) If E is an (A/M)-vector space of infinite rank, then R is a P-2-Bézout ring by Proposition 2.3. Conversely, assume that R is a P-2-Bézout ring. We claim that E is an (A/M)-vector space of infinite rank. Suppose, on the contrary, E is an (A/M)-vector space of finite rank. So, $E = Ax$ by Proposition 2.2 since M

is a finitely generated ideal of A. On the other hand, let $I := Aa$ be a principal prime ideal of A. Then, $Q := I \propto E = R(a,0) + R(0,x)$ is a finitely presented prime ideal of R. Indeed, consider the exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(v) \longrightarrow R \xrightarrow{v} R(a,0) \longrightarrow 0$$

where $v(\alpha, e) = (\alpha, e)(a, 0) = (\alpha a, 0)$ since $a \in M$. Clearly, $\mathrm{Ker}(v) = 0 \propto E$ since A is a domain and so $R(a,0)$ is a finitely presented ideal of R since E is a finitely generated A-module. Now, consider the exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(w) \longrightarrow R \xrightarrow{w} R(0,x) \longrightarrow 0$$

where $w(\alpha, e) = (\alpha, e)(0, x) = (0, \alpha x)$. Clearly, $\mathrm{Ker}(w) = \mathrm{Ann}_A(x) \propto E$ which is finitely generated ideal of R by hypothesis, so $R(0,x)$ is a finitely presented ideal of R. Then, $R(a,0)+R(0,x)$ is a finitely presented ideal of R by [8, Corollary 2.1.3] since $R(a,0) \cap R(0,x) = 0$. So, $Q = R(b,e)$ for some $b \in I$ and $e \in E$ since R is a P-2-Bézout ring. Hence, $Aa = Ab$ and $E = Ax = Ae$ (since $bE = 0$) and so $b = ua$ for some invertible element u of A (since A is a domain). Then, $R(b,e) = R(au,e) = R(u,0)(a,u^{-1}e) = R(a,u^{-1}e)$ since $(u,0)$ is an invertible element of R by [9, Theorem 25.1 (6)]. Therefore, there exists $(c,f) \in R$ such that $(a,0) = (c,f)(a,u^{-1}e) = (ca, cu^{-1}e)$, since $(a,0) \in Q$. Hence, $ca = a$ in A and $cu^{-1}e = 0$ in E and so $c = 1$ (since A is a domain) and $e = 0$ in E which means that $E = Ax = Ae = 0$, a contradiction since $E \neq 0$. Therefore, E is an (A/M)-vector space of infinite rank. □

The following Corollary is an immediate consequence of Theorem 2.1 and of Proposition 2.2.

Corollary 2.1. *Let K be a field, E a nonzero K-vector space and let $R := K \propto E$ be the trivial ring extension of K by E. Then, R is a P-2-Bézout ring if and only if E is an K-vector space of infinite rank or of rank one.*

Now, we show that the hypothesis "E is an (A/M)-vector space with infinite rank" is necessary in Proposition 2.3.

Example 2.2. Let A be a discrete valuation domain with maximal ideal M (for instance $A := \mathbb{Z}_{(2)}$ and $M := 2\mathbb{Z}_{(2)}$) and let $R := A \propto (A/M)$. Then, R is not a P-2-Bézout ring.

Proof: We claim that R is not a P-2-Bézout ring. On the contrary, let $x \in A$ such that $(\bar{x})$ is a basis of the (A/M)-vector space (A/M) and let $J := M \propto A/M = R(m,0) + R(0,\bar{x})$. Then, J is a finitely presented prime ideal of R by [10, Theorem 2.6 (2)] since A is a coherent domain. By the same reasoning as in Theorem 2.1 (2) we show that $A/M = 0$. This is a contradiction since M is a maximal ideal of A. Hence, R is not a P-2-Bézout ring. $\square$

Now, we study the homomorphic image of P-2-Bézout rings.

Theorem 2.2. *Let R be a P-2-Bézout ring and let I be a finitely presented ideal of R. Then R/I is a P-2-Bézout ring.*

Proof: Let Q be a finitely presented prime ideal of R/I. Then, $Q = P/I$ for some prime ideal P of R. Moreover Q is a finitely presented R-module by [8, Theorem 2.1.8 (2)] since I is a finitely generated ideal of R. On the other hand, the exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow P \longrightarrow P/I \longrightarrow 0$$

shows that P is a finitely presented ideal of R by [8, Theorem 2.1.2 (1)] since I is a finitely presented ideal of R and P/I is a finitely presented R-module. Hence, P is a principal ideal of R that is $P := Ra$ for some $a \in P$ since R is a P-2-Bézout ring. Therefore, $P/I = (R/I)(a+I)$ is a principal ideal of R/I and this completes the proof of Theorem 2.2. $\square$

Corollary 2.2. *Let R be a domain and $R[X]$ be the polynomial ring over R. If $R[X]$ is a P-2-Bézout ring, then so is R.*

Proof: It is clear that $XR[X] \cong R[X]$ (since X is regular) and so (X) is finitely presented. Therefore, R is a P-2-Bézout ring by Theorem 2.2 since $R[X]/(X) \cong R$. $\square$

Now, we prove that the condition "I is finitely presented" is necessary in Theorem 2.2.

Example 2.3. Let (A, M) be a local coherent ring with a finitely generated maximal ideal M, which is non-principal (for example, $A := \mathbb{R} \propto \mathbb{C}$, $M := 0 \propto \mathbb{C}$). Let E be an A/M-vector space with infinite rank and let $R := A \propto E$ be the trivial ring extension of A by E. Let $I := 0 \propto E$. Then:

(1) R is a P-2-Bézout ring.
(2) $R/(0 \propto E) \cong A$ is not a P-2-Bézout ring.
(3) The ideal I is not a finitely presented ideal of R.

Proof: (1) It follows from Proposition 2.3.

(2) We have that $R/(0 \propto E) \cong A$ is not a P-2-Bézout ring, since $M := 0 \propto \mathbb{C}$ is a finitely presented prime ideal of A (since A is a coherent ring) which is not principal.

(3) The ideal I is not a finitely presented ideal of R since E is an (A/M)-vector space with infinite rank. $\square$

Next, we study the transfer of P-2-Bézout and 2-Bézout properties to direct products.

Theorem 2.3. *Let $(R_i)_{i=1,\dots,n}$ be a family of rings. Then*

(1) $\prod_{i=1}^n R_i$ *is P-2-Bézout if and only if R_i is P-2-Bézout for each $i \in \{1,\dots,n\}$.*
(2) $\prod_{i=1}^n R_i$ *is 2-Bézout if and only if R_i is 2-Bézout for each $i \in \{1,\dots,n\}$.*

We need the following lemma before proving Theorem 2.3:

Lemma 2.1 ([13, Lemma 2.5 (1)]). *Let $(R_i)_{i=1,2}$ be a family of rings and E_i an R_i-module for $i=1,2$. Then $E_1 \times E_2$ is a finitely presented $R_1 \times R_2$-module if and only if E_i is a finitely presented R_i-module for $i=1,2$.*

Proof: (1) By induction on n, it suffices to prove the assertion for $n=2$. Since a prime ideal of $R_1 \times R_2$ is of the form $P_1 \times R_2$ or $R_1 \times P_2$, where P_i is a prime ideal of R_i for $i=1,2$ and a principal ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$, where I_i is a principal ideal of R_i for $i=1,2$, the conclusion follows easily from Lemma 2.1.

(2) This is straightforward. □

Theorem 2.3 enriches the literature with examples of non-2-Bézout P-2-Bézout rings and examples of non-P-Bézout P-2-Bézout rings.

Example 2.4. Let R_1 be a non-2-Bézout P-2-Bézout ring see [6, Example 2.3], R_2 a Bézout ring and $R := R_1 \times R_2$. Then:

(1) R is a P-2-Bézout ring by Theorem 2.3 (1).
(2) R is not a 2-Bézout ring by Theorem 2.3 (2).

Example 2.5. Let R_1 be a non-P-Bézout P-2-Bézout ring, see [6, Example 2.5], R_2 a Bézout ring and $R := R_1 \times R_2$. Then:

(1) R is a P-2-Bézout ring by Theorem 2.3 (1).
(2) R is not a P-Bézout ring by [3, Proposition 2.8].

Open problem. Is then P-2-Bézout property stable by localizations?

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