Comment.Math.Univ.Carolin. 62,3 (2021) 347–360

The Golomb space is topologically rigid

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Abstract. The Golomb space $\mathbb{N}_\tau$ is the set $\mathbb{N}$ of positive integers endowed with the topology τ generated by the base consisting of arithmetic progressions $\{a+bn\colon n\ge 0\}$ with coprime a,b. We prove that the Golomb space $\mathbb{N}_\tau$ is topologically rigid in the sense that its homeomorphism group is trivial. This resolves a problem posed by T. Banakh at Mathoverflow in 2017.

Keywords: Golomb topology; topologically rigid space

Classification: 11A99, 54G15

1. Introduction

In the AMS Meeting announcement [3] M. Brown introduced an amusing topology τ on the set $\mathbb{N}$ of positive integers turning it into a connected Hausdorff space. The topology τ is generated by the base consisting of arithmetic progressions $a+b\mathbb{N}_0 := \{a+bn\colon n\in\mathbb{N}_0\}$ with coprime parameters $a,b\in\mathbb{N}$. Here by $\mathbb{N}_0=\{0\}\cup\mathbb{N}$ we denote the set of nonnegative integer numbers.

In [15] the topology τ is called the *relatively prime integer topology*. This topology was popularized by S. Golomb in [7], [8], who observed that the classical Dirichlet theorem on primes in arithmetic progressions is equivalent to the density of the set Π of prime numbers in the topological space $(\mathbb{N},\tau)$. As a by-product of such popularization efforts, the topological space $\mathbb{N}_\tau := (\mathbb{N},\tau)$ is known in general topology as the *Golomb space*, see [16], [17].

The topological structure of the Golomb space $\mathbb{N}_\tau$ was studied by T. Banakh, J. Mioduszewski and S. Turek in [2], who proved that the space $\mathbb{N}_\tau$ is not topologically homogeneous (by showing that 1 is a fixed point of any homeomorphism of $\mathbb{N}$). Motivated by this results, the authors of [2] posed a problem of the topological rigidity of the Golomb space. This problem was also repeated by the first author at `Mathoverflow`, see [1]. A topological space X is defined to be *topologically rigid* if its homeomorphism group is trivial.

The main result of this note is the following theorem answering the above problem.

DOI 10.14712/1213-7243.2021.023

Theorem 1. *The Golomb space $\mathbb{N}_\tau$ is topologically rigid.*

The proof of this theorem will be presented in Section 5 after some preparatory work made in Sections 3 and 4. The idea of the proof belongs to the second author who studied in [13] the rigidity properties of the Golomb topology on a Dedekind ring with removed zero, and established in [13, Theorem 6.7] that the homeomorphism group of the Golomb topology on $\mathbb{Z} \setminus \{0\}$ consists of two homeomorphisms. The proof of Theorem 1 is a modified (and simplified) version of the proof of Theorem 6.7 given in [13]. It should be mentioned that the Golomb topology on Dedekind rings with removed zero was studied by J. Knopfmacher, Š. Porubský in [11], P. L. Clark, N. Lebowitz-Lockard, P. Pollack in [4], and D. Spirito in [13], [14].

2. Preliminaries and notations

In this section we fix some notation and recall some known results on the Golomb topology. For a subset A of a topological space X, by $\bar{A}$ we denote the closure of A in X.

A *poset* is a set X endowed with a partial order "$\le$". A subset L of a partially ordered set $(X, \le)$ is called

- *linearly ordered* (or else a *chain*) if any points $x, y \in L$ are *comparable* in the sense that $x \le y$ or $y \le x$;
- an *antichain* if any two distinct elements $x, y \in A$ are not comparable.

By Π we denote the set of prime numbers. For a number $x \in \mathbb{N}$ we denote by Π_x the set of all prime divisors of x. Two numbers $x, y \in \mathbb{N}$ are *coprime* if and only if $\Pi_x \cap \Pi_y = \emptyset$. For a number $x \in \mathbb{N}$ let $x^{\mathbb{N}} := \{x^n : n \in \mathbb{N}\}$ be the set of all powers of x.

For a number $x \in \mathbb{N}$ and a prime number p let $l_p(x)$ be the largest integer number such that $p^{l_p(x)}$ divides x. The function $l_p(x)$ plays the role of logarithm with base p.

The following formula for the closures of basic open sets in the Golomb topology was established in [2, 2.2].

Lemma 2 (T. Banakh, J. Mioduszewski, S. Turek)**.** *For any $a, b \in \mathbb{N}$*

$$\overline{a + b\mathbb{N}_0} = \mathbb{N} \cap \bigcap_{p \in \Pi_b} \big(p\mathbb{N} \cup (a + p^{l_p(b)}\mathbb{Z})\big).$$

Also we shall heavily exploit the following lemma, proved in [2, 5.1].

Lemma 3 (T. Banakh, J. Mioduszewski, S. Turek)**.** *Each homeomorphism $h\colon \mathbb{N}_\tau \longrightarrow \mathbb{N}_\tau$ of the Golomb space has the following properties:*

(1) $h(1) = 1$;
(2) $h(\Pi) = \Pi$;
(3) $\Pi_{h(x)} = h(\Pi_x)$ *for every* $x \in \mathbb{N}$;
(4) $h(x^{\mathbb{N}}) = h(x)^{\mathbb{N}}$ *for every* $x \in \mathbb{N}$.

Let p be a prime number and $k \in \mathbb{N}$. Let $\mathbb{Z}$ be the ring of integer numbers, $\mathbb{Z}_{p^k}$ be the residue ring $\mathbb{Z}/p^k\mathbb{Z}$, and $\mathbb{Z}_{p^k}^{\times}$ be the multiplicative group of invertible elements of the ring $\mathbb{Z}_{p^k}$. It is well-known that $|\mathbb{Z}_{p^k}^{\times}| = \varphi(p^k) = p^{k-1}(p-1)$. The structure of the group $\mathbb{Z}_{p^k}^{\times}$ was described by Gauss in [6, art. 52–56] (see also Theorems 2 and 2' in Chapter 4 of [9]).

Lemma 4 (C. F. Gauss)**.** *Let p be a prime number and $k \in \mathbb{N}$.*

(1) *If p is odd, then the group $\mathbb{Z}_{p^k}^{\times}$ is cyclic.*
(2) *If $p = 2$ and $k \geq 2$, then the element $-1 + 2^k\mathbb{Z}$ generates a two-element cyclic group C_2 in $\mathbb{Z}_{2^k}^{\times}$ and the element $5+2^k\mathbb{Z}$ generates a cyclic subgroup $C_{2^{k-2}}$ of order 2^{k-2} in $\mathbb{Z}_{2^k}^{\times}$ such that $\mathbb{Z}_{2^k}^{\times} = C_2 \oplus C_{2^{k-2}}$.*

Lemma 5. *If H is a non-cyclic subgroup of the multiplicative group $\mathbb{Z}_{2^k}^{\times}$ for some $k \geq 3$, then H contains the Boolean subgroup*

$$V = \{1 + 2^k\mathbb{Z}, -1 + 2^k\mathbb{Z}, 1 + 2^{k-1} + 2^k\mathbb{Z}, -1 + 2^{k-1} + 2^k\mathbb{Z}\}.$$

PROOF: Observe that the multiplicative group $\mathbb{Z}_{2^k}^{\times}$ has order 2^{k-1}, which implies that the order of every element of $\mathbb{Z}_{2^k}^{\times}$ is a power of 2. The Gauss Lemma 4 implies that the multiplicative group $\mathbb{Z}_{2^k}^{\times}$ has exactly 4 elements of order less than or equal to 2 and those elements form the 4-element Boolean subgroup V.

Applying the Frobenius–Stickelberger theorem 4.2.6, see [12], we conclude that the finite subgroup $H \subseteq \mathbb{Z}_{2^k}^{\times}$ is the direct sum of finite cyclic groups whose orders are powers of 2. Since H is not cyclic, at least two cyclic groups in this direct sum are not trivial, which implies that H contains at least four element of order less than or equal to 2. Taking into account that the elements of the subgroup V are the only elements of order less than or equal to 2 in the group $\mathbb{Z}_{2^k}^{\times}$, we conclude that $V \subseteq H$. □

3. Golomb topology versus the p-adic topologies on $\mathbb{N}$

Let p be any prime number. Let us recall that the *p-adic* topology on $\mathbb{Z}$ is generated by the base consisting of the sets $x + p^n\mathbb{Z}$, where $x \in \mathbb{Z}$ and $n \in \mathbb{N}$. This topology induces the *p-adic* topology on the subset $\mathbb{N}$ of $\mathbb{Z}$. It is generated by the base consisting of the sets $x + p^n\mathbb{N}_0$ where $x, n \in \mathbb{N}$. It is easy to see that $\mathbb{N}$ endowed with the p-adic topology is a regular second-countable space without isolated points. So, by Sierpiński theorem, see [5, 6.2.A (d)], this space is homeomorphic to the space of rationals and hence is topologically homogeneous. Consequently, any nonempty open subspace of $\mathbb{N}$ with the p-adic topology (in particular, $\mathbb{N} \setminus p\mathbb{N}$) also is homeomorphic to $\mathbb{Q}$ and hence is topologically homogeneous.

The following lemma is a special case of Proposition 3.1 in [13].

Lemma 6. *For any clopen subset Ω of $\mathbb{N}_\tau \setminus p\mathbb{N}$, and any $x \in \Omega$, there exists $n \in \mathbb{N}$ such that $x + p^n\mathbb{N}_0 \subseteq \Omega$.*

Proof: Since the set $p\mathbb{N}$ is closed in $\mathbb{N}_\tau$, the set Ω is open in $\mathbb{N}_\tau$ and hence $x + p^n b\mathbb{N}_0 \subseteq \Omega$ for some $n \in \mathbb{N}$ and $b \in \mathbb{N}$ which is coprime with px. We claim that $x + p^n\mathbb{N}_0 \subseteq \Omega$. To derive a contradiction, assume that $x + p^n\mathbb{N}_0 \setminus \Omega$ contains some number y. Since Ω is closed in $\mathbb{N}_\tau \setminus p\mathbb{N}$, there exist $m \geq n$ and $d \in \mathbb{N}$ such that d is coprime with p and y, and $(y + p^m d\mathbb{N}_0) \cap \Omega = \emptyset$. It follows that $y + p^m\mathbb{N}_0 \subseteq (x + p^n\mathbb{N}_0) + p^m\mathbb{N}_0 \subseteq x + p^n\mathbb{N}_0$. Since $p \notin \Pi_b \cup \Pi_d$, we can apply the Chinese remainder theorem [10, 3.12] and conclude that $\emptyset \neq (y + p^m\mathbb{N}) \cap \bigcap_{q \in \Pi_b \cup \Pi_d} q\mathbb{N}$. Applying Lemma 2 and taking into account that the set Ω is clopen in $\mathbb{N}_\tau \setminus p\mathbb{N}$, we conclude that

$$\begin{aligned}
\emptyset &\neq (y + p^m\mathbb{N}_0) \cap \Big(\bigcap_{q \in \Pi_b \cup \Pi_d} q\mathbb{N} \Big) \\
&= (x + p^n\mathbb{N}_0) \cap \Big(\bigcap_{q \in \Pi_b} q\mathbb{N} \Big) \cap (y + p^m\mathbb{N}_0) \cap \Big(\bigcap_{q \in \Pi_d} q\mathbb{N} \Big) \\
&\subseteq \overline{x + p^n b\mathbb{N}_0} \cap \overline{y + p^m d\mathbb{N}_0} \subseteq \overline{\Omega} \cap \overline{(\mathbb{N} \setminus p\mathbb{N}) \setminus \Omega} \subseteq p\mathbb{N},
\end{aligned}$$

which is not possible as the sets $x + p^n\mathbb{N}_0$ and $p\mathbb{N}$ are disjoint. This contradiction shows that $x + p^n\mathbb{N}_0 \subseteq \Omega$. □

A subset of a topological space is *clopen* if it is closed and open. By the *zero-dimensional reflection* of a topological space X we understand the space X endowed with the topology generated by the base consisting of clopen subsets of the space X.

Lemma 7. *The p-adic topology on $\mathbb{N} \setminus p\mathbb{N}$ coincides with the zero-dimensional reflection of the subspace $\mathbb{N}_\tau \setminus p\mathbb{N}$ of the Golomb space $\mathbb{N}_\tau$.*

Proof: Lemma 6 implies that the p-adic topology τ_p on $\mathbb{N} \setminus p\mathbb{N}$ is stronger than the topology ζ of zero-dimensional reflection on $\mathbb{N}_\tau \setminus p\mathbb{N}$. To see that the τ_p coincides with ζ, it suffices to show that for every $x \in \mathbb{N} \setminus p\mathbb{N}$ and $n \in \mathbb{N}$ the basic open set $\mathbb{N} \cap (x + p^n\mathbb{Z})$ in the p-adic topology is clopen in the subspace topology of $\mathbb{N}_\tau \setminus p\mathbb{N} \subset \mathbb{N}_\tau$. By the definition, the set $\mathbb{N} \cap (x + p^n\mathbb{Z})$ is open in the Golomb topology. To see that it is closed in $\mathbb{N}_\tau \setminus p\mathbb{N}$, take any point $y \in (\mathbb{N}\setminus p\mathbb{N})\setminus(x+p^n\mathbb{Z})$ and observe that the Golomb-open neighborhood $y + p^n\mathbb{N}_0$ of y is disjoint with the set $\mathbb{N} \cap (x + p^n\mathbb{Z})$. $\square$

For every prime number p, consider the countable family

$$\mathcal{X}_p = \big\{\overline{a^{\mathbb{N}}}\colon a \in \mathbb{N} \setminus p\mathbb{N},\ a \neq 1\big\},$$

where the closure $\overline{a^{\mathbb{N}}}$ is taken in the p-adic topology on $\mathbb{N} \setminus p\mathbb{N}$, which coincides with the topology of zero-dimensional reflection of the Golomb topology on $\mathbb{N}\setminus p\mathbb{N}$ according to Lemma 7.

The family $\mathcal{X}_p$ is endowed with the partial order "$\le$" defined by $X \le Y$ if and only if $Y \subseteq X$. So, $\mathcal{X}_p$ is a poset carrying the partial order of reverse inclusion.

Lemma 8. *For any prime number p, any homeomorphism h of the Golomb space $\mathbb{N}_\tau$ induces an order isomorphism*

$$h\colon \mathcal{X}_p \longrightarrow \mathcal{X}_{h(p)}, \qquad h\colon \overline{a^{\mathbb{N}}} \mapsto h(\overline{a^{\mathbb{N}}}) = \overline{h(a)^{\mathbb{N}}}$$

of the posets $\mathcal{X}_p$ and $\mathcal{X}_{h(p)}$.

Proof: By Lemma 3, $h(1) = 1$ and $h(p)$ is a prime number. First we show that $h(p\mathbb{N}) = h(p)\mathbb{N}$. Indeed, for any $x \in p\mathbb{N}$ we have $p \in \Pi_x$ and by Lemma 3, $h(p) \in h(\Pi_x) = \Pi_{h(x)}$ and hence $h(x) \in h(p)\mathbb{N}$ and $h(p\mathbb{N}) \subseteq h(p)\mathbb{N}$. Applying the same argument to the homeomorphism h^{-1}, we obtain $h^{-1}(h(p)\mathbb{N}) \subseteq p\mathbb{N}$, which implies the desired equality $h(p\mathbb{N}) = h(p)\mathbb{N}$. The bijectivity of h ensures that h maps homeomorphically the space $\mathbb{N}_\tau \setminus p\mathbb{N}$ onto the space $\mathbb{N}_\tau \setminus h(p)\mathbb{N}$.

Then h also is a homeomorphism of the spaces $\mathbb{N} \setminus p\mathbb{N}$ and $\mathbb{N} \setminus h(p)\mathbb{N}$ endowed with the zero-dimensional reflections of their subspace topologies inherited from the Golomb topology of $\mathbb{N}_\tau$. By Lemma 7, these reflection topologies on $\mathbb{N} \setminus p\mathbb{N}$ and $\mathbb{N} \setminus h(p)\mathbb{N}$ coincide with the p-adic and $h(p)$-adic topologies on $\mathbb{N} \setminus p\mathbb{N}$ and $\mathbb{N} \setminus h(p)\mathbb{N}$, respectively.

By Lemma 3, for any $a \in \mathbb{N} \setminus (\{1\} \cup p\mathbb{N})$ we have

$$h(a)^{\mathbb{N}} = h(a^{\mathbb{N}}) \subseteq h(\mathbb{N} \setminus p\mathbb{N}) = \mathbb{N} \setminus h(p)\mathbb{N}$$

and by the fact that $h\colon \mathbb{N} \setminus p\mathbb{N} \longrightarrow \mathbb{N} \setminus h(p)\mathbb{N}$ is a homeomorphism in the topologies of zero-dimensional reflections, we get $h(\overline{a^{\mathbb{N}}}) = \overline{h(a^{\mathbb{N}})} = \overline{h(a)^{\mathbb{N}}}$. The same argument applies to the homeomorphism h^{-1}. This implies that

$$h\colon \mathcal{X}_p \longrightarrow \mathcal{X}_{h(p)}, \qquad h\colon \overline{a^{\mathbb{N}}} \mapsto h(\overline{a^{\mathbb{N}}}) = \overline{h(a)^{\mathbb{N}}},$$

is a well-defined bijection. It is clear that this bijection preserves the inclusion order and hence it is an order isomorphism between the posets $\mathcal{X}_p$ and $\mathcal{X}_{h(p)}$. $\square$

4. The order structure of the posets $\mathcal{X}_p$

In this section, given a prime number p, we investigate the order-theoretic structure of the poset $\mathcal{X}_p$.

For every $n \in \mathbb{N}$ denote by $\pi_n \colon \mathbb{N} \longrightarrow \mathbb{Z}_{p^n}$ the homomorphism assigning to each number $x \in \mathbb{N}$ the residue class $x + p^n\mathbb{Z}$. Also for $n \le m$ let

$$\pi_{m,n} \colon \mathbb{Z}_{p^m} \longrightarrow \mathbb{Z}_{p^n}$$

be the ring homomorphism assigning to each residue class $x + p^m\mathbb{Z}$ the residue class $x+p^n\mathbb{Z}$. It is easy to see that $\pi_n = \pi_{m,n} \circ \pi_m$. Observe that the multiplicative group $\mathbb{Z}^\times_{p^n}$ of invertible elements of the ring $\mathbb{Z}_{p^n}$ coincides with the set $\mathbb{Z}_{p^n} \setminus p\mathbb{Z}_{p^n}$ and hence has cardinality $p^n - p^{n-1} = p^{n-1}(p-1)$. Observe that for every $a \in \mathbb{N} \setminus p\mathbb{Z}$ the set $\pi_n(a^{\mathbb{N}}) = \pi_n(a)^{\mathbb{N}}$ is a multiplicative subgroup of the finite group $\mathbb{Z}^\times_{p^n}$.

First we establish the structure of the elements $\overline{a^{\mathbb{N}}}$ of the family $\mathcal{X}_p$.

Lemma 9. *If for some $a \in \mathbb{N} \setminus p\mathbb{Z}$ and $n \in \mathbb{N}$ the element $\pi_n(a)$ has order greater than or equal to $\max\{p,3\}$ in the multiplicative group $\mathbb{Z}^\times_{p^n}$, then $\overline{a^{\mathbb{N}}} = \pi_n^{-1}(\pi_n(a)^{\mathbb{N}})$.*

Proof: Let $B = b^{\mathbb{N}}$ be the cyclic group generated by the element $b = \pi_n(a)$ in the multiplicative group $\mathbb{Z}^\times_{p^n}$. Since $|\mathbb{Z}^\times_{p^n}| = p^{n-1}(p-1)$, and b has order greater than or equal to $\max\{p,3\}$, the cardinality of the group B is equal to $p^k d$ for some $k \in \{1,\dots,n-1\}$ and some divisor d of the number $p-1$. Moreover, if $p=2$, then $2^k \ge 3$ and hence $k \ge 2$ and $n \ge 3$.

For any number $m \ge n$, consider the quotient homomorphism

$$\pi_{m,n} \colon \mathbb{Z}_{p^m} \longrightarrow \mathbb{Z}_{p^n}, \qquad \pi_{m,n} \colon x + p^m\mathbb{Z} \mapsto x + p^n\mathbb{Z}.$$

We claim that the subgroup $H = \pi_{m,n}^{-1}(B)$ of the multiplicative group $\mathbb{Z}^\times_{p^m}$ is cyclic. For odd p this follows from the cyclicity of the group $\mathbb{Z}^\times_{p^n}$, see Lemma 4.

For $p=2$, by Lemma 4, the multiplicative group $\mathbb{Z}^\times_{2^m}$ is isomorphic to the additive group $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$. Assuming that H is not cyclic and applying Lemma 5, we conclude that H contains the 4-element Boolean subgroup

$$V = \{1+2^m\mathbb{Z}, -1+2^m\mathbb{Z}, 1+2^{m-1}+2^m\mathbb{Z}, -1+2^{m-1}+2^m\mathbb{Z}\}$$

of $\mathbb{Z}^\times_{2^m}$. Then $B = \pi_{m,n}(H) \supseteq \pi_{m,n}(V) \ni -1+2^n\mathbb{Z}$. Taking into account that $-1+2^n\mathbb{Z}$ has order 2 in the cyclic group B, we conclude that $-1+2^n\mathbb{Z} = a^{2^{k-1}} + 2^n\mathbb{Z}$. Since $k \ge 2$, the odd number $c = a^{2^{k-2}}$ is well-defined and $c^2 + 4\mathbb{Z} = a^{2^{k-1}} + 4\mathbb{Z} = -1 + 4\mathbb{Z}$, which is not possible (as squares of odd numbers are equal to 1 modulo 4). This contradiction shows that the group H is cyclic.

By [12, 1.5.5], the number of generators of the cyclic group H can be calculated using the Euler totient function as

$$\begin{aligned}\varphi(|H|) &= \varphi(p^{m-n}|B|) = \varphi(p^{m-n}p^k d) = \varphi(p^{m-n+k})\varphi(d)\\ &= p^{m-n+k-1}(p-1)\varphi(d) = p^{m-n}\varphi(p^k)\varphi(d) = p^{m-n}\varphi(p^k d)\\ &= p^{m-n}\varphi(|B|),\end{aligned}$$

which implies that for every generator g of the group B, every element of the set $\pi_{m,n}^{-1}(g)$ is a generator of the group H. In particular, the element $\pi_m(a) \in \pi_{m,n}^{-1}(\pi_n(a))$ is a generator of the group H. By the definition of p-adic topology,

$$\begin{aligned}\overline{a^{\mathbb{N}}} &= \bigcap_{m\ge n} \pi_m^{-1}(\pi_m(a)^{\mathbb{N}}) = \bigcap_{m\ge n} \pi_m^{-1}(\pi_{m,n}^{-1}(B))\\ &= \bigcap_{m\ge n} \pi_n^{-1}(B) = \pi_n^{-1}(B) = \pi_n^{-1}(\pi_n(a)^{\mathbb{N}}).\end{aligned}$$

□

Lemma 10. (1) *For every $X \in \mathcal{X}_p$ there exists $n \in \mathbb{N}$ and a cyclic subgroup H of the multiplicative group $\mathbb{Z}_{p^n}^{\times}$ such that $X = \pi_n^{-1}(H)$ and $|H| \ge \max\{p, 3\}$.*

(2) *For every $n \in \mathbb{N}$ and cyclic subgroup H of $\mathbb{Z}_{p^n}^{\times}$ of order $|H| \ge \max\{p, 3\}$, there exists a number $a \in \mathbb{N}\setminus p\mathbb{N}$ such that $\pi_n^{-1}(H) = \overline{a^{\mathbb{N}}} \in \mathcal{X}_p$.*

Proof: (1) Given any $X \in \mathcal{X}_p$, find a number $a \in \mathbb{N}\setminus(\{1\}\cup p\mathbb{N})$ such that $X = \overline{a^{\mathbb{N}}}$. Choose any $n \in \mathbb{N}$ with $p^n > a^p$ and observe that the cyclic subgroup $H \subseteq \mathbb{Z}_{p^n}^{\times}$, generated by the element $\pi_n(a) = a + p^n\mathbb{Z}$, has order $|H| \ge p+1 \ge \max\{p, 3\}$.

(2) Fix $n \in \mathbb{N}$ and a cyclic subgroup H of $\mathbb{Z}_{p^n}^{\times}$ of order $|H| \ge \max\{p, 3\}$. Find a number $a \in \mathbb{N}$ such that the residue class $\pi_n(a) = a + p^n\mathbb{Z}$ is a generator of the cyclic group H. Then $\pi_n(a)$ has order $|H| \ge \max\{p, 3\}$, Lemma 9 ensures that $\pi_n^{-1}(H) = \pi_n^{-1}(\pi_n(a)^{\mathbb{N}}) = \overline{a^{\mathbb{N}}} \in \mathcal{X}_p$. □

For any $X \in \mathcal{X}_p$, let

$$n(X) = \min\{n \in \mathbb{N}\colon X = \pi_n^{-1}(\pi_n(X)),\ |\pi_n(X)| \ge \max\{p, 3\}\}.$$

Lemmas 9 and 10 imply that the number $n(X)$ is well-defined and $\pi_{n(X)}(X)$ is a cyclic subgroup of order greater than or equal to $\max\{p, 3\}$ in the multiplicative group $\mathbb{Z}_{p^{n(X)}}^{\times}$. Let $i(X)$ be the index of the cyclic subgroup $\pi_{n(X)}(X)$ in $\mathbb{Z}_{p^{n(X)}}^{\times}$.

Lemma 11. *Let $p = 2$, $a > 1$ be an odd integer, and $X = \overline{a^{\mathbb{N}}}$ be the closure of the set $a^{\mathbb{N}}$ in the 2-adic topology of $\mathbb{N}\setminus 2\mathbb{N}$. The cyclic subgroup $\pi_{n(X)}(X)$ of $\mathbb{Z}_{2^{n(X)}}^{\times}$ has order 4 and index $i(X) = 2^{n(X)-3} \ge 2$.*

Proof: By definition of $n(X)$ and Lemma 9, $n(X)$ is the smallest number such that the cyclic subgroup $\pi_{n(X)}(X) = \pi_{n(X)}(a^{\mathbb{N}})$ of $\mathbb{Z}^\times_{2^{n(X)}}$ has order greater than or equal to 3. Then $|\pi_{n(X)}(a^{\mathbb{N}})| = 2^k$ for some $k \geq 2$. If $k \neq 2$, then we can consider the projection $\pi_{n(X)-1}(X) = \pi_{n(X),n(X)-1}(\pi_{n(X)}(X))$ and conclude that $|\pi_{n(X)-1}(X)| \geq |\pi_{n(X)}(X)|/2 \geq 2^{k-1} \geq 4 \geq 3$ (since the homomorphism $\pi_{n(X),n(X)-1}\colon \mathbb{Z}_{2^{n(X)}} \twoheadrightarrow \mathbb{Z}_{2^{n(X)-1}}$ has kernel of cardinality 2), but this contradicts the minimality of $n(X)$. This contradiction shows that $|\pi_{n(X)}(X)| = 4$.

The group $\mathbb{Z}^\times_{2^{n(X)}}$ has cardinality $|\mathbb{Z}^\times_{2^{n(X)}}| \geq |\pi_{n(X)}(X)| = 4$ and therefore $n(X) \geq 3$. By Lemma 4 (2), the multiplicative group $\mathbb{Z}^\times_{2^{n(X)}}$ is not cyclic, which implies $\pi_{n(X)}(X) \neq \mathbb{Z}^\times_{2^{n(X)}}$ and hence $i(X) \geq |\mathbb{Z}^\times_{2^{n(X)}}/\pi_{n(X)}(X)| = 2^{n(X)-3} \geq 2$. □

Lemma 12. *For any odd prime number p and two sets $X, Y \in \mathcal{X}_p$, the inclusion $X \subseteq Y$ holds if and only if $i(Y)$ divides $i(X)$.*

Proof: Let $m = \max\{n(X), n(Y)\}$. Then $X = \pi_m^{-1}(\pi_m(X))$, $Y = \pi_m^{-1}(\pi_m(Y))$ and $\pi_m(X), \pi_m(Y)$ are subgroups of the multiplicative group $\mathbb{Z}^\times_{p^m}$, which is cyclic by the Gauss Lemma 4 (1). It follows that the subgroups $\pi_m(X)$ and $\pi_m(Y)$ have indexes $i(X)$ and $i(Y)$ in $\mathbb{Z}^\times_{p^m}$, respectively. Let g be a generator of the cyclic group $\mathbb{Z}^\times_{p^m}$. It follows that the subgroups $\pi_m(X)$ and $\pi_m(Y)$ are generated by the elements $g^{i(X)}$ and $g^{i(Y)}$, respectively. Now we see that $X \subseteq Y$ if and only if $\pi_m(X) \subseteq \pi_m(Y)$ if and only if $g^{i(X)} \in (g^{i(Y)})^{\mathbb{N}}$ if and only if $i(Y)$ divides $i(X)$. □

Lemma 13. *For any odd prime number p, any $n \in \mathbb{N}$, and the number $a = 1+p^n$ we have $\overline{a^{\mathbb{N}}} = 1 + p^n\mathbb{N}_0$ and $i(\overline{a^{\mathbb{N}}}) = p^{n-1}(p-1)$.*

Proof: Observe that for any $k < p$ we have $a^k = (1+p^n)^k \in 1 + kp^n + p^{n+1}\mathbb{Z} \neq 1+p^{n+1}\mathbb{Z}$ and $a^p = (1+p^n)^p \in 1+p^{n+1}\mathbb{Z}$, which means that the element $\pi_{n+1}(a)$ has order p in the group $\mathbb{Z}^\times_{p^{n+1}}$. By Lemma 9,

$$\overline{a^{\mathbb{N}}} = \pi_{n+1}^{-1}(\{a^k + p^{n+1}\mathbb{Z}\colon 0 \leq k < p\}) = \bigcup_{k=0}^{p-1}(a^k + p^{n+1}\mathbb{N}_0) = 1 + p^n\mathbb{N}_0.$$

Also $i(\overline{a^{\mathbb{N}}}) = |\mathbb{Z}^\times_{p^{n+1}}|/p = p^{n-1}(p-1)$. □

Lemmas 10 and 12 imply that for an odd prime number p, the poset $\mathcal{X}_p$ is order isomorphic to the set

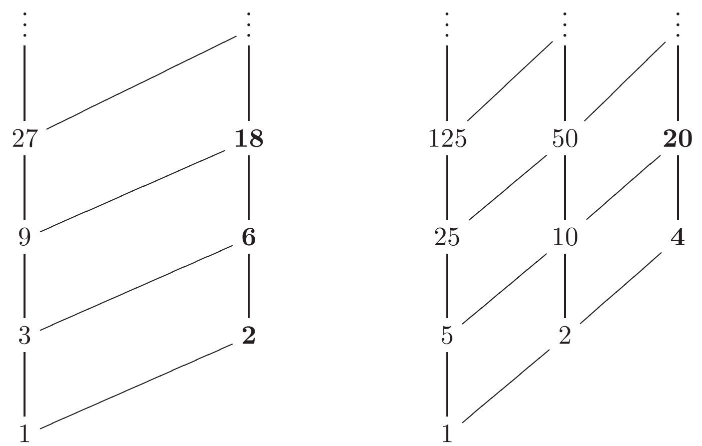
$$\mathcal{D}_p = \{d \in \mathbb{N}\colon d \text{ divides } p^n(p-1) \text{ for some } n \in \mathbb{N}_0\},$$

endowed with the divisibility relation.

An element t of a partially ordered set $(X, \leq)$ is called $\uparrow$-*chain* if its upper set ${\uparrow}t = \{x \in X\colon x \geq t\}$ is a chain. It is easy to see that the set of $\uparrow$-chain elements of the poset $\mathcal{D}_p$ coincides with the set $\{p^n(p-1)\colon n \in \mathbb{N}_0\}$ and hence is a well-ordered chain with the smallest element $(p-1)$.

Below on the Hasse diagrams of the posets $\mathcal{D}_3$ and $\mathcal{D}_5$ (showing that these posets are not order isomorphic) the $\uparrow$-chain elements are drawn with the bold font.

$\mathcal{D}_3$ $\mathcal{D}_5$



Lemmas 12 and 13 and the isomorphness of the posets $\mathcal{X}_p$ and $\mathcal{D}_p$ imply the following lemma.

Lemma 14. *For an odd prime number p, the family $\{1+p^n\mathbb{N}_0\colon n \in \mathbb{N}\}$ coincides with the well-ordered set of $\uparrow$-chain elements of the poset $\mathcal{X}_p$.*

Now we reveal the order structure of the poset $\mathcal{X}_2$. This poset consists of the closures $\overline{a^{\mathbb{N}}}$ in the 2-adic topology of the sets $a^{\mathbb{N}}$ for odd numbers $a > 1$.

Lemma 15. *Let $a > 1$ be an odd integer and $X = \overline{a^{\mathbb{N}}}$ be the closure of $a^{\mathbb{N}}$ in the 2-adic topology on $\mathbb{N} \setminus 2\mathbb{N}$.*

(1) *If $a \in 1+4\mathbb{N}$, then $\overline{a^{\mathbb{N}}} = 1 + 2^{n(X)-2}\mathbb{N}_0$.*

(2) *If $a \in 3+4\mathbb{N}_0$, then $\overline{a^{\mathbb{N}}} = (1+2^{n(X)-1}\mathbb{N}_0) \cup (-1+2^{n(X)-2}+2^{n(X)-1}\mathbb{N}_0)$.*

In both cases, $i(X) = 2^{n(X)-3} \geq 2$.

Proof: By Lemma 11, the projection $C_X := \pi_{n(X)}(X) = \pi_{n(X)}(a^{\mathbb{N}})$ is a cyclic subgroup of order 4 and index $i(X) = 2^{n(X)-3} \geq 2$ in the group $\mathbb{Z}^{\times}_{2^{n(X)}}$.

By Lemma 4 (2), the coset $5 + 2^{n(X)}\mathbb{Z}$ generates a maximal cyclic subgroup

$$M_X = \{1 + 4k + 2^{n(X)}\mathbb{Z}\colon 0 \leq k < 2^{n(X)-2}\}$$

of cardinality $2^{n(X)-2}$ in $\mathbb{Z}^\times_{2^{n(X)}}$. If $a \in 1+4\mathbb{N}$, the subgroup generated by $\pi_{n(X)}(a)$ is contained in M_X. Then $C_X = \{1 + k \cdot 2^{n(X)-2} + 2^{n(X)}\mathbb{Z} \colon 0 \le k < 4\}$ and $X = \pi^{-1}_{n(X)}(C_X) = 1 + 2^{n(X)-2}\mathbb{N}_0$.

If $a \in 3 + 4\mathbb{N}_0$, then C_X is not contained in M_X. By the Gauss Lemma 4 (2), there are two cyclic subgroups of $\mathbb{Z}^\times_{2^{n(X)}}$ of order 4: one generated by $g = (5 + 2^{n(X)}\mathbb{Z})^{n(X)-2}$ (which is contained in M_X) and the other is generated by $-g$, which is not contained in M_X but contains $-1 + 2^{n(X)}$. Therefore, C_X must be equal to $C_X = \{(-1)^k + k \cdot 2^{n(X)-2} + 2^{n(X)}\mathbb{Z} \colon 0 \le k < 4\}$ and

$$\begin{aligned} X = \pi^{-1}_{n(X)}(C_X) &= \bigcup_{k=0}^{3} \pi^{-1}_{n(X)}\big((-1)^k + k \cdot 2^{n(X)-2} + 2^{n(X)}\mathbb{Z}\big) \\ &= (1 + 2^{n(X)-1}\mathbb{N}_0) \cup (-1 + 2^{n(X)-2} + 2^{n(X)-1}\mathbb{N}_0). \end{aligned}$$

□

Lemma 16. *For every $n \ge 2$,*

1. *the set $X = \overline{(1+2^n)^{\mathbb{N}}} \in \mathcal{X}_2$ coincides with $1+2^n\mathbb{N}_0$ and has $i(X) = 2^{n-1}$;*
2. *the set $Y = \overline{(-1+2^n)^{\mathbb{N}}} \in \mathcal{X}_2$ coincides with $(1 + 2^{n+1}\mathbb{N}_0) \cup (2^n - 1 + 2^{n+1}\mathbb{N}_0)$ and has $i(Y) = 2^{n-1}$.*

Proof: (1) Observe that for every positive $k < 4$ we have $(1+2^n)^k \in 1 + k2^n + 2^{n+2}\mathbb{Z} \neq 1 + 2^{n+2}\mathbb{Z}$ and $(1+2^n)^4 \in 1 + 2^{n+2}\mathbb{Z}$, which means that the element $(1+2^n) + 2^{n+2}\mathbb{Z}$ has order 4 in the group $\mathbb{Z}^\times_{2^{n+2}}$. Then the element $X = \overline{(1+2^n)^{\mathbb{N}}} \in \mathcal{X}_2$ has $n(X) = n+2$ and hence $X = 1 + 2^n\mathbb{N}_0$ and $i(X) = 2^{n(X)-3} = 2^{n-1}$ by Lemma 15.

(2) Also for every positive $k < 4$ we have $(-1+2^n)^k \in (-1)^k(1 - k2^n) + 2^{n+2}\mathbb{Z} \neq 1 + 2^{n+2}\mathbb{Z}$ and $(-1+2^n)^4 \in 1 + 2^{n+2}\mathbb{Z}$, which means that the element $(-1+2^n) + 2^{n+2}\mathbb{Z}$ has order 4 in the group $\mathbb{Z}^\times_{2^{n+2}}$. Then the element $Y = \overline{(-1+2^n)^{\mathbb{N}}} \in \mathcal{X}_2$ has $n(Y) = n+2$ and hence $Y = (1+2^{n+1}\mathbb{N}_0) \cup (2^n - 1 + 2^{n+1}\mathbb{N}_0)$ and $i(Y) = 2^{n(Y)-3} = 2^{n-1}$ by Lemma 15. □

Lemma 17. *For distinct sets $X, Y \in \mathcal{X}_2$, the strict embedding $X \subset Y$ holds if and only if $X \subseteq 1 + 4\mathbb{N}_0$ and $i(Y) < i(X)$.*

Proof: If $X \subseteq 1+4\mathbb{N}_0$, then by Lemma 15, $X = 1+2^{n(X)-2}\mathbb{N}_0$. If $i(Y) < i(X)$, then $n(Y) < n(X)$ (see Lemma 15). If $Y \subseteq 1 + 4\mathbb{N}_0$, then Lemma 15 implies

$$X = 1 + 2^{n(X)-2}\mathbb{N}_0 \subset 1 + 2^{n(Y)-2}\mathbb{N}_0 = Y.$$

If $Y \not\subseteq 1 + 4\mathbb{N}_0$, then Lemma 15 ensures that

$$X = 1 + 2^{n(X)-2}\mathbb{N}_0 \subset 1 + (2^{n(Y)-1}\mathbb{N}_0) \cup (-1 + 2^{n(Y)-2} + 2^{n(Y)-1}\mathbb{N}_0) = Y.$$

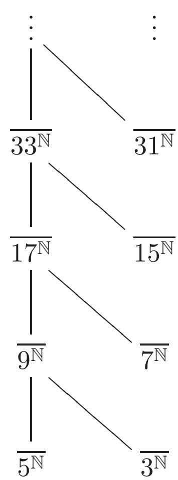
In both cases we have the strict embedding $X \subset Y$.

Conversely, assume that $X \subset Y$. We should prove that $X \subseteq 1+4\mathbb{N}_0$ and $i(Y) < i(X)$. To derive a contradiction, assume that $X \not\subseteq 1+4\mathbb{N}_0$. Applying Lemma 15 and taking into account that $X \subset Y$, we conclude that $X = (1+2^{n(X)-1}\mathbb{N}_0) \cup (-1+2^{n(X)-2}+2^{n(X)-1}\mathbb{N}_0)$, $Y = (1+2^{n(Y)-1}\mathbb{N}_0)\cup(-1+2^{n(Y)-2}+2^{n(Y)-1}\mathbb{N}_0)$ and $n(X) > n(Y)$. Then $-1+2^{n(X)-2} \in X \subseteq Y$ implies that $-1+2^{n(X)-2}$ belongs either to $1+2^{n(Y)-1}\mathbb{N}_0$ or to $-1+2^{n(Y)-2}+2^{n(Y)-1}\mathbb{N}_0$. In the first case we conclude that $2 \in 2^{n(Y)-1}\mathbb{Z}$ and hence $n(Y) \le 2$, which contradicts Lemma 11. In the second case, we obtain that $2^{n(X)-2} \in 2^{n(Y)-2}+2^{n(Y)-1}\mathbb{N}_0$. Since $n(X) > n(Y)$, this implies $2^{n(Y)-2} \in 2^{n(Y)-1}\mathbb{Z}$, which is the final contradiction showing that $X \subseteq 1+4\mathbb{N}_0$. Then $X = 1+2^{n(X)-2}\mathbb{N}_0$ according to Lemma 15.

Next, we prove that $i(Y) < i(X)$. By Lemma 15, two cases are possible: $Y = 1+2^{n(Y)-2}\mathbb{N}_0$ or $Y = (1+2^{n(Y)-1}\mathbb{N}_0)\cup(-1+2^{n(Y)-2}+2^{n(Y)-1}\mathbb{N}_0)$. In both cases the strict inclusion $1+2^{n(X)-2}\mathbb{N}_0 = X \subset Y$ implies that $n(X) > n(Y)$ and hence $i(X) = 2^{n(X)-3} > 2^{n(Y)-3} = i(Y)$. □

Lemmas 16 and 17 imply:

Lemma 18. *The family* $\min \mathcal{X}_2 = \{X \in \mathcal{X}_2 \colon X \not\subseteq 1+8\mathbb{N}_0\}$ *coincides with the set of minimal elements of the poset* $\mathcal{X}_2$ *and the set* $\mathcal{X}_2 \setminus \min \mathcal{X}_2 = \{X \in \mathcal{X}_2 \colon X \subseteq 1+8\mathbb{N}_0\}$ *is well-ordered and coincides with the set* $\{1+2^n\mathbb{N}_0 \colon n \ge 3\}$.



The Hasse diagram of the poset $\mathcal{X}_2$.

Lemma 19. *For any homeomorphism h of the Golomb space $\mathbb{N}_\tau$ and any $n \in \{1,2,3\}$ we have $h(n) = n$.*

Proof: 1. The equality $h(1) = 1$ follows from Lemma 3 (1).

2. By Lemma 8, h induces an order isomorphism of the posets $\mathcal{X}_2$ and $\mathcal{X}_{h(2)}$. By Lemmas 16 and 18, the set $\{\overline{(-1+2^n)^{\mathbb{N}}}\colon n \geq 2\}$ is an infinite antichain in the poset $\mathcal{X}_2$. Consequently, the poset $\mathcal{X}_{h(2)}$ also contains an infinite antichain. On the other hand, for any odd prime number p the poset $\mathcal{X}_p$ is order-isomorphic to the poset $\mathcal{D}_p$, which contain no infinite antichains. Consequently, $\mathcal{X}_{h(2)}$ cannot be order isomorphic to $\mathcal{X}_p$, and hence $h(2) = 2$.

3. By Lemma 3 (2), $h(3)$ is a prime number, not equal to $h(2) = 2$. By Lemma 8, h induces an order isomorphism of the posets $\mathcal{X}_3$ and $\mathcal{X}_{h(3)}$. Then the posets $\mathcal{D}_3$ and $\mathcal{D}_{h(3)}$ also are order isomorphic. The smallest $\uparrow$-chain element of the poset $\mathcal{D}_3$ is 2 and the set ${\downarrow}2 = \{d \in \mathcal{D}_3 \colon d \text{ divides } 2\}$ has cardinality 2. On the other hand, the smallest $\uparrow$-chain element of the poset $\mathcal{D}_{h(3)}$ is $h(3)-1$. Since the sets $\mathcal{D}_3$ and $\mathcal{D}_{h(3)}$ are order-isomorphic, the set ${\downarrow}(h(3)-1) = \{d \in \mathcal{D}_{h(3)}\colon d \text{ divides } h(3)-1\}$ has cardinality 2, which means that the number $h(3)-1$ is prime. Observing that 3 is a unique odd prime number p such that $p-1$ is prime, we conclude that $h(3) = 3$. □

Lemma 20. *For any homeomorphism h of the Golomb space $\mathbb{N}_\tau$, and any prime number p we have $h(1+p^n\mathbb{N}_0) = 1 + h(p)^n\mathbb{N}_0$ for all $n \in \mathbb{N}$.*

Proof: By Lemma 8, the homeomorphism h induces an order isomorphism of the posets $\mathcal{X}_p$ and $\mathcal{X}_{h(p)}$.

If $p = 2$, then $h(p) = 2$ by Lemma 19. Lemma 3 implies $h(2\mathbb{N}) = h(2)\cdot\mathbb{N} = 2\mathbb{N}$ and hence $h(1+2\mathbb{N}_0) = h(\mathbb{N}\setminus 2\mathbb{N}) = \mathbb{N}\setminus h(2\mathbb{N}) = 1+2\mathbb{N}_0$. By Lemma 8, h induces an order automorphism of the poset $\mathcal{X}_2$ and hence h is identity on the well-ordered set $\{1+2^n\mathbb{N}_0\colon n \geq 3\}$ of non-minimal elements of $\mathcal{X}_2$, see Lemma 18. Consequently, $h(1+2^n\mathbb{N}_0) = 1+2^n\mathbb{N}_0$ for all $n \geq 3$.

Next, we show that $h(1+4\mathbb{N}_0) = 1+4\mathbb{N}_0$. Observe that for the smallest non-minimal element $\overline{9^{\mathbb{N}}} = 1+8\mathbb{N}_0$ of $\mathcal{X}_2$ there are only two elements, $\overline{5^{\mathbb{N}}} = 1+4\mathbb{N}_0$ and $\overline{3^{\mathbb{N}}} = (1+8\mathbb{N}_0)\cup(3+8\mathbb{N}_0)$, which are strictly smaller than $\overline{9^{\mathbb{N}}}$ in the poset $\mathcal{X}_2$. Then $h(\overline{5^{\mathbb{N}}}) \in \{\overline{3^{\mathbb{N}}}, \overline{5^{\mathbb{N}}}\}$. By Lemma 19, $h(3) = 3$ and hence $h(\overline{3^{\mathbb{N}}}) = \overline{3^{\mathbb{N}}}$, which implies that $h(1+4\mathbb{N}_0) = h(\overline{5^{\mathbb{N}}}) = \overline{5^{\mathbb{N}}} = 1+4\mathbb{N}_0$.

Now assume that p is an odd prime number. Since $h(2) = 2$, the prime number $h(p) \neq h(2) = 2$ is odd. By Lemma 14, the well-ordered sets $\{1+p^n\mathbb{N}_0\colon n \in \mathbb{N}\}$ and $\{1+h(p)^n\mathbb{N}_0\colon n \in \mathbb{N}\}$ coincide with the sets of $\uparrow$-chain elements of the posets $\mathcal{X}_p$ and $\mathcal{X}_{h(p)}$, respectively. Taking into account that h is an order isomorphism, we conclude that $h(1+p^n\mathbb{N}_0) = 1+h(p)^n\mathbb{N}_0$ for every $n \in \mathbb{N}$. □

5. Proof of Theorem 1

In this section we present the proof of Theorem 1. Given any homeomorphism h of the Golomb space $\mathbb{N}_\tau$, we need to prove that $h(n) = n$ for all $n \in \mathbb{N}$. This equality will be proved by induction.

For $n \le 3$ the equality $h(n) = n$ is proved in Lemma 19. Assume that for some number $n \ge 4$ we have proved that $h(k) = k$ for all $k < n$. For every prime number p let α_p be the largest integer number such that p^{α_p} divides $n-1$ (so, $\alpha_p = l_p(n-1)$). For every $p \in \Pi_{n-1}$ we have $p \le n-1$ and hence $h(p) = p$ (by the inductive hypothesis). Then $h(\Pi_{n-1}) = \Pi_{n-1}$ and $h(\Pi \setminus \Pi_{n-1}) = \Pi \setminus \Pi_{n-1}$.

Observe that n is the unique element of the set

$$\bigcap_{p\in\Pi} (1 + p^{\alpha_p}\mathbb{N}_0) \setminus (1 + p^{\alpha_p+1}\mathbb{N}_0).$$

By Lemma 20, $h(n)$ coincides with the unique element of the set

$$\begin{aligned}
&\bigcap_{p\in\Pi} (1 + h(p)^{\alpha_p}\mathbb{N}_0) \setminus (1 + h(p)^{\alpha_p+1}\mathbb{N}_0)\\
&\quad= \Big(\bigcap_{p\in\Pi_{n-1}} (1{+}h(p)^{\alpha_p}\mathbb{N}_0) \setminus (1{+}h(p)^{\alpha_p+1}\mathbb{N}_0)\Big) \cap \Big(\bigcap_{p\in\Pi\setminus\Pi_{n-1}} \mathbb{N} \setminus (1{+}h(p)\mathbb{N}_0)\Big)\\
&\quad= \Big(\bigcap_{p\in\Pi_{n-1}} (1 + p^{\alpha_p}\mathbb{N}_0) \setminus (1 + p^{\alpha_p+1}\mathbb{N}_0)\Big) \cap \Big(\bigcap_{p\in\Pi\setminus\Pi_{n-1}} \mathbb{N} \setminus (1 + p\mathbb{N}_0)\Big) = \{n\}
\end{aligned}$$

and hence $h(n) = n$. □

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(*Received* January 13, 2020*, revised* December 9, 2020)