On a class of variational problems with linear growth and radial symmetry

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Abstract. We discuss variational problems on two-dimensional domains with energy densities of linear growth and with radially symmetric data.

The smoothness of generalized minimizers is established under rather weak ellipticity assumptions. Further results concern the radial symmetry of solutions as well as a precise description of their behavior near the boundary.

Keywords: linear growth problem; symmetric solutions in 2D; existence of solutions in 2D; uniqueness solution in 2D; (non-)attainment of boundary data

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1. Introduction

Inspired by the fundamental work of M. Giaquinta, G. Modica and J. Souček ([17], [18]) we here discuss the particular minimization problem

$$J[w] := \int_\Omega g(|\nabla w|)\,\mathrm{d}x \to \min \qquad \text{in } u_0 + W_0^{1,1}(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is the annulus $\{x \in \mathbb{R}^2 \colon \varrho_1 < |x| < \varrho_2\}$ with radii $0 < \varrho_1 < \varrho_2 < \infty$. The function u_0 is radially symmetric, which means

$$u_0(x) = \hat{u}_0(|x|), \quad m_i = \hat{u}_0(\varrho_i), \qquad i = 1, 2, \tag{1.2}$$

reflecting the fact that we want to minimize the functional J among functions with constant values on the circles $|x_i| = \varrho_i$, $i = 1, 2$. Moreover, we assume that $g(|\nabla u|)$ is of linear growth with respect to $|\nabla u|$.

The purpose of our note is threefold.

(1) We give a general regularity theory for the minimizing problem (1.1), (1.2). In particular, we exclude the occurrence of an autonomous counterpart of the famous singular example of M. Giaquinta, G. Modica and J. Souček, see [18], see also the twodimensional variant given in [7]. Note that we establish the

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smoothness of solutions up to the boundary which essentially differs from the attainment of the boundary data (compare (1.16)).

(2) We allow a wide range of ellipticity since we do not require a balancing condition like

$$\frac{g''(s)}{g''(t)} \leq C \qquad \text{for all } s \geq 1 \text{ and } t \in \left[\frac{s}{2}, 2s\right], \tag{1.3}$$

which is a part of the main assumption in [4]. In fact, this condition is used for the construction of barriers such that the attainment of boundary data can be proved as in [4] supposing (1.9) of that paper.

We like to point out, that the construction of barriers, i.e. the balancing condition in connection with (1.9) of [4], serves as the main tool in [4] for reducing the problem to the analysis of an ordinary differential equation with an explicit representation of the solution. For this reason we could not adapt these kind of arguments to the proof of Theorem 1.1. Let us postpone a more detailed discussion to Remark 3.1.

Since our arguments leading to the regularity of solutions do not incorporate some detailed estimates concerning the first derivative of the energy density, we also do not impose an analogue to (1.7) of Theorem 1.1 given in [14].

(3) Following [10], it is easily shown that boundary data are respected at least for $|x| = \varrho_2$ which gives the uniqueness of solutions.

Moreover, the possible non-attainment of the boundary data in the radially symmetric case has a complete interpretation.

Remark 1.1. Regardless of the fundamental difficulties sketched in Remark 3.1, large parts of our note could be presented in a one-dimensional setting by exploiting the radial symmetry of the problem under consideration.

While the main arguments could be translated following the same lines as in two dimensions, we would be faced with the difficulty of finding precise citations.

For this reason and since the primal problem under consideration is given in the two-dimensional setting, we like to keep this as the general framework.

Of course we first have to introduce the problem more precisely.

In what follows $g\colon [0,\infty) \to [0,\infty)$ is a function of class $C^2([0,\infty))$ satisfying (with suitable constants a, $A > 0$, b, $B \in \mathbb{R}$)

$$at - b \leq g(t) \leq At + B \qquad \text{for all } t \geq 0 \tag{1.4}$$

as well as

$$0 = g(0) = g'(0). \tag{1.5}$$

Let us require for the moment that we just have the inequality

$$\nu_1(1+t)^{-\mu} \le g''(t) \le \nu_2(1+t)^{-1}, \qquad t \ge 0, \tag{1.6}$$

for some exponent $\mu > 1$, ν_1, ν_2 denoting positive constants. This type of μ-elliptic integrand occurs as a special case of the densities discussed, for instance, in [8] and a series of further papers.

Observe that the minimal surface case is included with the choice $g(t) = \sqrt{1+t^2}-1$ leading to (1.6) with $\mu = 3$. Other examples are ($\mu > 1$, $k > 1$, compare Section 4)

$$\begin{aligned} \Phi_\mu(t) &:= (\mu-1)\int_0^t\int_0^s (1+r)^{-\mu}\,\mathrm{d}r\,\mathrm{d}s, \\ &= \begin{cases} t - \dfrac{1}{2-\mu}(1+t)^{2-\mu} - \dfrac{1}{\mu-2} & \text{if } \mu \neq 2, \\ t - \ln(1+t) & \text{if } \mu = 2, \end{cases} \end{aligned} \tag{1.7}$$

$$\tilde{g}_k(t) := (1+t^k)^{1/k} - 1, \qquad t \ge 0, \tag{1.8}$$

where in the latter case we have (1.6) with $\mu = k+1$. We also note that in recent years the example from (1.8) becomes more and more popular and in some sense serves as a model for strain-limiting elastic models with linear growth, see, for instance, [3], [13], [11] and [12].

Associated to our density is the strictly convex integrand

$$G\colon \mathbb{R}^2 \to [0,\infty), \quad G(p) := g(|p|), \qquad p \in \mathbb{R}^2,$$

being of linear growth and satisfying the common condition of μ-ellipticity

$$\nu_1(1+|p|)^{-\mu}|q|^2 \le D^2G(p)(q,q) \le \nu_2(1+|p|)^{-1}|q|^2. \tag{1.9}$$

In fact, (1.9) follows from the formula

$$D^2G(p)(q,q) = \frac{1}{p}g'(|p|)\Big[|q|^2 - \frac{(p\cdot q)^2}{|p|^2}\Big] + g''(|p|)\frac{(p\cdot q)^2}{|p|^2} \tag{1.10}$$

in combination with (1.6).

Let us return to our variational problem (1.1). As a matter of fact, the existence of a solution in the subclass $u_0 + W_0^{1,1}(\Omega)$ of the non-reflexive Sobolev space $W^{1,1}(\Omega)$, (see, e.g., [1] for a definition of the various Sobolev classes $W^{k,p}(\Omega)$ and their local variants) cannot be guaranteed. Therefore one has to pass to a suitable relaxed version of (1.1). This approach to linear growth problems is nowadays standard and outlined, for example, in the monographs [2], [21] and [19], [20]. A comprehensive survey of the topic including the historical background is also presented in the more recent paper [5].

A relaxed version of (1.1) is given by

$$
\begin{aligned}
K[w] := & \int_\Omega G(\nabla^a w)\,\mathrm{d}x + \int_\Omega G_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}|\nabla^s w| \\
& + \int_{\partial\Omega} G_\infty\big((u_0 - w)\mathcal{N}\big)\,\mathrm{d}\mathcal{H}^1 \to \min \qquad \text{in } \mathrm{BV}(\Omega),
\end{aligned} \tag{1.11}
$$

where $\mathcal{N}$ is the outward unit normal to $\partial\Omega$, G_∞ is the recession function of G, and $\nabla^a w$, $\nabla^s w$ denote the regular and the singular part of ∇w with respect to the Lebesgue measure. For a definition of the space $\mathrm{BV}(\Omega)$ we refer to [2] or [21].

From the convexity of G together with the linear growth condition we obtain the boundedness of DG, moreover,

$$
g'_\infty := \lim_{t\to\infty} g'(t) = \lim_{t\to\infty} \frac{g(t)}{t}
$$

exists in $(0,\infty)$ and the recession function is given by

$$
G_\infty(p) = g'_\infty |p|, \qquad p \in \mathbb{R}^2.
$$

Thus (1.11) simply reads

$$
\begin{aligned}
K[w] = & \int_\Omega g(|\nabla^a w|)\,\mathrm{d}x + g'_\infty |\nabla^s w|(\Omega) \\
& + g'_\infty \int_{\partial\Omega} |u_0 - w|\,\mathrm{d}\mathcal{H}^1 \to \min \qquad \text{in } \mathrm{BV}(\Omega),
\end{aligned} \tag{1.12}
$$

and clearly it holds

$$
K[w] = J[w], \qquad \text{whenever } w \in u_0 + W^{1,1}_0(\Omega).
$$

We summarize some known results in the following proposition.

Proposition 1.1. *Let the conditions* (1.2), (1.4)–(1.6) *hold for some exponent* $\mu > 1$. *Then we have:*

(1) *The functional K is lower semicontinuous with respect to convergence in $L^1(\Omega)$.*
(2) *Problem* (1.12) *admits at least one solution* $u \in \mathrm{BV}(\Omega)$.
(3) *It holds that*

$$
\inf_{u_0 + W^{1,1}_0(\Omega)} J = \inf_{\mathrm{BV}(\Omega)} K.
$$

(4) *A function $u \in \mathrm{BV}(\Omega)$ is K-minimizing if and only if $u \in \mathcal{M}$, where*

$$
\begin{aligned}
\mathcal{M} := \{ v \in L^1(\Omega) \colon\ & v \textit{ is a } L^1(\Omega)\textit{-cluster point} \\
& \textit{of some } J\textit{-minimizing sequence from } u_0 + W^{1,1}_0(\Omega) \}.
\end{aligned}
$$

(5) *Suppose that* (1.12) *admits a solution* $u \in \mathrm{BV}(\Omega) \cap C^1(\Omega)$. *Then any solution* v *is of the form* $v = u + c$ *for some* $c \in \mathbb{R}$. *Moreover, it holds* $u(x) = \hat{u}(|x|)$.

(6) *For any* K*-minimizer* u *we have*

$$\sup_{\Omega} |u| \leq \max\{|m_1|, |m_2|\}.$$

In fact, the proposition is based on classical results as the representation formula of C. Goffman and J. Serrin in [22] and Reshetnyak's continuity theorem in [23]. We refer to [7], Appendix A, for a detailed discussion of (3) and (4) which in particular leads to uniqueness Theorem A.9 stated there, hence to (5). Note that a variant of the mentioned Theorem A.9 is also given in [6], Corollary 2.5. Finally, the last claim is due to Corollary 1 of [9].

Remark 1.2. On account of particular relevance for our note let us shortly recall one way to prove the radial symmetry of general minimizers. We start with a suitable regularization u_δ and immediately have this property for u_δ. Then, passing to the limit, the symmetry is carried over to any cluster point u^*. If there exists any generalized solution which is of class C^1, then uniqueness up to a constant gives the radial symmetry of any generalized minimizer. In fact, the reasoning of [7], Section 4.4, shows that C^1 may even be replaced by partial regularity.

Part (5) of Proposition 1.1 raises the first challenging question under which conditions a regular solution $u \in \mathrm{BV}(\Omega) \cap C^1(\Omega) \subset W^{1,1}(\Omega)$ exists which is immediately leading to the second question, if this minimizer takes the boundary values u_0 thereby solving (1.1).

Roughly speaking, we have a positive answer to the first problem provided that

$$\mu < 3, \tag{1.13}$$

see, e.g., [7] or [5], and from the work [4] by L. Beck, M. Bulíček and E. Maringová we deduce that $u = u_0$ on $\partial\Omega$, if (1.13) is replaced by the requirement $\mu < 2$ and if the second inequality in (1.6) is replaced by the condition $g''(t) \leq \nu_2(1+t)^{-\mu}$.

In the situation at hand we neither require any upper bound on the exponent μ nor a balancing condition in the sense of (1.3) still giving a positive answer to the existence of a smooth K-minimizer.

We just need the limitation (1.15) for the range of anisotropy admissible in the behavior of g'', which is quite similar to the superlinear analogue $q < p + 2$ in the case of anisotropic growth conditions (see [7] for an overview and a list of references).

Let us now state our main results.

Theorem 1.1. *Suppose that* $\mu \in (1, \infty)$, *let* (1.2), (1.4), (1.5) *hold and replace* (1.6) *by the condition*

$$\nu_1(1+t)^{-\mu} \le g''(t) \le v_2(1+t)^{-\overline{\mu}}, \qquad t \ge 0, \tag{1.14}$$

for some exponent $\overline{\mu} \in [1, \mu]$ *such that*

$$\mu - \overline{\mu} < 2. \tag{1.15}$$

Then the relaxed problem (1.12) *admits a solution*

$$u \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega}) \cap C^1(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$$

which in addition is of the form $u(x) = \hat{u}(|x|)$. *Moreover, the solution is unique up to additive constants.*

Remark 1.3. (1) In the case μ, $\overline{\mu} > 1$ we deduce from (1.14) and (1.5) the inequality

$$c\Phi_\mu(t) \le g(t) \le C\Phi_{\overline{\mu}}(t)$$

with $\Phi_{\dots}$ defined in (1.7), which means that (1.4) automatically holds.

(2) Note that in particular the one parameter family of energy densities given in (1.8) and suitable generalizations are covered by our considerations.

(3) We may take any function $\psi(t)$ satisfying for some constants $c_1, c_2 \in \mathbb{R}$

$$c_1(1+t)^{-\mu} \le \psi(t) \le c_2(1+t)^{-\overline{\mu}}, \qquad t \ge 0,$$

and obtain a function

$$\Psi(t) = \int_0^t \int_0^s \psi(r)\,\mathrm{d}r\,\mathrm{d}s$$

which clearly satisfies (1.14) but in general violates a balancing condition like given in (1.8) of [4].

We do not know if the solution u takes the boundary values u_0 for $|x| = \varrho_1$. However, the following theorem yields a complete description of the boundary behavior.

Theorem 1.2. *The minimizer given in Theorem* 1.1 *in fact is the unique solution of problem* (1.12). *Moreover, we have:*

(1) *The minimizer respects the boundary data for* $|x| = \varrho_2$, *hence it is the solution of the minimizing problem*

$$\begin{gathered}\int_\Omega g(|\nabla w|)\,\mathrm{d}x + g'_\infty \int_{|x|=\varrho_1} |w - m_1|\,\mathrm{d}\mathcal{H}^1 \to \min \\ w \in W^{1,1}(\Omega), \quad w = m_2 \quad \text{on} \;\; \{|x| = \varrho_2\}.\end{gathered} \tag{1.16}$$

(2) *Suppose that m_2 is fixed, abbreviate $m = m_1$ and let $u_m(x) = \hat{u}_m(|x|)$ denote the unique solution of* (1.16).

Suppose without loss of generality that $m < m_2$. Then we have

(a) *for any $\varrho \in (\varrho_1, \varrho_2)$ it holds that $\hat{u}_m(\varrho) \geq m$;*

(b) *as a function depending on m, the quantity $\hat{u}_m(\varrho_1)$ is a non-decreasing function, i.e.*

$$\zeta_1 < \zeta_2 \Rightarrow \hat{u}_{\zeta_1}(\varrho_1) \leq \hat{u}_{\zeta_2}(\varrho_1).$$

As a corollary we obtain in particular:

Corollary 1.1. *With the notation of Theorem* 1.2 *we suppose that there exists $m < m_2$ such that the boundary data are not attained for $|x| = \varrho_1$.*

Then for any $\zeta \leq m$ we have $u_\zeta \equiv u_m$.

2. Proof of Theorem 1.1

We proceed by induction showing that the statements of the theorem hold provided $\mu \in (1, k)$ for some $k \geq 2$.

Let in the beginning $k = 3$. From (1.14) we immediately get (1.9) (recall (1.10)) and from Theorem 4.32 in [7] we deduce the existence of a unique (up to constants) generalized minimizer u of class $\mathrm{BV}(\Omega) \cap C^1(\Omega) \subset W^{1,1}(\Omega)$. Alternatively, we can quote Theorem 4.16 from this reference observing that Assumption 4.11 trivially holds for the situation at hand.

Proposition 1.1 (5), implies that $u(x) = \hat{u}(|x|)$ with $\hat{u} \in W^{1,1}(\varrho_1, \varrho_2) \subset C^0([\varrho_1, \varrho_2])$, see [15, Chapter 2], hence $u \in C^0(\overline{\Omega})$. In order to show

$$u \in W^{2,2}_{\mathrm{loc}}(\Omega) \tag{2.1}$$

it is sufficient to prove uniform local $W^{2,2}$-bounds for the solutions u_δ of the regularized problem

$$J_\delta[w] := \frac{\delta}{2} \int_\Omega |\nabla w|^2 \, \mathrm{d}x + \int_\Omega g(|\nabla w|) \, \mathrm{d}x \to \min \qquad \text{in } \ u_0 + W^{1,2}_0(\Omega).$$

To this purpose we just quote Lemma 4.19, i), in [7] choosing the exponent s so large that the left hand side of the Caccioppoli inequality is bounded from below by

$$\alpha \int_\Omega \eta^2 |\nabla^2 u_\delta|^2 \, \mathrm{d}x, \qquad \alpha \ \text{ denoting a suitable uniform constant.}$$

On the right hand side we observe Theorem 4.25 from [7], which gives the desired uniform bound for $u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega)$ leading to (2.1).

Suppose next that $k \geq 3$ and that Theorem 1.1 is true for exponents $\mu \in (1, k)$. We then claim the validity of Theorem 1.1 for

$$\mu \in [k, k+1). \tag{2.2}$$

So let the density g satisfy (1.14) with exponent $\overline{\mu} \in [1, \mu]$ such that (1.15) is true. We choose

$$\tau \in (\mu - 2, \min\{k, \overline{\mu}\}) \tag{2.3}$$

and observe the inequalities

$$\tau < k, \qquad \tau < \overline{\mu}, \qquad \mu - \tau < 2. \tag{2.4}$$

For $\delta \in (0, 1)$ we introduce the density

$$g_\delta(t) := \delta \Phi_\tau(t) + g(t), \qquad t \geq 0, \tag{2.5}$$

with function Φ_τ from (1.7). Moreover, we let

$$G_\delta(p) := g_\delta(|p|), \qquad p \in \mathbb{R}^2. \tag{2.6}$$

Then it holds

$$g_\delta''(t) = \delta(\tau - 1)(1+t)^{-\tau} + g''(t)$$

and the second inequality in (2.4) together with (1.14) shows

$$c_1(\delta)(1+t)^{-\tau} \leq g_\delta''(t) \leq c_2(\delta)(1+t)^{-\tau} \tag{2.7}$$

with constants $c_i(\delta) > 0$. Recalling the first inequality in (2.4) and observing (2.7) our inductive hypothesis applies to the regularized problem

$$K_\delta[w] \to \min \qquad \text{in } \mathrm{BV}(\Omega), \tag{2.8}$$

where K_δ is defined according to (1.11) and (1.12) with G and g replaced by G_δ and g_δ, respectively. Let

$$u_\delta \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega}) \cap C^1(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega) \tag{2.9}$$

denote the unique (up to constants) solution to (2.8) which additionally satisfies $u_\delta(x) = \hat{u}_\delta(|x|)$. The regularity properties of u_δ stated in (2.9) are in turn sufficient to derive the Caccioppoli inequality from Lemma 4.19, ii), in [7], i.e. it holds

$$\begin{aligned}(2.10)\qquad &\int_\Omega D^2G_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\Gamma_\delta^s\eta^{2l}\,\mathrm{d}x\\ &\qquad\leq c\int_\Omega D^2G_\delta(\nabla u_\delta)(\nabla\eta,\nabla\eta)\eta^{2l-2}|\nabla u_\delta|^2\Gamma_\delta^s\,\mathrm{d}x\end{aligned}$$

for any $s\geq 0$, $l\in\mathbb{N}$ and $\eta\in C_0^1(\Omega)$, $0\leq\eta\leq 1$, where we have abbreviated $\Gamma_\delta := 1+|\nabla u_\delta|^2$ and the sum is taken with respect to the index γ.

Letting $\eta(x)=\hat\eta(|x|)$ we set

$$p=\hat u_\delta'(|x|)\frac{x}{|x|},\qquad q=\hat\eta'(|x|)\frac{x}{|x|},$$

and observe that we have in (1.10)

$$\Big[|q|^2-\frac{(p\cdot q)^2}{|p|^2}\Big]=0.$$

This reduces (2.10) to the inequality

$$(2.11)\qquad \int_\Omega g_\delta''(|\nabla u_\delta|)|\nabla^2u_\delta|^2\eta^{2l}\Gamma_\delta^s\,\mathrm{d}x\leq c\int_\Omega g_\delta''(|\nabla u_\delta|)\Gamma_\delta^{s+1}|\nabla\eta|^2\eta^{2l-2}\,\mathrm{d}x$$

with constant $c>0$ not depending on δ. Applying (1.14) and recalling Definition 2.5 we arrive at (neglecting the δ-term on the left hand side of (2.11))

$$\begin{aligned}(2.12)\qquad &\int_\Omega\eta^{2l}|\nabla^2u_\delta|^2\Gamma_\delta^{s-\mu/2}\,\mathrm{d}x\\ &\qquad\leq c\Big[\delta\int_\Omega\eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{s+1-\tau/2}\,\mathrm{d}x+\int_\Omega\eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{s+1-\overline{\mu}/2}\,\mathrm{d}x\Big].\end{aligned}$$

Next we choose $\varphi:=u_\delta\Gamma_\delta^{\alpha/2}\eta^{2l}$ as admissible (recall (2.9)) test function in the Euler equation

$$(2.13)\qquad 0=\int_\Omega DG_\delta(\nabla u_\delta)\cdot\nabla\varphi\,\mathrm{d}x,$$

where η and l are as above and $\alpha\geq 0$ is some number to be fixed later. With this choice (2.13) gives

$$\begin{aligned}(2.14)\qquad &\int_\Omega DG_\delta(\nabla u_\delta)\cdot\nabla u_\delta\Gamma_\delta^{\alpha/2}\eta^{2l}\,\mathrm{d}x\\ &\quad=-\int_\Omega DG_\delta(\nabla u_\delta)\cdot\nabla\eta\eta^{2l-1}2lu_\delta\Gamma_\delta^{\alpha/2}\,\mathrm{d}x\\ &\qquad-\int_\Omega DG_\delta(\nabla u_\delta)\cdot\nabla\Gamma_\delta^{\alpha/2}\eta^{2l}u_\delta\,\mathrm{d}x=:T_1+T_2.\end{aligned}$$

We have

$$DG_\delta(\nabla u_\delta)\cdot\nabla u_\delta = g_\delta'(t)t \geq g'(t)t, \qquad t := |\nabla u_\delta|,$$

and from the first inequality in (1.14) we get

$$g'(t) \geq c\int_0^t (1+s)^{-\mu}\,\mathrm{d}s \geq c[1-(1+t)^{1-\mu}]$$

and in conclusion

$$DG_\delta(\nabla u_\delta)\cdot\nabla u_\delta \geq c[|\nabla u_\delta| - 1],$$

where as usual the value of c may vary from line to line. Therefore we get

$$\text{left hand side of (2.14)} \geq c\Big[\int_\Omega \Gamma_\delta^{(\alpha+1)/2}\eta^{2l}\,\mathrm{d}x - \int_\Omega \eta^{2l}\Gamma_\delta^{\alpha/2}\,\mathrm{d}x\Big]. \tag{2.15}$$

For T_1, T_2 on the right hand side of (2.14) we use, see Proposition 1.1 (6)

$$\sup_\Omega |u_\delta| \leq \max\{|m_1|, |m_2|\},$$

as well as the uniform boundedness of

$$DG_\delta(p) = g_\delta'(|p|)\frac{p}{|p|},$$

which is immediate by the definition of g_δ and the properties of g. We obtain

$$|T_1| \leq c\int_\Omega |\nabla\eta|\eta^{2l-1}\Gamma_\delta^{\alpha/2}\,\mathrm{d}x,$$
$$|T_2| \leq c\int_\Omega \eta^{2l}|\nabla^2 u_\delta|\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x.$$

Returning to (2.14), using (2.15) and the inequalities for T_i, it is shown ($c = c(l)$) that

$$\begin{aligned}\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x \leq c\Big[&\int_\Omega \eta^{2l}\Gamma_\delta^{\alpha/2}\,\mathrm{d}x + \int_\Omega |\nabla\eta|\eta^{2l-1}\Gamma_\delta^{\alpha/2}\,\mathrm{d}x \\ &+ \int_\Omega \eta^{2l}|\nabla^2 u_\delta|\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x\Big] \\ =:\ & c[S_1+S_2+S_3].\end{aligned} \tag{2.16}$$

To the quantities S_i, $i = 1, 2, 3$, we apply Young's inequality:

$$\begin{aligned}
S_1 &\le \varepsilon \int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x + c(\varepsilon)\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x,\\
S_2 &\le \varepsilon \int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x + c(\varepsilon)\int_\Omega \eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x,\\
S_3 &\le \varepsilon \int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x + c(\varepsilon)\int_\Omega \eta^{2l}|\nabla^2 u_\delta|^2\Gamma_\delta^{(\alpha-3)/2}\,\mathrm{d}x.
\end{aligned}$$

For ε sufficiently small we obtain from (2.16)

$$\begin{aligned}
&\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x\\
&\qquad\le c\Big[\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha-3)/2}|\nabla^2 u_\delta|^2\,\mathrm{d}x + \int_\Omega \eta^{2l-2}[\eta^2+|\nabla\eta|^2]\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x\Big].
\end{aligned}\tag{2.17}$$

In a final step we estimate

$$\begin{aligned}
\int_\Omega \eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x &= \int_\Omega \eta^{2l-2}\Gamma_\delta^{(\alpha+1)/4}|\nabla\eta|^2\Gamma_\delta^{(\alpha-3)/4}\,\mathrm{d}x\\
&\le \varepsilon\int_\Omega \eta^{4l-4}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x + c(\varepsilon)\int_\Omega |\nabla\eta|^4\Gamma_\delta^{(\alpha-3)/2}\,\mathrm{d}x\\
&\le \varepsilon\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x + c(\varepsilon)\int_\Omega |\nabla\eta|^4\Gamma_\delta^{(\alpha-3)/2}\,\mathrm{d}x,
\end{aligned}$$

where we have used $\eta^{4l-4} \le \eta^{2l}$, $l \ge 2$, on account of $0 \le \eta \le 1$. Clearly it holds

$$\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha-1)/2}\,\mathrm{d}x \le \varepsilon\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x + c(\varepsilon)\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha-3)/2}\,\mathrm{d}x,$$

and (2.17) implies

$$\begin{aligned}
&\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha+1)/2}\,\mathrm{d}x\\
&\qquad\le c\Big[\int_\Omega \eta^{2l}\Gamma_\delta^{(\alpha-3)/2}|\nabla^2 u_\delta|^2\,\mathrm{d}x + \int_\Omega [\,|\nabla\eta|^4+\eta^{2l}]\Gamma_\delta^{(\alpha-3)/2}\,\mathrm{d}x\Big].
\end{aligned}\tag{2.18}$$

Let us choose $\alpha = 3$ in (2.18) yielding

$$\int_\Omega \eta^{2l}\Gamma_\delta^2\,\mathrm{d}x \le c\Big[\int_\Omega \eta^{2l}|\nabla^2 u_\delta|^2\,\mathrm{d}x + c(\eta)\Big].\tag{2.19}$$

On the right hand side of (2.19) we apply (2.12) for the choice $s = \mu/2$ yielding

$$\int_\Omega \eta^{2l}\Gamma_\delta^2 \,\mathrm{d}x \le c\Big[\delta \int_\Omega \eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{\mu/2+1-\tau/2}\,\mathrm{d}x + \int_\Omega \eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^{\mu/2+1-\overline{\mu}/2}\,\mathrm{d}x + c(\eta)\Big]. \tag{2.20}$$

Moreover, (2.4) implies

$$\frac{\mu}{2} - \frac{\tau}{2} + 1 < 2$$

and from (1.15) it follows

$$\frac{\mu}{2} - \frac{\overline{\mu}}{2} + 1 < 2,$$

thus we have to handle terms like

$$\int_\Omega \eta^{2l-2}\Gamma_\delta^p|\nabla\eta|^2\,\mathrm{d}x$$

with exponent $p \in (1,2)$ on the right hand side of (2.20). Evidently it holds for l sufficiently large

$$\int_\Omega \eta^{2l-2}|\nabla\eta|^2\Gamma_\delta^p\,\mathrm{d}x \le \varepsilon \int_\Omega \eta^{2l}\Gamma_\delta^2\,\mathrm{d}x + c(\varepsilon,\eta)$$

and therefore (2.20) implies

$$|\nabla u_\delta| \in L^4_{\mathrm{loc}}(\Omega) \qquad \text{uniformly in } \delta. \tag{2.21}$$

Next we let $\alpha = 7$ in (2.18) and $s = 2 + \mu/2$ in (2.12). Taking into account (2.21) a repetition of the preceeding calculations leads to $|\nabla u_\delta| \in L^8_{\mathrm{loc}}(\Omega)$ and by iteration we find for any $q < \infty$

$$|\nabla u_\delta| \in L^q_{\mathrm{loc}}(\Omega) \qquad \text{uniformly in } \delta. \tag{2.22}$$

With (2.22) we deduce from (2.12) with the choice $s = \mu/2$ uniform higher weak differentiability, i.e.

$$u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega) \qquad \text{uniformly in } \delta. \tag{2.23}$$

From

$$K[u_\delta] \le K_\delta[u_\delta] \le K_\delta[u_0] \le c(u_0) < \infty$$

together with

$$\|u_\delta\|_{L^\infty(\Omega)} \le \max\{|m_1|,|m_2|\}$$

it follows

$$\sup_{0<\delta<1} \|u_\delta\|_{W^{1,1}(\Omega)} < \infty,$$

hence there is a function $u \in \mathrm{BV}(\Omega)$ such that

$$u_\delta \to u \qquad \text{in } L^1(\Omega) \tag{2.24}$$

at least for a subsequence. We claim that u is K-minimizing. Let $v \in \mathrm{BV}(\Omega)$. By Proposition 1.1 (1), and (2.24) it holds

$$K[u] \le \lim_{\delta\to 0} K[u_\delta].$$

At the same time we have by the minimizing property of u_δ

$$K[u_\delta] \le K_\delta[u_\delta] \le K_\delta[v] \to K[v] \qquad \text{as } \delta \to 0,$$

which proves our claim. Obviously (2.24) implies the validity of (2.22) and (2.23) for the function u. Moreover, the radial symmetry of u_δ extends to u. Since, by embedding, $\hat{u} \in W^{2,2}_{\mathrm{loc}}(\varrho_1, \varrho_2)$ implies $u \in C^1(\Omega)$, the proof of Theorem 1.1 is completed. □

3. Proof of Theorem 1.2 and Corollary 1.1

Proof of Theorem 1.2: Suppose that $\tilde{u}$ is any given solution of (1.12). The first part of Theorem 1.1 guarantees that $\tilde{u}$ is sufficiently smooth such that any solution of (1.12) is of the form $\tilde{u} + c$, $c \in \mathbb{R}$.

In order to show uniqueness together with claim (1), we distinguish four different cases which correspond to the different scenarios for the comparison with a shifted function:

Case 1. The data are attained on the whole boundary $\partial\Omega$.

Then, if $\tilde{u} + c$, $c \in \mathbb{R}$, is a candidate for a possibly different minimizer, then on account of

$$0 = \int_{\partial\Omega} |\tilde{u} - u_0| \, \mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega} |\tilde{u} + c - u_0| \, \mathrm{d}\mathcal{H}^1 = |c| \mathcal{H}^1(\partial\Omega),$$

$c = 0$ is immediate, hence $\tilde{u}$ is the unique solution. (Case 1 corresponds to [5, Lemma 5.5].)

Case 2. Both for $|x| = \varrho_1$ and for $|x| = \varrho_2$ the solution $\tilde{u}$ does not attain the boundary data.

Following [10], we let for any $w \in \mathrm{BV}(\Omega)$

$$\begin{aligned}
\partial^w_+\Omega &:= \{x \in \partial\Omega \colon w(x) > u_0(x)\},\\
\partial^w_-\Omega &:= \{x \in \partial\Omega \colon w(x) < u_0(x)\},\\
\partial^w_0\Omega &:= \{x \in \partial\Omega \colon w(x) = u_0(x)\},
\end{aligned}$$

and observe that Theorem 2.4 of this reference just needs the hypothesis of the strict convexity of the linear growth energy density. Thus, Theorem 2.4 shows for the solution $\tilde{u}$

$$|\mathcal{H}^1(\partial_+^{\tilde{u}}\Omega) - \mathcal{H}^1(\partial_-^{\tilde{u}}\Omega)| \le \mathcal{H}^1(\partial_0^{\tilde{u}}\Omega). \tag{3.1}$$

Since the boundary data are completely ignored in the case under consideration, we have

$$\mathcal{H}^1(\partial_0^{\tilde{u}}\Omega) = 0 \qquad \text{and in conclusion} \qquad \mathcal{H}^1(\partial_+^{\tilde{u}}\Omega) = \mathcal{H}^1(\partial_-^{\tilde{u}}\Omega). \tag{3.2}$$

This, however, is not possible on account of

$$|\{|x| = \varrho_1\}| < |\{|x| = \varrho_2\}|. \tag{3.3}$$

Case 3. The boundary data are attained for $|x| = \varrho_1$, they are not attained for $|x| = \varrho_2$.

In this case

$$\partial_0^{\tilde{u}}\Omega = \{|x| = \varrho_1\}$$

gives a contradiction referring to (3.1) and (3.3).

Case 4. The boundary data are attained for $|x| = \varrho_2$, they are not attained for $|x| = \varrho_1$.

This case is possible and in accordance with our claim

$$\tilde{u} = m_2 \qquad \text{on } \{|x| = \varrho_2\} \text{ for any solution } \tilde{u} \text{ of (1.12).}$$

Since by Theorem 1.1 uniqueness up to additive constants holds true, we now even have the uniqueness of solutions on account of the attainment of the data for $|x| = \varrho_2$.

Next we prove our claim (2). In the following m_2 is fixed and we suppose by the first part of the theorem that for any solution under consideration we have

$$u(\varrho_2) = u_0(\varrho_2) = m_2 \qquad \text{and without lost of generality} \qquad m_2 > m_1 =: m.$$

In the case $m_2 < m_1$ the analogous arguments are obvious.

Let us define for any $w \in \mathrm{BV}(\Omega)$ satisfying $w = m_2$ for $|x| = \varrho_2$ and for any real number $\zeta < m_2$ the energies

$$\begin{aligned} K^0[w] &:= \int_\Omega g(|\nabla^a w|)\,\mathrm{d}x, \\ K_\zeta[w] &:= \int_\Omega g(|\nabla^a w|)\,\mathrm{d}x + g'_\infty |\nabla^s w|(\Omega) + g'_\infty \int_{|x|=\varrho_1} |w - \zeta|\,\mathrm{d}\mathcal{H}^1. \end{aligned}$$

By the first part and by Theorem 1.1, the unique solution $u_\zeta(x) = \hat{u}_\zeta(|x|)$ of the minimizing problem

$$K_\zeta[w] \to \min \qquad \text{in } \mathrm{BV}(\Omega)$$

in particular is of class $W^{1,1}(\Omega)$, hence

$$K_\zeta[u_\zeta] = \int_\Omega g(|\nabla u_\zeta|)\,\mathrm{d}x + g'_\infty \int_{|x|=\varrho_1} |u_\zeta - \zeta|\,\mathrm{d}\mathcal{H}^1 \tag{3.4}$$

and $K_\zeta[w]$ takes the form (3.4) whenever $w \in W^{1,1}(\Omega)$.

Establishing our claim (a) we suppose by contradiction that there exists $\varrho \in (\varrho_1, \varrho_2)$ such that $\hat{u}_\zeta(\varrho) < \zeta$.

Then the continuity of u_ζ yields a real number $\hat{\varrho} \in (\varrho, \varrho_2)$ such that $\hat{u}_\zeta(\hat{\varrho}) = \zeta$ and the choice

$$w_\zeta(x) := \begin{cases} u_\zeta(x) & \text{for } |x| \in (\hat{\varrho}, \varrho_2), \\ \zeta & \text{for } |x| \in (\varrho_1, \hat{\varrho}] \end{cases}$$

immediately contradicts the minimality of u_ζ.

In order to prove claim (b) we suppose that there exist real numbers

$$\zeta_1 < \zeta_2 \qquad \text{and} \qquad \hat{u}_{\zeta_2}(\varrho_1) = \zeta_2^{(+)} < \zeta_1^{(+)} = \hat{u}_{\zeta_1}(\varrho_1). \tag{3.5}$$

Part (a) shows that in this case we have

$$\zeta_1 < \zeta_2 \le \zeta_2^{(+)} < \zeta_1^{(+)} < m_2 \tag{3.6}$$

which guarantees the positive sign of the penalty terms below.

Note that, given two real numbers ξ, κ such that $m_2 \ge \xi \ge \kappa$, part (a) also implies the representation formula

$$\begin{aligned} K_\xi[u_\xi] &= K^0[u_\xi] + g'_\infty \int_{|x|=\varrho_1} |u_\xi - \xi|\,\mathrm{d}\mathcal{H}^{n-1} \\ &= K^0[u_\xi] + g'_\infty 2\pi\varrho_1(\hat{u}_\xi(\varrho_1) - \xi) \\ &= K^0[u_\xi] + g'_\infty 2\pi\varrho_1(\hat{u}_\xi(\varrho_1) - \kappa) - g'_\infty 2\pi\varrho_1(\xi - \kappa) \\ &= K_\kappa[u_\xi] - g'_\infty 2\pi\varrho_1(\xi - \kappa). \end{aligned} \tag{3.7}$$

Now we proceed by observing

$$\begin{aligned} K_{\zeta_1}[u_{\zeta_1}] &= K^0[u_{\zeta_1}] + g'_\infty 2\pi\varrho_1(\zeta_1^{(+)} - \zeta_1) \\ &= K^0[u_{\zeta_1}] + g'_\infty 2\pi\varrho_1(\zeta_1^{(+)} - \zeta_2) + g'_\infty 2\pi\varrho_1(\zeta_2 - \zeta_1) \\ &\ge K_{\zeta_2}[u_{\zeta_2}] + g'_\infty 2\pi\varrho_1(\zeta_2 - \zeta_1) \\ &= K_{\zeta_1}[u_{\zeta_2}], \end{aligned} \tag{3.8}$$

where we recall (3.6) for the discussion of the absolute values in the penalty term. Moreover, the inequality

$$K^0[u_{\zeta_1}] + g'_\infty 2\pi\varrho_1(\zeta_1^{(+)} - \zeta_2) = K_{\zeta_2}[u_{\zeta_1}] \geq K_{\zeta_2}[u_{\zeta_2}]$$

follows from the minimality of u_{ζ_2} and the last equality in (3.8) is due to (3.7).

Finally we observe that inequality (3.8) would give $u_{\zeta_1} = u_{\zeta_2}$ by uniqueness of minimizers which contradicts the hypothesis (3.5), i.e. we have a contradiction to $\zeta_2^{(+)} < \zeta_1^{(+)}$, and the proof of Theorem 1.2 is complete. □

Proof of Corollary 1.1: Using the notation of Theorem 1.1 we first recall two facts that are already established above:

(1) $\hat{u} \in W^{1,1}(\varrho_1, \varrho_2) \cap C^1(\varrho_1, \varrho_2)$;
(2) $\hat{u}(\varrho_2) = m_2$.

For the reader's convenience we sketch some general observations on the Euler equation which can also be found in [4]:

Given a test function $\eta \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \frac{g'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla\eta \, \mathrm{d}x = 0. \tag{3.9}$$

Inserting

$$\nabla u = \hat{u}'(|x|)\frac{x}{|x|},$$

and choosing $\eta(x) = \hat{\eta}(|x|)$ we obtain

$$0 = \int_\Omega g'(|\hat{u}'|)\frac{\hat{u}'(|x|)}{|\hat{u}'(|x|)|}\hat{\eta}'(|x|) \, \mathrm{d}x = 2\pi \int_{\varrho_1}^{\varrho_2} g'(|\hat{u}'|)\frac{\hat{u}'}{|\hat{u}'|}\hat{\eta}' r \, \mathrm{d}r. \tag{3.10}$$

Note that on account of (1.5) the expression $g'(t)/t$ is well defined in the limit $t \to 0$.

Using (3.10), Du Bois–Reymond's lemma as variant of the fundamental lemma implies the existence of a real number $\lambda \in \mathbb{R}$ such that

$$g'(|\hat{u}'|)\frac{\hat{u}'}{|\hat{u}'|} = \frac{\lambda}{r}. \tag{3.11}$$

If $\hat{u} \not\equiv 0$, then zeroes of $\hat{u}$ are excluded by (3.11) and supposing without loss of generality $m_1 < m_2$ we have $\hat{u}' > 0$ and (3.11) reduces to

$$g'(\hat{u}'(r)) = \frac{\lambda}{r} \qquad \text{for all } r \in (\varrho_1, \varrho_2). \tag{3.12}$$

By assumption g is a strictly convex function, i.e. g' is a strictly increasing function and we have that

$$g'\colon (0,\infty) \to (0, g'_\infty) \qquad \text{is one-to-one,}$$

hence we obtain from (3.12)

$$0 < \hat{u}'(r) = (g')^{-1}\Big(\frac{\lambda}{r}\Big) \qquad \text{for all } r \in (\varrho_1, \varrho_2). \tag{3.13}$$

Note the validity of (3.12) for all $r \in (\varrho_1, \varrho_2)$ and in conclusion the possible values of λ are given by

$$0 < \lambda \le \varrho_1 g'_\infty. \tag{3.14}$$

Finally we consider the possible range for realizing boundary data:

$$\Delta m(\lambda) := u(\varrho_2) - u(\varrho_1) = \int_{\varrho_1}^{\varrho_2} (g')^{-1}\Big(\frac{\lambda}{r}\Big)\, \mathrm{d}r,$$

where we note that $\Delta m(\lambda) \to 0$ as $\lambda \to 0$.

Now, on account of (3.14), for any $\varrho_1 < \hat{\varrho} < \varrho_2$ the function $(g')^{-1}(\lambda/r)$ is bounded in $(\hat{\varrho}, \varrho_2]$ with a constant not depending on λ, hence a critical behavior may just be expected at ϱ_1 in the limit $\lambda \to \varrho_1 g'_\infty$.

Summarizing these observation we obtain that, if

$$\lim_{\lambda \to \varrho_1 g'_\infty} \int_{\varrho_1}^{\varrho_2} (g')^{-1}\Big(\frac{\lambda}{r}\Big)\, \mathrm{d}r = \infty,$$

then $\Delta m(\lambda)$ takes any value in $(0,\infty)$ and for all $m_1 < m_2 \in \mathbb{R}$ problem (1.12) admits a solution taking the boundary data.

If

$$\lim_{\lambda \to \varrho_1 g'_\infty} \int_{\varrho_1}^{\varrho_2} (g')^{-1}\Big(\frac{\lambda}{r}\Big)\, \mathrm{d}r =: \Delta m_\infty < \infty,$$

then a solution taking the boundary data exists if and only if $m_2 - m_1 < \Delta m_\infty$.

At this point we note that, given $\zeta_1 < \zeta_2$ such that $\hat{u}_{\zeta_1}(\varrho_1) = \hat{u}_{\zeta_2}(\varrho_1)$, the arguments yielding monotonicity imply $\hat{u}_{\zeta_1} \equiv \hat{u}_{\zeta_2}$ (compare (3.5) and (3.6) in the case $\zeta_2^{(+)} \le \zeta_1^{(+)}$).

Let us finally suppose that $\zeta_1 < \zeta_2 < m_2 - \Delta m_\infty$ for some real number ζ. By the above considerations we have

$$\hat{u}_{\zeta_1}(\varrho_1) = \hat{u}_{\zeta_2}(\varrho_1) = m_2 - \Delta m_\infty,$$

and the limit number $m_2 - \Delta m_\infty$ serves as the boundary datum for $\hat{u}_{\zeta_1}$ as well as for $\hat{u}_{\zeta_2}$, which immediately gives the corollary. $\square$

Remark 3.1. Having (3.13) in mind, one may try to attack the problem via this explicit representation of the derivative of the solution. However, recall that (3.13) is derived by supposing the solution to be sufficiently regular. We also recall that the Giaquinta–Modica–Souček example of [18] is based on a contradiction to the assumption of Sobolev regularity.

Hence, the first step on the way towards the use of explicit solutions should be the construction of a suitable regularization.

In order to illustrate the main difficulty, let us concentrate on the minimal surface energy with standard quadratic regularization as a model case, i.e. we consider for fixed $\delta > 0$

$$g(t) = \sqrt{1+t^2}, \quad g_\delta(t) = g(t) + \frac{\delta}{2}t^2, \quad g_\delta'(t) = \frac{t}{\sqrt{1+t^2}} + \delta t, \qquad t \geq 0.$$

Exactly as in (3.13) we derive ($\hat{u}_\delta$ denoting the smooth solution of the regularized problem)

$$\hat{u}_\delta'(r) = (g_\delta')^{-1}\Big(\frac{\lambda}{r}\Big) \qquad \text{with one-to-one function } (g_\delta')^{-1} \colon (0,\infty) \to (0,\infty).$$

Note that we have the analogue of (3.12) again on (ϱ_1, ϱ_2) but in contrast to (3.14) any positive constant λ_δ provides a suitable choice.

Let us abbreviate for the moment ($r_\delta \in (\varrho_1, \varrho_2)$ fixed)

$$\tau_\delta = \hat{u}_\delta'(r_\delta) > 0, \qquad y_\delta = \frac{\lambda_\delta}{r_\delta} > 0, \qquad \text{i.e. by (3.12)} \;\; y_\delta = \frac{\tau_\delta}{\sqrt{1+\tau_\delta^2}} + \delta\tau_\delta,$$

hence we obtain

$$\tau_\delta^2(1-y_\delta^2) = y_\delta^2 - \Big[2\delta\tau_\delta^2\sqrt{1+\tau_\delta^2} + \delta^2\tau_\delta^2(1+\tau_\delta^2)\Big]. \tag{3.15}$$

Note that the left hand side of (3.15) proves the right hand side of (3.15) to be positive whenever $y_\delta < 1$ and to be negative whenever $y_\delta > 1$.

We estimate the zeroes with some asymptotic considerations (which can be validated by simple numerical experiments) and observe for small δ

$$y_\delta = 1 \Rightarrow \tau_\delta \sim \delta^{-1/3}. \tag{3.16}$$

We summarize that, if for any $\delta > 0$ there exists some $r_\delta \in (\varrho_1, \varrho_2)$ such that $\lambda_\delta = r_\delta$, then by (3.16) the regularizing sequence is not uniformly Lipschitz.

If we would have an initial condition like

$$\hat{u}_\delta'(\varrho_1) = c, \qquad c \text{ denoting a positive constant not depending on } \delta \tag{3.17}$$

for the ODE under consideration, then we would obtain the desired estimate $\lambda_\delta < \varrho_1$ on account of

$$\frac{\lambda_\delta}{\varrho_1} = g'_\delta\big(\hat{u}'_\delta(\varrho_1)\big) = \frac{c}{\sqrt{1+c^2}} + \delta c < 1$$

by choosing δ sufficiently small.

An estimate in the spirit of (3.17) in fact is the main result of [4], see (4.5), and established by the construction of barrier functions.

This is in accordance with the examples given in the next section. The first examples are explicitely solved with parameter $\lambda < \varrho_1$, hence $\hat{u}'(\varrho_1)$ remains bounded. In the third example λ has to pass to the limit ϱ_1 if $m_2 - m_1$ approaches (and crosses) the finite value Δm_∞.

It remains an open question, whether, for instance, a refined regularization process could eliminate the problems sketched above.

Concerning these remarks and the following examples we finally like to refer to the classical paper [16] by R. Finn in the minimal surface case.

4. Examples

We finally sketch three characteristic examples by presenting explicit solutions.

To this purpose we recall the one parameter family given in (1.7) (now denoted by g_μ)

$$g_\mu(t) := \begin{cases} t - \dfrac{1}{2-\mu}(1+t)^{2-\mu} - \dfrac{1}{\mu-2} & \text{if } \mu \neq 2, \\ t - \ln(1+t) & \text{if } \mu = 2. \end{cases}$$

Note that $g'_{\mu_\infty} = 1$ for any $\mu > 1$.

With this choice of g_μ, the Euler equation (3.13) reads as

$$\hat{u}' = \Big[\frac{r}{r-\lambda}\Big]^{1/(\mu-1)} - 1. \tag{4.1}$$

We note that the condition (1.9) of [4] motivates to consider examples choosing $1 < \mu < 2$, $\mu = 2$, $\mu > 3$, respectively.

(1) Suppose that $\mu = 3/2$, i.e. $1 < \mu < 2$. Then

$$\hat{u}(r) = 2\lambda \ln(r-\lambda) - \frac{\lambda^2}{r-\lambda} + c, \qquad c \in \mathbb{R},$$

provides an exact solution. With the notation from above we have

$$\Delta m(\lambda) := \Big[2\lambda\ln(\varrho_2-\lambda) - \frac{\lambda^2}{\varrho_2-\lambda}\Big] - \Big[2\lambda\ln(\varrho_1-\lambda) - \frac{\lambda^2}{\varrho_1-\lambda}\Big].$$

We see that in this case we have

$$\Delta m(\lambda) \to \infty \qquad \text{as } \lambda \to \varrho_1,$$

and with the right choice of the free parameters λ, c we obtain a smooth solution to (1.12) taking the boundary data.

(2) Consider the limit case $\mu = 2$. We then have

$$\hat{u}(r) = \lambda \ln(r - \lambda) + c$$

and as above we find for any given boundary data a solution realizing this data.

(3) In the case $\mu = 3$ we find as solution of the Euler equation

$$\hat{u}(r) = \sqrt{r^2 - r\lambda} - t + \frac{\lambda}{2} \ln \Big[\frac{2r - \lambda + 2\sqrt{r^2 - r\lambda}}{2} + c \Big].$$

Now we note that

$$\Delta m_\infty < \infty,$$

hence that boundary data cannot be attained if $\Delta m_\infty < m_2 - m_1$.

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References

[1] Adams R. A., *Sobolev Spaces*, Pure and Applied Mathematics, 65, Academic Press, New York, 1975.

[2] Ambrosio L., Fusco N., Pallara D., *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 224, The Clarendon Press, Oxford University Press, New York, 2000.

[3] Beck L., Bulíček M., Málek J., Süli E., *On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth*, Arch. Ration. Mech. Anal. **225** (2017), no. 2, 717–769.

[4] Beck L., Bulíček M., Maringová E., *Globally Lipschitz minimizers for variational problems with linear growth*, ESAIM Control Optim. Calc. Var. **24** (2018), no. 4, 1395–1403.

[5] Beck L., Schmidt T., *On the Dirichlet problem for variational integrals in BV*, J. Reine Angew. Math. **674** (2013), 113–194.

[6] Beck L., Schmidt T., *Convex duality and uniqueness for BV-minimizers*, J. Funct. Anal. **268** (2015), no. 10, 3061–3107.

[7] Bildhauer M., *Convex Variational Problems. Linear, Nearly Linear and Anisotropic Growth Conditions*, Lecture Notes in Mathematics, 1818, Springer, Berlin, 2003.

[8] Bildhauer M., Fuchs M., *On a class of variational integrals with linear growth satisfying the condition of μ-ellipticity*, Rend. Mat. Appl. **22** (2002), no. 7, 249–274.

[9] Bildhauer M., Fuchs M., *A geometric maximum principle for variational problems in spaces of vector valued functions of bounded variation*, Zap. Nauchn. Sem. S.-Peterburg. Otdel.

Mat. Inst. Steklov. (POMI) **385** (2010), Kraevye Zadachi Matematicheskoĭ Fiziki i Smezhnye Voprosy Teorii Funktsiĭ 41, 5–17, 234; reprinted in J. Math. Sci. (N.Y.) **178** (2011), no. 3, 235–242.

[10] Bildhauer M., Fuchs M., *Some remarks on the (non-)attainment of the boundary data for variational problems in the space BV*, J. Convex Anal. **25** (2018), no. 1, 219–223.

[11] Bulíček M., Málek J., Rajagopal K., Süli E., *On elastic solids with limiting small strain: modelling and analysis*, EMS Surv. Math. Sci. **1** (2014), no. 2, 283–332.

[12] Bulíček M., Málek J., Rajagopal K. R., Walton J. R., *Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies*, Calc. Var. Partial Differential Equations **54** (2015), no. 2, 2115–2147.

[13] Bulíček M., Málek J., Süli E., *Analysis and approximation of a strain-limiting nonlinear elastic model*, Math. Mech. Solids **20** (2015), no. 1, 92–118.

[14] Bulíček M., Maringová E., Stroffolini B., Verde A., *A boundary regularity result for minimizers of variational integrals with nonstandard growth*, Nonlinear Anal. **177** (2018), part A, 153–168.

[15] Buttazzo G., Giaquinta M., Hildebrandt S., *One-dimensional Variational Problems*, An Introduction, Oxford Lecture Series in Mathematics and Its Applications, 15, The Clarendon Press, Oxford University Press, New York, 1998.

[16] Finn R., *Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature*, J. Analyse Math. **14** (1965), 139–160.

[17] Giaquinta M., Modica G., Souček J., *Functionals with linear growth in the calculus of variations. I*, Comment. Math. Univ. Carolin. **20** (1979), no. 1, 143–156.

[18] Giaquinta M., Modica G., Souček J., *Functionals with linear growth in the calculus of variations. II*, Comment. Math. Univ. Carolin. **20** (1979), no. 1, 157–172.

[19] Giaquinta M., Modica G., Souček J., *Cartesian Currents in the Calculus of Variations. I. Cartesian Currents*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A Series of Modern Surveys in Mathematics, 37, Springer, Berlin, 1998.

[20] Giaquinta M., Modica G., Souček, J., *Cartesian Currents in the Calculus of Variations. II. Cartesian Currents*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A Series of Modern Surveys in Mathematics, 38, Springer, Berlin, 1998.

[21] Giusti E., *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, 80, Birkhäuser, Basel, 1984.

[22] Goffman C., Serrin J., *Sublinear functions of measures and variational integrals*, Duke Math. J. **31** (1964), 159–178.

[23] Reshetnyak Y., *Weak convergence of completely additive vector functions on a set*, Sibirsk. Maz. Ž. **9** (1968), 1386–1394; English translation: Sib. Math. J. **9** (1968), 1039–1045.

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