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Decomposition of Cartesian product of complete graphs into paths and stars with four edges

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Abstract. Let P_k and S_k denote a path and a star, respectively, on k vertices. We give necessary and sufficient conditions for the existence of a complete $\{P_5, S_5\}$-decomposition of Cartesian product of complete graphs.

Keywords: graph decomposition; path; star graph; product graph

Classification: 05C51, 05C70

1. Introduction

Unless stated otherwise, all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology, the readers are referred to J. A. Bondy and U. S. R. Murty, see [5]. Let P_k, S_k, C_k, K_k denote a path, star, cycle and complete graph, respectively, on k vertices, and let $K_{m,n}$ denote the complete bipartite graph containing m vertices in one partite set and n vertices in the other partite set. A graph whose vertex set is partitioned into subsets $V_1, \ldots, V_t$ with edge set $\bigcup_{i\neq j\in[t]} V_i \times V_j$ is a complete t-partite graph, denoted by $K_{n_1,\ldots,n_t}$, when $|V_i| = n_i$ for all i. For $G = K_{2n}$ or $K_{n,n}$, the graph $G - I$ denotes G with a 1-factor I removed. For any integer $\lambda > 0$, λG and $G(\lambda)$ respectively denote the graph consisting of λ edge-disjoint copies of G and a multigraph G with uniform edge multiplicity λ. Moreover $\upsilon(G)$ and $\varepsilon(G)$ denote the number of vertices and number, respectively, of edges in G. The *complement* of the graph G is denoted by $\overline{G}$. For two graphs G and H, we define their *Cartesian product*, denoted by $G \,\square\, H$, with vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and edge set

$$E(G \,\square\, H) = \{(g,h)(g',h')\colon\ g = g',\ hh' \in E(H),\ \text{or}\ gg' \in E(G),\ h = h'\}.$$

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It is well known that the Cartesian product is commutative and associative. For a graph G, if $E(G)$ can be partitioned into $E_1, \ldots, E_k$ such that the subgraph of G induced by E_i is H_i for all $1 \leq i \leq k$, then we say that $H_1, \ldots, H_k$ *decompose* G, and we write $G = H_1 \oplus \cdots \oplus H_k$, since $H_1, \ldots, H_k$ are edge-disjoint subgraphs of G. If for $1 \leq i \leq k$, $H_i \cong H$, we say that G has a H*-decomposition*. If G has a decomposition into p copies of H_1 and q copies of H_2, then we say that G has a $\{pH_1, qH_2\}$*-decomposition*. If such a decomposition exists for all values of p and q satisfying trivial necessary conditions, then we say that G has a $\{H_1, H_2\}_{\{p,q\}}$*-decomposition* or has a *complete* $\{H_1, H_2\}$*-decomposition*.

Study on $\{H_1, H_2\}_{\{p,q\}}$-decomposition of graphs is not new. A. A. Abueida et al. in [1], [3] completely determined the values of n for which $K_n(\lambda)$ admits a $\{pH_1, qH_2\}$-decomposition such that $H_1 \cup H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, where $t \in \{4, 5\}$. A. A. Abueida and M. Daven in [2] proved that there exists a $\{pK_k, qS_{k+1}\}$-decomposition of K_n for $k \geq 3$ and $n \equiv 0, 1 \pmod k$. A. A. Abueida and T. O'Neil in [4] proved that for $k \in \{3, 4, 5\}$, there exists a $\{pC_k, qS_k\}$-decomposition of $K_n(\lambda)$, whenever $n \geq k+1$ except for the ordered triples $(k, n, \lambda) \in \{(3, 4, 1),\ (4, 5, 1), (5, 6, 1),\ (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1)\}$. T.-W. Shyu in [9], [10] obtained a necessary and sufficient condition on (p, q) for the existence of $\{P_4, S_4\}_{\{p,q\}}$-decomposition of K_n and $K_{m,n}$. H. M. Priyadharsini and A. Muthusamy in [8] established necessary and sufficient conditions for the existence of the (G_n, H_n)-multidecomposition of $K_n(\lambda)$, where $G_n, H_n \in \{C_n, P_{n-1}, S_{n-1}\}$. A. P. Ezhilarasi and A. Muthusamy in [6] have obtained necessary and sufficient conditions for the existence of a decomposition of product graphs into paths and stars with three edges. S. Jeevadoss and A. Muthusamy in [7] have obtained necessary and sufficient conditions for $\{P_5, C_4\}_{\{p,q\}}$-decomposition of product graphs.

In this paper, we show that the necessary condition $mn(m+n-2) \equiv 0 \pmod 8$ is sufficient for the existence of a complete $\{P_5, S_5\}$-decomposition of $K_m \Box K_n$.

Notations. A star S_{k+1} with center at x_0 and end vertices $x_1, \ldots, x_k$ is denoted by $(x_0; x_1, \ldots, x_k)$ and a path on $k+1$ vertices $x_0, x_1, \ldots, x_k$ is denoted by $x_0 x_1 \cdots x_k$. We abbreviate the complete $\{P_{k+1}, S_{k+1}\}$-decomposition as $(4; p, q)$-decomposition. In a $(4; p, q)$-decomposition of a graph G, we mean p and q are integers with $0 \leq p, q \leq \varepsilon(G)/4$ and $p + q = \varepsilon(G)/4$.

To prove our results we state the following:

Theorem 1.1 ([10]). *Let $p, q \geq 0$, $m \geq k > 0$, be integers. There exists a $(k; p, q)$-decomposition of $K_{k,m}$ if and only if the following conditions are fulfilled:*

1. $k(p+q) = \varepsilon(K_{k,m})$;
2. $p \leq \left\lceil \frac{k}{2} \right\rceil - 1 \Rightarrow (p \equiv 0 \pmod 2) \wedge m \geq k + p)$;
3. $\left(\left\lceil \frac{k}{2} \right\rceil \leq p \leq k-1 \wedge k \equiv 1 \pmod 2) \wedge p \equiv 1 \pmod 2\right) \Rightarrow m \geq k+1$.

Theorem 1.2 ([10]). *Let $p, q \geq 0$, and $m > k > 0$, $n \geq 2$, be integers. There exists a $(k; p, q)$-decomposition of $K_{m,nk}$ if and only if $k(p+q) = \varepsilon(K_{m,nk})$.*

Theorem 1.3 ([10]). *Let $p, q \geq 0$, and $k > m > 0$, $n > 0$, be integers. There exists a $(k; p, q)$-decomposition of $K_{nk,m}$ if and only if the following conditions are fulfilled:*

1. *$k(p+q) = \varepsilon(K_{nk,m})$;*
2. *there is a $t \in \{0, \ldots, n\}$ such that $\lceil \frac{tk}{2} \rceil \leq p \leq tm$;*
3. *$(k \equiv 1 \pmod 2 \wedge n = 1) \Rightarrow p \equiv 0 \pmod 2$.*

Theorem 1.4 ([10]). *Let $p, q \geq 0$ and $n \geq 4k > 0$ be integers. There exists a $(k; p, q)$-decomposition of K_n if and only if $k(p+q) = \varepsilon(K_n)$.*

Remark 1.1. If G and H each have a $(4; p, q)$-decomposition, then $G \cup H$ has such a decomposition. In this paper, we denote $G \cup H$ as $G \oplus H$.

Remark 1.2. If two stars S_5^1 and S_5^2 with distinct centers share at least two pendant vertices, then $S_5^1 \oplus S_5^2$ can be decomposed into $2P_5$. i.e. if $S_5^1 = (x_0; y_0, \boldsymbol{y_1}, \boldsymbol{y_2}, y_3)$ and $S_5^2 = (y_4; y_0, \boldsymbol{y_1}, \boldsymbol{x_1}, x_2)$ are two stars, then the $2P_5$ are $P_5^1 = \boldsymbol{y_2 x_0 y_1 y_4 x_1}$, $P_5^2 = y_3 x_0 y_0 y_4 x_2$ (one can easily understand that the edges of stars with bold vertices and ordinary vertices give a required number of paths from stars). We denote such a pair of star as $\{(x_0; y_0, \boldsymbol{y_1}, \boldsymbol{y_2}, y_3), (y_4; y_0, \boldsymbol{y_1}, \boldsymbol{x_1}, x_2)\}$.

Example 1.1. There exists a $(4; p, q)$-decomposition of K_8.

Solution: Let $V(K_8) = \{x_1, x_2, \ldots, x_8\}$. First we decompose K_8 into $\{2P_5, 5S_5\}$ as follows:

$$x_7x_1x_8x_6x_2,\ x_2x_7x_8x_4x_3,\ (x_5; x_2, x_1, x_7, x_8),\ \{(x_3; \boldsymbol{x_1}, \boldsymbol{x_7}, x_5, x_8),$$
$$(x_4; x_1, x_5, \boldsymbol{x_6}, \boldsymbol{x_7})\},\ \{(x_2; x_1, \boldsymbol{x_3}, \boldsymbol{x_4}, x_8),\ (x_6; x_5, \boldsymbol{x_3}, \boldsymbol{x_7}, x_1)\}.$$

Now, we decompose the first $2P_5$ and a S_5 into $3P_5$ as follows:

$$\{x_2x_5x_7x_1x_8, x_1x_5x_8x_6x_2, x_2x_7x_8x_4x_3\}.$$

Hence from the above decompositions and Remark 1.2 we have a $(4; p, q)$-decomposition of K_8 except for the values $p = 0, 1$. For $p = 0, 1$, we have the following sets of paths and stars: $\{(x_1; x_5, x_6, x_7, x_8)$, $(x_2; x_1, x_3, x_4, x_8)$, $(x_3; x_1, x_4, x_5, x_8)$, $(x_4; x_1, x_5, x_6, x_8)$, $(x_5; x_2, x_6, x_7, x_8)$, $(x_6; x_2, x_3, x_7, x_8)$, $(x_7; x_2, x_3, x_4, x_8)\}$ and $\{x_7x_1x_8x_6x_2$, $(x_2; x_1, x_3, x_4, x_8)$, $(x_3; x_1, x_4, x_5, x_8)$, $(x_4; x_1, x_5, x_6, x_8)$, $(x_5; x_2, x_1, x_7, x_8)$, $(x_6; x_5, x_3, x_7, x_1)$, $(x_7; x_2, x_3, x_4, x_8)\}$. $\square$

Example 1.2. There exists a $(4; p, q)$-decomposition of K_9.

Solution: Let $V(K_9) = \{x_1, x_2, \cdots, x_9\}$ and $G = K_9$. Then $G = K_8 \oplus (x_9; x_1, x_2, x_3, x_4) \oplus (x_9; x_5, x_6, x_7, x_8)$ and by Example 1.1, K_9 has a $(4; p, q)$-decomposition except for the values $p = 8$ and 9. For $p = 8, 9$, we have the following sets of paths and stars: $\{x_7x_1x_8x_6x_2,\ x_2x_7x_8x_4x_3,\ x_4x_2x_1x_6x_5,\ x_3x_2x_8x_5x_1,\ x_2x_5x_7x_6x_3,\ x_1x_3x_5x_4x_6,\ x_1x_4x_7x_9x_6,\ x_5x_9x_8x_3x_7,\ (x_9; x_1, x_2, x_3, x_4)\}$ and $\{x_7x_1x_8x_6x_2,\ x_2x_7x_8x_4x_3,\ x_4x_2x_1x_6x_5,\ x_2x_5x_7x_6x_3,\ x_1x_3x_5x_4x_6,\ x_1x_4x_7x_9x_6,\ x_5x_9x_8x_3x_7,\ x_2x_9x_1x_5x_8,\ x_8x_2x_3x_9x_4\}$. □

Example 1.3. There exists a $(4; p, q)$-decomposition of $K_{6,6}$.

Solution: Let $V(K_{6,6}) = \{x_1, x_2, \ldots, x_6\} \cup \{y_1, y_2, \ldots, y_6\}$. First we decompose $K_{6,6}$ into $\{0P_5, 9S_5\}$ and $\{P_5, 9S_5\}$ as follows:

$$\begin{array}{rc}
 & \{(x_1; y_1, y_2, y_3, y_4),\ \{(x_2; y_1, \boldsymbol{y_2}, \boldsymbol{y_5}, y_6),\ (x_3; \boldsymbol{y_5}, \boldsymbol{y_4}, y_3, y_6)\}, \\
 & \{(y_1; \boldsymbol{x_3}, \boldsymbol{x_4}, x_5, x_6),\ (y_3; x_2, \boldsymbol{x_4}, \boldsymbol{x_5}, x_6)\}, \\
 & \{(y_2; \boldsymbol{x_3}, \boldsymbol{x_4}, x_5, x_6),\ (y_5; x_1, \boldsymbol{x_4}, \boldsymbol{x_5}, x_6)\}, \\
 & \{(y_4; \boldsymbol{x_2}, \boldsymbol{x_4}, x_5, x_6),\ (y_6; x_1, \boldsymbol{x_4}, \boldsymbol{x_5}, x_6)\}\} \\
\text{and} & \{y_1x_1y_2x_2y_5,\ \{(x_2; \boldsymbol{y_1}, \boldsymbol{y_3}, y_4, y_6),\ (x_3; \boldsymbol{y_3}, \boldsymbol{y_4}, y_5, y_6)\}, \\
 & \{(y_1; \boldsymbol{x_3}, \boldsymbol{x_4}, x_5, x_6),\ (y_3; x_1, \boldsymbol{x_4}, \boldsymbol{x_5}, x_6)\}, \\
 & \{(y_4; \boldsymbol{x_1}, \boldsymbol{x_4}, x_5, x_6),\ (y_2; x_3, \boldsymbol{x_4}, \boldsymbol{x_5}, x_6)\}, \\
 & \{(y_5; \boldsymbol{x_1}, \boldsymbol{x_4}, x_5, x_6),\ (y_6; x_1, \boldsymbol{x_4}, \boldsymbol{x_5}, x_6)\}\}.
\end{array}$$

By Remark 1.2, we obtain a required even number of paths from $\{0P_5, 9S_5\}$ and a required odd number of paths from $\{P_5, 8S_5\}$. □

2. $(4; p, q)$-decomposition of $K_m \Box K_n$

In this section we investigate the existence of $(4; p, q)$-decomposition of Cartesian product of complete graphs. To prove our results we need the following lemmas.

Lemma 2.1. *There exists a $(4; p, q)$-decomposition of $K_4 \Box K_2$ with $p \geq 2$.*

Proof: Let $V(K_4 \Box K_2) = \{x_{i,j} \colon 1 \leq i \leq 4,\ 1 \leq j \leq 2\}$. First we decompose $K_4 \Box K_2$ into $\{2P_5, 2S_5\}$ as follows:

$$x_{2,1}x_{4,1}x_{3,1}x_{3,2}x_{2,2},\ x_{3,1}x_{2,1}x_{2,2}x_{1,2}x_{3,2},$$
$$\{(x_{1,1}; x_{3,1}, x_{4,1}, \boldsymbol{x_{2,1}}, \boldsymbol{x_{1,2}}),\ (x_{4,2}; \boldsymbol{x_{1,2}}, \boldsymbol{x_{2,2}}, x_{3,2}, x_{4,1})\}.$$

By Remark 1.2, we have a $\{4P_5, 0S_5\}$-decomposition of $K_4 \Box K_2$ from $\{2P_5, 2S_5\}$. Now, the $\{3P_5, S_5\}$-decomposition of $K_4 \Box K_2$ is given by $x_{1,2}x_{2,2}x_{2,1}x_{4,1}x_{3,1}$, $x_{1,2}x_{4,2}x_{3,2}x_{3,1}x_{2,1}$, $x_{1,2}x_{3,2}x_{2,2}x_{4,2}x_{4,1}$, $(x_{1,1}; x_{1,2}, x_{3,1}, x_{4,1}, x_{2,1})$. □

Lemma 2.2. *There exists a $(4; p, q)$-decomposition of $K_6 \Box K_2$, $p \neq 0$.*

Proof: Let $V(K_6\Box K_2) = \{x_{i,j}\colon 1 \le i \le 6,\ 1 \le j \le 2\}$. First we decompose $K_6\Box K_2$ into $\{P_5, 8S_5\}$ and $\{2P_5, 7S_5\}$ as follows:

$$\begin{gathered}\{x_{5,1}x_{2,1}x_{4,1}x_{4,2}x_{3,2},\ \{(x_{1,1}; x_{2,1}, x_{3,1}, \boldsymbol{x_{4,1}}, \boldsymbol{x_{1,2}}),\ (x_{2,2}; x_{2,1}, \boldsymbol{x_{1,2}}, \boldsymbol{x_{3,2}}, x_{4,2})\},\\ \{(x_{3,1}; x_{3,2}, x_{2,1}, \boldsymbol{x_{4,1}}, \boldsymbol{x_{6,1}}),\ (x_{6,2}; \boldsymbol{x_{6,1}}, \boldsymbol{x_{2,2}}, x_{3,2}, x_{4,2})\},\\ (x_{5,1}; x_{5,2}, x_{1,1}, x_{3,1}, x_{4,1}),\ (x_{6,1}; x_{2,1}, x_{1,1}, x_{4,1}, x_{5,1}),\\ (x_{1,2}; x_{3,2}, x_{4,2}, x_{5,2}, x_{6,2}),\ (x_{5,2}; x_{2,2}, x_{3,2}, x_{4,2}, x_{6,2})\}\\ \text{and} \qquad \{x_{5,1}x_{2,1}x_{4,1}x_{4,2}x_{3,2},\ x_{1,1}x_{3,1}x_{4,1}x_{5,1}x_{5,2},\\ \{(x_{1,1}; x_{2,1}, x_{4,1}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{1,2}}),\ (x_{2,2}; x_{2,1}, \boldsymbol{x_{1,2}}, \boldsymbol{x_{3,2}}, x_{4,2})\},\\ \{(x_{3,1}; x_{3,2}, x_{2,1}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{6,1}}),\ (x_{6,2}; \boldsymbol{x_{6,1}}, \boldsymbol{x_{2,2}}, x_{3,2}, x_{4,2})\},\\ (x_{6,1}; x_{2,1}, x_{1,1}, x_{4,1}, x_{5,1}),\ (x_{1,2}; x_{3,2}, x_{4,2}, x_{5,2}, x_{6,2}),\ (x_{5,2}; x_{2,2}, x_{3,2}, x_{4,2}, x_{6,2})\}.\end{gathered}$$

By Remark 1.2, we obtain a required even number of paths from $\{2P_5, 7S_5\}$ except $p = 8$ and we obtain a required odd number of paths from $\{P_5, 8S_5\}$ except $p = 7, 9$. Now,

$$\begin{gathered}\{x_{5,2}x_{4,2}x_{2,2}x_{1,2}x_{3,2},\ x_{3,2}x_{6,2}x_{4,2}x_{1,2}x_{5,2},\ x_{3,2}x_{2,2}x_{6,2}x_{1,2}x_{1,1},\\ x_{4,1}x_{5,1}x_{3,1}x_{2,1}x_{2,2},\ x_{6,1}x_{2,1}x_{5,1}x_{1,1}x_{3,1},\ x_{3,1}x_{3,2}x_{4,2}x_{4,1}x_{2,1},\\ x_{2,1}x_{1,1}x_{4,1}x_{3,1}x_{6,1},\ \{(x_{6,1}; x_{6,2}, x_{1,1}, \boldsymbol{x_{4,1}}, \boldsymbol{x_{5,1}}), (x_{5,2}; \boldsymbol{x_{5,1}}, \boldsymbol{x_{2,2}}, x_{3,2}, x_{6,2})\}\}\\ \text{and} \qquad \{x_{5,1}x_{2,1}x_{4,1}x_{4,2}x_{3,2},\ x_{4,2}x_{2,2}x_{1,2}x_{1,1}x_{3,1},\ x_{2,1}x_{1,1}x_{4,1}x_{3,1}x_{6,1},\\ x_{6,1}x_{6,2}x_{2,2}x_{5,2}x_{4,2},\ x_{3,2}x_{1,2}x_{4,2}x_{6,2}x_{5,2},\ x_{4,1}x_{5,1}x_{5,2}x_{1,2}x_{6,2},\\ x_{6,2}x_{3,2}x_{3,1}x_{5,1}x_{1,1},\ x_{5,2}x_{3,2}x_{2,2}x_{2,1}x_{3,1},\ (x_{6,1}; x_{2,1}, x_{1,1}, x_{4,1}, x_{5,1})\}\end{gathered}$$

gives the remaining number of paths and stars of $K_6\Box K_2$. □

Lemma 2.3. *There exists a $(4; p, q)$-decomposition of $K_8\Box K_2$.*

Proof: Let $V(K_8\Box K_2) = \{x_{i,j}\colon 1 \le i \le 8,\ 1 \le j \le 2\}$ and K_2^i (K_8^j, respectively) be K_2 in the i^{th} row (K_8 in the j^{th} column, respectively) of $K_8\Box K_2$. We can write $K_8\Box K_2 = G_1 \oplus G_2$, where $G_1 = K_8^1 \oplus K_2^1 \oplus K_2^3 \oplus \cdots \oplus K_2^7$ and $G_2 = K_8^2 \oplus K_2^2 \oplus K_2^4 \oplus \cdots \oplus K_2^8$. Since $G_1 \cong G_2$, it is enough to prove without loss of generality that G_1 has a $(4; p, q)$-decomposition. First decompose G_1 into $\{0P_5, 8S_5\}$ as follows:

$$\begin{gathered}\{(x_{1,1}; x_{1,2}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{7,1}}, x_{8,1}),\ (x_{3,1}; x_{3,2}, \boldsymbol{x_{4,1}}, \boldsymbol{x_{7,1}}, x_{8,1})\},\\ \{(x_{5,1}; x_{5,2}, \boldsymbol{x_{3,1}}, \boldsymbol{x_{6,1}}, x_{8,1}),\ (x_{7,1}; x_{7,2}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{6,1}}, x_{8,1})\},\\ (x_{1,1}; x_{2,1}, x_{3,1}, x_{4,1}, x_{6,1}),\ (x_{4,1}; x_{2,1}, x_{5,1}, x_{7,1}, x_{8,1}),\\ (x_{2,1}; x_{3,1}, x_{5,1}, x_{7,1}, x_{8,1}),\ (x_{6,1}; x_{2,1}, x_{3,1}, x_{4,1}, x_{8,1}).\end{gathered}$$

Now, we decompose the last $4S_5$ into either $\{1P_5, 3S_5\}$, $\{2P_5, 2S_5\}$, $\{3P_5, S_5\}$ or $\{4P_5\}$ as follows:

$$\begin{gathered}\{x_{4,1}x_{5,1}x_{2,1}x_{3,1}x_{1,1},\ (x_{2,1}; x_{1,1}, x_{6,1}, x_{7,1}, x_{8,1}),\\ (x_{4,1}; x_{1,1}, x_{2,1}, x_{7,1}, x_{8,1}),\ (x_{6,1}; x_{1,1}, x_{3,1}, x_{4,1}, x_{8,1})\}\end{gathered}$$

$$\{x_{3,1}x_{1,1}x_{6,1}x_{8,1}x_{2,1},\ x_{7,1}x_{2,1}x_{3,1}x_{6,1}x_{4,1},$$
$$(x_{2,1};x_{1,1},x_{4,1},x_{5,1},x_{6,1}),\ (x_{4,1};x_{1,1},x_{5,1},x_{7,1},x_{8,1})\},$$
$$\{x_{2,1}x_{1,1}x_{6,1}x_{4,1}x_{5,1},\ x_{7,1}x_{4,1}x_{8,1}x_{6,1}x_{3,1},$$
$$x_{3,1}x_{1,1}x_{4,1}x_{2,1}x_{6,1},\ (x_{2,1};x_{3,1},x_{5,1},x_{7,1},x_{8,1})\}$$
$$\text{or}\qquad \{x_{2,1}x_{1,1}x_{6,1}x_{4,1}x_{5,1},\ x_{3,1}x_{1,1}x_{4,1}x_{2,1}x_{8,1},$$
$$x_{6,1}x_{2,1}x_{7,1}x_{4,1}x_{8,1},\ x_{8,1}x_{6,1}x_{3,1}x_{2,1}x_{5,1}\}.$$

Now, from $\{4P_5\}$ and the paired stars given above we can obtain an even number of paths and from $\{3P_5, S_5\}$ and the paired stars given above we can obtain an odd number of paths (see Remark 1.2). □

Lemma 2.4. *There exists a* $(4;p,q)$*-decomposition of* $K_{10}\Box K_2$.

Proof: Let $V(K_{10}\Box K_2) = \{x_{i,j}\colon 1 \le i \le 10,\ 1 \le j \le 2\}$. We can write $K_{10}\Box K_2 = (K_6\Box K_2) \oplus (K_4\Box K_2) \oplus 2K_{6,4}$. By Lemmas 2.1 and 2.2, $K_4\Box K_2$ has a $(4;p,q)$-decomposition with $p \ge 2$ and $K_6\Box K_2$ has a $(4;p,q)$-decomposition with $p \ne 0$. Also, by Theorem 1.1, $K_{6,4}$ has a $(4;p,q)$-decomposition. Hence by Remark 1.1, $K_{10}\Box K_2$ has a $(4;p,q)$-decomposition with $p \ge 3$. Now, the following $\{25S_5\}$ gives us the $\{0P_5, 25S_5\}$ and $\{2P_5, 23S_5\}$-decomposition of $K_{10}\Box K_2$ (use Remark 1.2)

$$(x_{8,1};x_{1,1},x_{7,1},x_{9,1},x_{10,1}),\ (x_{9,1};x_{2,1},x_{4,1},x_{7,1},x_{10,1}),\ (x_{10,1};x_{2,1},x_{4,1},x_{5,1},x_{7,1}),$$
$$\{(x_{2,1};\boldsymbol{x_{5,1}},\boldsymbol{x_{6,1}},x_{4,1},x_{2,2}),\ (x_{3,1};x_{4,1},x_{5,1},\boldsymbol{x_{6,1}},\boldsymbol{x_{3,2}})\},$$
$$(x_{1,1};x_{5,1},x_{6,1},x_{9,1},x_{1,2}),\ (x_{4,2};x_{2,2},x_{3,2},x_{9,2},x_{4,1}),\ (x_{5,2};x_{1,2},x_{2,2},x_{3,2},x_{5,1}),$$
$$(x_{6,2};x_{1,2},x_{2,2},x_{3,2},x_{6,1}),\ (x_{7,2};x_{8,2},x_{9,2},x_{10,2},x_{7,1}),\ (x_{8,2};x_{1,2},x_{9,2},x_{10,2},x_{8,1}),$$
$$(x_{9,2};x_{1,2},x_{2,2},x_{10,2},x_{9,1}),\ (x_{10,2};x_{2,2},x_{4,2},x_{5,2},x_{10,1}),$$
$$(x_{1,j};x_{3,j},x_{4,j},x_{7,j},x_{10,j}),\ (x_{3,j};x_{7,j},x_{8,j},x_{9,j},x_{10,j}),\ (x_{2,j};x_{1,j},x_{3,j},x_{8,j},x_{7,j}),$$
$$(x_{4,j};x_{5,j},x_{6,j},x_{7,j},x_{8,j}),\ (x_{5,j};x_{6,j},x_{7,j},x_{8,j},x_{9,j}),\ (x_{6,j};x_{7,j},x_{8,j},x_{9,j},x_{10,j}),$$

$j = 1,2$. For $p = 1$, decompose the first $3S_5$ into $\{P_5, 2S_5\}$ as follows:

$$\{x_{1,1}x_{8,1}x_{7,1}x_{10,1}x_{5,1},(x_{9,1};x_{2,1},x_{4,1},x_{7,1},x_{8,1}),(x_{10,1};x_{2,1},x_{4,1},x_{8,1},x_{9,1})\}.$$

This $\{P_5, 2S_5\}$ together with the remaining stars in the above $\{25S_5\}$ will give a required decomposition of $K_{10}\Box K_2$. □

Lemma 2.5. *There exists a* $(4;p,q)$*-decomposition of* $K_{12}\Box K_2$.

Proof: Let $V(K_{12}\Box K_2) = \{x_{i,j}\colon 1 \le i \le 12,\ 1 \le j \le 2\}$. We can write $K_{12}\Box K_2 = G \oplus (K_8\Box K_2)$, where $G = (K_{12}\Box K_2)\backslash E(K_8\Box K_2)$ and $G = (K_4\Box K_2) \oplus 2K_{8,4}$. By Theorem 1.1 and Lemma 2.1, $K_{8,4}$ has a $(4;p,q)$-decomposition and $K_4\Box K_2$ has a $(4;p,q)$-decomposition with $p \ge 2$. Hence by Remark 1.1, G has a $(4;p,q)$-decomposition with $p \ge 2$. Now, for $p = 0$ we have the following $20S_5$ of G

$$(x_{1,1}; x_{2,1}, x_{11,1}, x_{12,1}, x_{1,2}),\ (x_{2,1}; x_{3,1}, x_{4,1}, x_{11,1}, x_{12,1}),$$
$$(x_{3,1}; x_{1,1}, x_{4,1}, x_{11,1}, x_{12,1}),\ (x_{4,1}; x_{4,2}, x_{1,1}, x_{11,1}, x_{12,1}),$$
$$(x_{1,2}; x_{2,2}, x_{3,2}, x_{11,2}, x_{12,2}),\ (x_{2,2}; x_{2,1}, x_{3,2}, x_{11,2}, x_{12,2}),$$
$$(x_{3,2}; x_{3,1}, x_{4,2}, x_{11,2}, x_{12,2}),\ (x_{4,2}; x_{1,2}, x_{2,2}, x_{11,2}, x_{12,2}),$$
$$(x_{i,j}; x_{1,j}, x_{2,j}, x_{3,j}, x_{4,j})$$

for $5 \le i \le 10$ and $j = 1, 2$. For $p = 1$, decompose the first $4S_5$ into $\{P_5, 3S_5\}$ as follows:

$$\{x_{11,1}x_{2,1}x_{12,1}x_{1,1}x_{1,2},\ (x_{1,1}; x_{2,1}, x_{3,1}, x_{4,1}, x_{11,1}),$$
$$(x_{3,1}; x_{2,1}, x_{4,1}, x_{11,1}, x_{12,1}),\ (x_{4,1}; x_{4,2}, x_{2,1}, x_{11,1}, x_{12,1})\}.$$

This $\{P_5, 3S_5\}$ together with the remaining stars in the above stars will give a required decomposition of G. Now, by Remark 1.1, $K_{12}\Box K_2$ has a $(4; p, q)$-decomposition. $\square$

Lemma 2.6. *There exists a* $(4; p, q)$*-decomposition of* $K_{14}\Box K_2$*.*

Proof: Let $V(K_{14}\Box K_2) = \{x_{i,j} \colon 1 \le i \le 14,\ 1 \le j \le 2\}$. We can write $K_{14}\Box K_2 = (K_8\Box K_2) \oplus (K_6\Box K_2) \oplus 2K_{8,6}$. By Theorem 1.2 and Lemmas 2.3 and 2.2, $K_{8,6}$ and $K_8\Box K_2$ each have a $(4; p, q)$-decomposition and $K_6\Box K_2$ has a $(4; p, q)$-decomposition with $p \neq 0$. Hence by Remark 1.1, $K_{14}\Box K_2$ has a $(4; p, q)$-decomposition with $p \neq 0$. Now, consider $K_{14}\Box K_2$ as $K_{10}\Box K_2 \oplus G$, where $G = (K_{14}\Box K_2)\backslash E(K_{10}\Box K_2)$. Since $K_{10}\Box K_2$ has a $(4; p, q)$-decomposition (by Lemma 2.4), it is enough to prove that G has a $\{24S_5\}$-decomposition and the required $\{24S_5\}$-decomposition is as follows:

$$(x_{1,1}; x_{2,1}, x_{13,1}, x_{14,1}, x_{1,2}),\ (x_{2,1}; x_{3,1}, x_{4,1}, x_{13,1}, x_{14,1}),$$
$$(x_{3,1}; x_{1,1}, x_{4,1}, x_{13,1}, x_{14,1}),\ (x_{4,1}; x_{4,2}, x_{1,1}, x_{13,1}, x_{14,1}),$$
$$(x_{1,2}; x_{2,2}, x_{3,2}, x_{13,2}, x_{14,2}),\ (x_{2,2}; x_{2,1}, x_{3,2}, x_{13,2}, x_{14,2}),$$
$$(x_{3,2}; x_{3,1}, x_{4,2}, x_{13,2}, x_{14,2}),\ (x_{4,2}; x_{1,2}, x_{2,2}, x_{13,2}, x_{14,2}),\ (x_{i,j}; x_{1,j}, x_{2,j}, x_{3,j}, x_{4,j})$$

for $5 \le i \le 12$ and $j = 1, 2$. Hence $K_{14}\Box K_2$ has a $(4; p, q)$-decomposition. $\square$

Lemma 2.7. *There exists a* $(4; p, q)$*-decomposition of* $K_4\Box K_4$*.*

Proof: Let $V(K_4\Box K_4) = \{x_{i,j} \colon 1 \le i, j \le 4\}$. First we decompose $K_4\Box K_4$ into $\{0P_5, 12S_5\}$ and $\{P_5, 11S_5\}$ as follows:

$$\{(x_{2,3}; x_{2,1}, x_{2,2}, x_{3,3}, x_{4,3}),\ (x_{4,4}; x_{4,1}, x_{4,3}, x_{3,4}, x_{1,4}),$$
$$\{(x_{1,1}; \boldsymbol{x_{3,1}}, \boldsymbol{x_{2,1}}, x_{1,2}, x_{1,4}),\ (x_{2,4}; x_{1,4}, \boldsymbol{x_{2,1}}, \boldsymbol{x_{2,3}}, x_{4,4})\},$$
$$\{(x_{1,2}; x_{3,2}, x_{2,2}, \boldsymbol{x_{1,3}}, \boldsymbol{x_{1,4}}),\ (x_{3,4}; \boldsymbol{x_{1,4}}, \boldsymbol{x_{2,4}}, x_{3,3}, x_{3,2})\},$$
$$\{(x_{1,3}; \boldsymbol{x_{1,4}}, \boldsymbol{x_{1,1}}, x_{2,3}, x_{4,3}),\ (x_{4,1}; \boldsymbol{x_{1,1}}, \boldsymbol{x_{2,1}}, x_{4,2}, x_{4,3})\},$$
$$\{(x_{2,2}; x_{2,1}, \boldsymbol{x_{2,4}}, \boldsymbol{x_{3,2}}, x_{4,2}),\ (x_{3,1}; x_{2,1}, x_{4,1}, \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,4}})\},$$
$$\{(x_{3,3}; x_{3,1}, x_{3,2}, \boldsymbol{x_{1,3}}, \boldsymbol{x_{4,3}}),\ (x_{4,2}; x_{1,2}, x_{3,2}, \boldsymbol{x_{4,3}}, \boldsymbol{x_{4,4}})\}\}$$
$$\text{and} \qquad \{x_{2,1}x_{2,3}x_{4,3}x_{4,4}x_{4,2},$$

$$\begin{gathered}\{(x_{1,1};\boldsymbol{x_{3,1}},\boldsymbol{x_{2,1}},x_{1,2},x_{1,4}),\ (x_{2,4};x_{1,4},\boldsymbol{x_{2,1}},\boldsymbol{x_{2,3}},x_{2,2})\},\\ \{(x_{1,2};x_{3,2},x_{2,2},\boldsymbol{x_{1,3}},\boldsymbol{x_{1,4}}),\ (x_{3,4};\boldsymbol{x_{1,4}},\boldsymbol{x_{2,4}},x_{3,3},x_{3,2})\},\\ \{(x_{1,3};\boldsymbol{x_{1,4}},\boldsymbol{x_{1,1}},x_{2,3},x_{4,3}),\ (x_{4,1};\boldsymbol{x_{1,1}},\boldsymbol{x_{2,1}},x_{3,1},x_{4,3})\},\\ \{(x_{2,2};x_{2,1},\boldsymbol{x_{2,3}},\boldsymbol{x_{3,2}},x_{4,2}),\ (x_{3,1};x_{2,1},x_{3,3},\boldsymbol{x_{3,2}},\boldsymbol{x_{3,4}})\},\\ \{(x_{3,3};x_{2,3},x_{3,2},\boldsymbol{x_{1,3}},\boldsymbol{x_{4,3}}),\ (x_{4,2};x_{1,2},x_{3,2},\boldsymbol{x_{4,3}},\boldsymbol{x_{4,1}})\},\\ (x_{4,4};x_{4,1},x_{1,4},x_{2,4},x_{3,4})\}.\end{gathered}$$

By Remark 1.2, we obtain a required even number of paths from $\{0P_5, 12S_5\}$ except $p = 12$ and we obtain a required odd number of paths from $\{P_5, 11S_5\}$. For $p = 12$, the required paths are

$x_{1,4}x_{4,4}x_{4,1}x_{3,1}x_{3,2}$, $x_{4,4}x_{4,2}x_{3,2}x_{3,4}x_{2,4}$, $x_{4,4}x_{2,4}x_{2,1}x_{2,3}x_{2,2}$, $x_{2,2}x_{2,4}x_{2,3}x_{3,3}x_{1,3}$, $x_{2,4}x_{1,4}x_{1,1}x_{3,1}x_{3,4}$, $x_{1,4}x_{1,2}x_{3,2}x_{3,3}x_{3,1}$, $x_{3,1}x_{2,1}x_{1,1}x_{1,2}x_{1,3}$, $x_{2,1}x_{4,1}x_{1,1}x_{1,3}x_{2,3}$, $x_{2,3}x_{4,3}x_{1,3}x_{1,4}x_{3,4}$, $x_{2,1}x_{2,2}x_{4,2}x_{4,3}x_{4,4}$, $x_{3,2}x_{2,2}x_{1,2}x_{4,2}x_{4,1}$, $x_{4,1}x_{4,3}x_{3,3}x_{3,4}x_{4,4}$.

□

Lemma 2.8. *There exists a* $(4;p,q)$*-decomposition of* $K_4\Box K_6$.

Proof: Let $V(K_4\Box K_6) = \{x_{i,j}\colon\ 1 \le i \le 4,\ 1 \le j \le 6\}$. First we decompose $K_4\Box K_6$ into $\{0P_5, 24S_5\}$ as follows:

$$\begin{gathered}\{(x_{3,2};\boldsymbol{x_{1,2}},\boldsymbol{x_{4,2}},x_{3,1},x_{3,4}),\ (x_{4,1};x_{2,1},x_{3,1},\boldsymbol{x_{4,2}},\boldsymbol{x_{4,3}})\},\\ \{(x_{2,2};x_{2,3},\boldsymbol{x_{2,4}},\boldsymbol{x_{2,5}},x_{4,2}),\ (x_{2,6};x_{1,6},\boldsymbol{x_{2,1}},\boldsymbol{x_{2,4}},x_{2,3})\},\\ \{(x_{3,1};x_{2,1},\boldsymbol{x_{3,4}},\boldsymbol{x_{3,5}},x_{3,6}),\ (x_{3,3};x_{3,2},\boldsymbol{x_{2,3}},\boldsymbol{x_{3,5}},x_{3,6})\},\\ \{(x_{4,4};x_{4,2},x_{4,3},\boldsymbol{x_{4,1}},\boldsymbol{x_{2,4}}),\ (x_{4,5};x_{2,5},\boldsymbol{x_{3,5}},\boldsymbol{x_{4,1}},x_{4,3})\},\\ \{(x_{1,1};\boldsymbol{x_{1,3}},\boldsymbol{x_{1,4}},x_{4,1},x_{1,2}),\ (x_{1,5};x_{1,2},\boldsymbol{x_{1,3}},\boldsymbol{x_{3,5}},x_{4,5})\},\\ \{(x_{3,3};\boldsymbol{x_{1,3}},\boldsymbol{x_{3,4}},x_{4,3},x_{3,1}),\ (x_{2,3};x_{2,1},\boldsymbol{x_{2,4}},\boldsymbol{x_{1,3}},x_{4,3})\},\\ \{(x_{2,4};x_{2,1},\boldsymbol{x_{2,5}},\boldsymbol{x_{1,4}},x_{3,4}),\ (x_{3,5};x_{3,2},x_{3,4},\boldsymbol{x_{3,6}},\boldsymbol{x_{2,5}})\},\\ \{(x_{2,2};x_{1,2},\boldsymbol{x_{3,2}},\boldsymbol{x_{2,6}},x_{2,1}),\ (x_{2,5};x_{1,5},x_{2,1},\boldsymbol{x_{2,3}},\boldsymbol{x_{2,6}})\},\\ \{(x_{4,4};x_{1,4},\boldsymbol{x_{4,5}},\boldsymbol{x_{4,6}},x_{3,4}),\ (x_{3,6};x_{2,6},\boldsymbol{x_{3,2}},\boldsymbol{x_{4,6}},x_{3,4})\},\\ \{(x_{1,1};x_{2,1},\boldsymbol{x_{3,1}},\boldsymbol{x_{1,5}},x_{1,6}),\ (x_{1,4};x_{1,2},x_{1,6},\boldsymbol{x_{3,4}},\boldsymbol{x_{1,5}})\},\\ \{(x_{4,2};x_{1,2},\boldsymbol{x_{4,3}},\boldsymbol{x_{4,5}},x_{4,6}),\ (x_{1,3};x_{1,2},x_{1,4},\boldsymbol{x_{1,6}},\boldsymbol{x_{4,3}})\},\\ (x_{1,6};x_{1,2},x_{1,5},x_{3,6},x_{4,6}),\ (x_{4,6};x_{2,6},x_{4,1},x_{4,3},x_{4,5}).\end{gathered}$$

By Remark 1.2, we obtain a required even number of paths from the paired stars except $p = 24$. For $p = 24$, the $18P_5$ can be obtained from the first nine paired stars (see Remark 1.2) and the remaining paths can be obtained from the last $6S_5$ as follows:

$$\begin{gathered}\{x_{3,1}x_{1,1}x_{1,6}x_{1,4}x_{1,5},\ x_{2,1}x_{1,1}x_{1,5}x_{1,6}x_{3,6},\ x_{4,3}x_{4,2}x_{4,5}x_{4,6}x_{4,1},\\ x_{2,6}x_{4,6}x_{1,6}x_{1,3}x_{1,4},\ x_{3,4}x_{1,4}x_{1,2}x_{1,3}x_{4,3},\ x_{4,3}x_{4,6}x_{4,2}x_{1,2}x_{1,6}\}.\end{gathered}$$

To get an odd number of paths we decompose the last $6S_5$ into either $\{P_5, 5S_5\}$, $\{3P_5, 3S_5\}$ or $\{5P_5, S_5\}$ as follows:

$$\{x_{1,5}x_{1,6}x_{1,2}x_{1,3}x_{4,3},\ (x_{1,6}; x_{1,4}, x_{1,3}, x_{3,6}, x_{4,6}),\ (x_{4,6}; x_{2,6}, x_{4,1}, x_{4,3}, x_{4,5}),$$
$$(x_{4,2}; x_{1,2}, x_{4,3}, x_{4,5}, x_{4,6}),\ (x_{1,4}; x_{1,2}, x_{1,3}, x_{3,4}, x_{1,5}),\ (x_{1,1}; x_{2,1}, x_{3,1}, x_{1,5}, x_{1,6})\},$$
$$\{x_{2,1}x_{1,1}x_{1,6}x_{1,3}x_{4,3},\ x_{4,3}x_{4,2}x_{4,5}x_{4,6}x_{4,1},\ x_{3,1}x_{1,1}x_{1,5}x_{1,6}x_{3,6},$$
$$(x_{1,2}; x_{4,2}, x_{1,3}, x_{1,4}, x_{1,6}),\ (x_{1,4}; x_{1,6}, x_{1,3}, x_{3,4}, x_{1,5}),\ (x_{4,6}; x_{2,6}, x_{4,2}, x_{4,3}, x_{1,6})\}$$
$$\text{or}\qquad \{x_{3,4}x_{1,4}x_{1,2}x_{1,3}x_{4,3},\ x_{4,2}x_{1,2}x_{1,6}x_{1,3}x_{1,4},\ x_{3,1}x_{1,1}x_{1,6}x_{1,4}x_{1,5},$$
$$x_{2,1}x_{1,1}x_{1,5}x_{1,6}x_{3,6},\ x_{4,3}x_{4,2}x_{4,5}x_{4,6}x_{4,1},\ (x_{4,6}; x_{2,6}, x_{4,2}, x_{4,3}, x_{1,6})\}.$$

Now, the remaining number of paths can be obtained from the first nine paired stars (see Remark 1.2). Hence $K_4\square K_6$ has a $(4;p,q)$-decomposition. □

Lemma 2.9. *There exists a $(4;p,q)$-decomposition of $K_6\square K_6$.*

Proof: Let $V(K_6\square K_6) = \{x_{i,j}\colon 1 \le i,\ j \le 6\}$. Now, we can write $K_6\square K_6 = (K_4\square K_6) \oplus (K_2\square K_6) \oplus 6K_{4,2}$. By Lemma 2.8 and Theorem 1.3, $K_4\square K_6$ and $K_{4,2}$ each have a $(4;p,q)$-decomposition. Also, $K_2\square K_6$ ($\cong K_6\square K_2$) has a $(4;p,q)$-decomposition with $p \ne 0$, by Lemma 2.2. Hence $K_6\square K_6$ has a $(4;p,q)$-decomposition with $p \ne 0$. For $p = 0$, we have the following $\{45S_5\}$.

$$(x_{1,1}; x_{1,2}, x_{1,3}, x_{2,1}, x_{3,1}),\ (x_{1,1}; x_{1,4}, x_{1,5}, x_{4,1}, x_{6,1}),\ (x_{6,1}; x_{5,1}, x_{4,1}, x_{6,2}, x_{6,3}),$$
$$(x_{3,4}; x_{3,3}, x_{3,5}, x_{2,4}, x_{4,4}),\ (x_{6,6}; x_{5,6}, x_{4,6}, x_{6,4}, x_{6,5}),\ (x_{2,2}; x_{2,1}, x_{2,3}, x_{1,2}, x_{3,2}),$$
$$(x_{1,6}; x_{1,5}, x_{1,4}, x_{2,6}, x_{3,6}),\ (x_{4,4}; x_{4,3}, x_{4,5}, x_{6,4}, x_{1,4}),\ (x_{6,2}; x_{5,2}, x_{4,2}, x_{6,3}, x_{6,4}),$$
$$(x_{6,6}; x_{6,1}, x_{6,2}, x_{1,6}, x_{2,6}),\ (x_{2,5}; x_{2,4}, x_{2,6}, x_{1,5}, x_{3,5}),\ (x_{3,4}; x_{3,2}, x_{3,6}, x_{1,4}, x_{5,4}),$$
$$(x_{1,6}; x_{1,1}, x_{1,3}, x_{4,6}, x_{5,6}),\ (x_{2,2}; x_{2,4}, x_{2,6}, x_{4,2}, x_{6,2}),\ (x_{5,5}; x_{5,1}, x_{5,4}, x_{4,5}, x_{1,5}),$$
$$(x_{1,3}; x_{1,4}, x_{1,5}, x_{3,3}, x_{4,3}),\ (x_{2,5}; x_{2,2}, x_{2,3}, x_{4,5}, x_{6,5}),\ (x_{6,4}; x_{6,1}, x_{6,3}, x_{3,4}, x_{1,4}),$$
$$(x_{2,1}; x_{2,6}, x_{2,5}, x_{6,1}, x_{5,1}),\ (x_{5,5}; x_{3,5}, x_{2,5}, x_{5,2}, x_{5,3}),\ (x_{1,2}; x_{1,3}, x_{1,6}, x_{5,2}, x_{6,2}),$$
$$(x_{6,3}; x_{5,3}, x_{1,3}, x_{6,5}, x_{6,6}),\ (x_{3,5}; x_{3,1}, x_{3,6}, x_{4,5}, x_{6,5}),\ (x_{3,3}; x_{3,1}, x_{3,2}, x_{5,3}, x_{6,3}),$$
$$(x_{4,4}; x_{2,4}, x_{5,4}, x_{4,1}, x_{4,6}),\ (x_{1,4}; x_{1,2}, x_{1,5}, x_{2,4}, x_{5,4}),\ (x_{4,2}; x_{1,2}, x_{3,2}, x_{4,3}, x_{4,4}),$$
$$(x_{3,3}; x_{2,3}, x_{4,3}, x_{3,5}, x_{3,6}),\ (x_{1,5}; x_{1,2}, x_{3,5}, x_{4,5}, x_{6,5}),\ (x_{2,4}; x_{2,1}, x_{2,6}, x_{5,4}, x_{6,4}),$$
$$(x_{2,3}; x_{1,3}, x_{6,3}, x_{2,1}, x_{2,4}),\ (x_{3,6}; x_{3,2}, x_{4,6}, x_{5,6}, x_{6,6}),\ (x_{5,4}; x_{5,1}, x_{5,2}, x_{5,6}, x_{6,4}),$$
$$(x_{5,2}; x_{4,2}, x_{3,2}, x_{2,2}, x_{5,3}),\ (x_{4,3}; x_{4,1}, x_{4,5}, x_{2,3}, x_{6,3}),\ (x_{6,5}; x_{6,1}, x_{6,2}, x_{6,4}, x_{5,5}),$$
$$(x_{4,5}; x_{4,6}, x_{4,1}, x_{4,2}, x_{6,5}),\ (x_{5,3}; x_{4,3}, x_{1,3}, x_{2,3}, x_{5,4}),\ (x_{3,1}; x_{2,1}, x_{3,4}, x_{3,6}, x_{6,1}),$$
$$(x_{4,6}; x_{4,1}, x_{4,2}, x_{4,3}, x_{5,6}),\ (x_{3,2}; x_{3,1}, x_{3,5}, x_{1,2}, x_{6,2}),\ (x_{5,6}; x_{5,1}, x_{5,2}, x_{5,3}, x_{5,5}),$$
$$(x_{2,6}; x_{2,3}, x_{3,6}, x_{4,6}, x_{5,6}),\ (x_{4,1}; x_{2,1}, x_{3,1}, x_{5,1}, x_{4,2}),\ (x_{5,1}; x_{3,1}, x_{1,1}, x_{5,2}, x_{5,3}).$$

□

Lemma 2.10. *There exists a $(4;p,q)$-decomposition of $K_5\square K_5$.*

Proof: Let $V(K_5\square K_5) = \{x_{i,j}\colon 1 \le i, j \le 5\}$. First we decompose $K_5\square K_5$ into $\{0P_5, 25S_5\}$ as follows:

$$\{(x_{1,1}; \boldsymbol{x_{2,1}}, \boldsymbol{x_{1,3}}, x_{3,1}, x_{1,5}),\ (x_{1,4}; \boldsymbol{x_{1,3}}, \boldsymbol{x_{3,4}}, x_{1,5}, x_{5,4})\},$$
$$\{(x_{1,1}; x_{1,2}, \boldsymbol{x_{1,4}}, \boldsymbol{x_{4,1}}, x_{5,1}),\ (x_{2,1}; \boldsymbol{x_{3,1}}, \boldsymbol{x_{4,1}}, x_{5,1}, x_{2,5})\},$$
$$\{(x_{5,5}; x_{1,5}, x_{2,5}, \boldsymbol{x_{5,4}}, \boldsymbol{x_{4,5}}),\ (x_{3,5}; x_{2,5}, \boldsymbol{x_{4,5}}, \boldsymbol{x_{3,4}}, x_{3,1})\},$$

$$\{(x_{3,3}; \boldsymbol{x_{5,3}}, \boldsymbol{x_{3,2}}, x_{3,4}, x_{3,5}),\ (x_{3,1}; x_{4,1}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{3,2}}, x_{3,4})\},$$
$$\{(x_{2,2}; x_{2,1}, \boldsymbol{x_{2,3}}, \boldsymbol{x_{4,2}}, x_{5,2}),\ (x_{1,2}; x_{1,3}, \boldsymbol{x_{1,4}}, \boldsymbol{x_{4,2}}, x_{5,2})\},$$
$$\{(x_{3,3}; x_{1,3}, \boldsymbol{x_{2,3}}, \boldsymbol{x_{4,3}}, x_{3,1}),\ (x_{5,3}; x_{5,1}, \boldsymbol{x_{5,4}}, \boldsymbol{x_{2,3}}, x_{1,3})\},$$
$$\{(x_{2,2}; x_{1,2}, \boldsymbol{x_{3,2}}, \boldsymbol{x_{2,4}}, x_{2,5}),\ (x_{2,3}; x_{2,1}, \boldsymbol{x_{1,3}}, \boldsymbol{x_{2,4}}, x_{2,5})\},$$
$$\{(x_{4,4}; x_{1,4}, \boldsymbol{x_{4,2}}, \boldsymbol{x_{3,4}}, x_{5,4}),\ (x_{2,4}; \boldsymbol{x_{2,5}}, \boldsymbol{x_{3,4}}, x_{1,4}, x_{2,1})\},$$
$$\{(x_{5,5}; x_{5,1}, \boldsymbol{x_{5,2}}, \boldsymbol{x_{5,3}}, x_{3,5}),\ (x_{5,4}; x_{2,4}, \boldsymbol{x_{3,4}}, \boldsymbol{x_{5,2}}, x_{5,1})\},$$
$$\{(x_{3,2}; x_{1,2}, x_{4,2}, \boldsymbol{x_{3,4}}, \boldsymbol{x_{3,5}}),\ (x_{1,5}; x_{1,3}, x_{1,2}, \boldsymbol{x_{2,5}}, \boldsymbol{x_{3,5}})\},$$
$$\{(x_{5,2}; x_{4,2}, x_{3,2}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{5,3}}),\ (x_{4,3}; x_{4,2}, x_{2,3}, \boldsymbol{x_{1,3}}, \boldsymbol{x_{5,3}})\},$$
$$(x_{4,4}; x_{4,1}, x_{2,4}, x_{4,3}, x_{4,5}),\ (x_{4,5}; x_{4,2}, x_{4,3}, x_{1,5}, x_{2,5}),\ (x_{4,1}; x_{4,2}, x_{4,3}, x_{4,5}, x_{5,1}).$$

Now, we decompose the last $3S_5$ into either $\{1P_5, 2S_5\}$, $\{2P_5, 1S_5\}$ or $\{3P_5\}$ as follows:

$$\{x_{2,4}x_{4,4}x_{4,3}x_{4,5}x_{4,1},\ (x_{4,5}; x_{4,2}, x_{4,4}, x_{1,5}, x_{2,5}),\ (x_{4,1}; x_{4,2}, x_{4,3}, x_{4,4}, x_{5,1})\},$$
$$\{x_{2,4}x_{4,4}x_{4,3}x_{4,1}x_{4,2},\ x_{4,2}x_{4,5}x_{4,4}x_{4,1}x_{5,1},\ (x_{4,5}; x_{4,1}, x_{4,3}, x_{1,5}, x_{2,5})\}$$
$$\text{or}\qquad \{x_{2,4}x_{4,4}x_{4,1}x_{4,5}x_{4,3},\ x_{2,5}x_{4,5}x_{4,4}x_{4,3}x_{4,1},\ x_{1,5}x_{4,5}x_{4,2}x_{4,1}x_{5,1}\}.$$

Now, from $\{2P_5, 1S_5\}$ and the paired stars given above we can obtain an even number of paths and from $\{3P_5\}$ and the paired stars given above we can obtain an odd number of paths (see Remark 1.2). □

Lemma 2.11. *There exists a $(4; p, q)$-decomposition of $K_3 \Box K_7$.*

Proof: Let $V(K_3 \Box K_7) = \{x_{i,j} \colon 1 \le i \le 3,\ 1 \le j \le 7\}$ and K_7^i (K_3^j, respectively) be a K_7 in the i^{th} row (K_3 in the j^{th} column, respectively) of $K_3 \Box K_7$. For $i = 1, 2, 3$, let $F_i = \{x_{i,1}x_{i+1,1}, \ldots, x_{i,7}x_{i+1,7}\}$, where the first coordinate of the subscripts of x are taken modulo 3 with residues $1, 2, 3$. We can write $K_3 \Box K_7 = G_1 \oplus G_2 \oplus G_3$, where $G_i = F_i \oplus K_7^i$. Since $G_1 \cong G_2 \cong G_3$, it is enough to prove without loss of generality that G_1 has a $(4; p, q)$-decomposition. Now, G_1 has a $(4; p, q)$-decomposition as follows:

1. For $p = 0$, $q = 7$, the required stars are $(x_{1,1}; x_{2,1}, x_{1,2}, x_{1,3}, x_{1,4})$, $(x_{1,2}; x_{2,2}, x_{1,5}, x_{1,3}, x_{1,4})$, $(x_{1,3}; x_{2,3}, x_{1,4}, x_{1,5}, x_{1,6})$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$, $(x_{1,5}; x_{2,5}, x_{1,1}, x_{1,6}, x_{1,7})$, $(x_{1,6}; x_{2,6}, x_{1,1}, x_{1,2}, x_{1,7})$, $(x_{1,7}; x_{2,7}, x_{1,1}, x_{1,3}, x_{1,2})$.
2. For $p = 1$, $q = 6$, the required path and stars are $x_{2,1}x_{1,1}x_{1,4}x_{1,3}x_{1,2}$, $(x_{1,2}; x_{2,2}, x_{1,5}, x_{1,1}, x_{1,4})$, $(x_{1,3}; x_{2,3}, x_{1,1}, x_{1,5}, x_{1,6})$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$, $(x_{1,5}; x_{2,5}, x_{1,1}, x_{1,6}, x_{1,7})$, $(x_{1,6}; x_{2,6}, x_{1,1}, x_{1,2}, x_{1,7})$, $(x_{1,7}; x_{2,7}, x_{1,1}, x_{1,3}, x_{1,2})$.
3. For $p = 2$, $q = 5$, the required paths and stars are $x_{2,1}x_{1,1}x_{1,4}x_{1,3}x_{1,2}$, $x_{2,3}x_{1,3}x_{1,1}x_{1,6}x_{1,5}$, $(x_{1,2}; x_{2,2}, x_{1,5}, x_{1,1}, x_{1,4})$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$, $(x_{1,5}; x_{2,5}, x_{1,1}, x_{1,3}, x_{1,7})$, $(x_{1,6}; x_{2,6}, x_{1,3}, x_{1,2}, x_{1,7})$, $(x_{1,7}; x_{2,7}, x_{1,1}, x_{1,3}, x_{1,2})$.
4. For $p = 3$, $q = 4$, the required paths and stars are $x_{2,1}x_{1,1}x_{1,4}x_{1,3}x_{1,2}$, $x_{2,3}x_{1,3}x_{1,1}x_{1,2}x_{1,4}$, $x_{1,1}x_{1,6}x_{1,5}x_{1,2}x_{2,2}$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$, $(x_{1,5}; x_{2,5}, x_{1,1}, x_{1,3}, x_{1,7})$, $(x_{1,6}; x_{2,6}, x_{1,3}, x_{1,2}, x_{1,7})$, $(x_{1,7}; x_{2,7}, x_{1,1}, x_{1,3}, x_{1,2})$.

5. For $p = 4$, $q = 3$, the required paths and stars are $x_{2,7}x_{1,7}x_{1,1}x_{1,4}x_{1,3}$, $x_{2,3}x_{1,3}x_{1,7}x_{1,2}x_{1,5}$, $x_{2,2}x_{1,2}x_{1,1}x_{1,6}x_{1,5}$, $x_{2,1}x_{1,1}x_{1,3}x_{1,2}x_{1,4}$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$, $(x_{1,5}; x_{2,5}, x_{1,1}, x_{1,3}, x_{1,7})$, $(x_{1,6}; x_{2,6}, x_{1,3}, x_{1,2}, x_{1,7})$.

6. For $p = 5$, $q = 2$, the required paths and stars are $x_{2,1}x_{1,1}x_{1,4}x_{1,3}x_{1,2}$, $x_{2,3}x_{1,3}x_{1,1}x_{1,2}x_{1,4}$, $x_{1,1}x_{1,6}x_{1,5}x_{1,2}x_{2,2}$, $x_{2,5}x_{1,5}x_{1,7}x_{1,6}x_{1,2}$, $x_{2,6}x_{1,6}x_{1,3}x_{1,5}x_{1,1}$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$, $(x_{1,7}; x_{2,7}, x_{1,1}, x_{1,3}, x_{1,2})$.

7. For $p = 6$, $q = 1$, the require paths and stars are $x_{2,7}x_{1,7}x_{1,1}x_{1,4}x_{1,3}$, $x_{2,3}x_{1,3}x_{1,7}x_{1,2}x_{1,5}$, $x_{2,2}x_{1,2}x_{1,1}x_{1,6}x_{1,5}$, $x_{2,1}x_{1,1}x_{1,3}x_{1,2}x_{1,4}$, $x_{2,5}x_{1,5}x_{1,7}x_{1,6}x_{1,2}$, $x_{2,6}x_{1,6}x_{1,3}x_{1,5}x_{1,1}$, $(x_{1,4}; x_{2,4}, x_{1,6}, x_{1,7}, x_{1,5})$.

8. For $p = 7$, $q = 0$, the required paths are $x_{2,1}x_{1,1}x_{1,2}x_{1,3}x_{1,4}$, $x_{2,2}x_{1,2}x_{1,4}x_{1,6}x_{1,7}$, $x_{2,3}x_{1,3}x_{1,1}x_{1,7}x_{1,5}$, $x_{2,4}x_{1,4}x_{1,1}x_{1,5}x_{1,3}$, $x_{2,5}x_{1,5}x_{1,2}x_{1,6}x_{1,1}$, $x_{2,6}x_{1,6}x_{1,3}x_{1,7}x_{1,2}$, $x_{2,7}x_{1,7}x_{1,4}x_{1,5}x_{1,6}$.

Hence by Remark 1.1, $K_3 \square K_7$ has a $(4; p, q)$-decomposition. $\square$

Lemma 2.12. *There exists a $(4; p, q)$-decomposition of $K_3 \square K_8$.*

Proof: Let $V(K_3 \square K_8) = \{x_{i,j} \colon 1 \le i \le 3,\ 1 \le j \le 8\}$ and K_8^i (K_3^j, respectively) be a K_8 in the i^{th} row (K_3 in the j^{th} column, respectively) of $K_3 \square K_8$. For $i = 1, 2, 3$, let $F_i = \{x_{i,1}x_{i+1,1}, \ldots, x_{i,8}x_{i+1,8}\}$, where the first subscripts of x are taken modulo 3 with residues $1, 2, 3$. We can write $K_3 \square K_8 = G_1 \oplus G_2 \oplus G_3$, where $G_i = F_i \oplus K_8^i$. Since $G_1 \cong G_2 \cong G_3$, it is enough to prove without loss of generality that G_1 has a $(4; p, q)$-decomposition. Now,

$$G_1 = F_1' \oplus K_7^1 \oplus (x_{1,8}; x_{2,8}, x_{1,1}, x_{1,3}, x_{1,2}) \oplus (x_{1,8}; x_{1,4}, x_{1,5}, x_{1,6}, x_{1,7}),$$

where $F_1' = \{x_{i,1}x_{i+1,1}, \ldots, x_{i,7}x_{i+1,7}\}$ and it has a $(4; p, q)$-decomposition except for the values $p = 8$ and 9 (see Lemma 2.11). For $p = 8, 9$, we have the following sets of paths and stars:

$$\begin{gathered}\{x_{2,1}x_{1,1}x_{1,2}x_{1,3}x_{1,4},\ x_{2,2}x_{1,2}x_{1,4}x_{1,6}x_{1,7},\ x_{2,3}x_{1,3}x_{1,1}x_{1,7}x_{1,5},\\ x_{2,4}x_{1,4}x_{1,1}x_{1,5}x_{1,3},\ x_{1,2}x_{1,6}x_{1,1}x_{1,8}x_{2,8},\ x_{2,5}x_{1,5}x_{1,2}x_{1,8}x_{1,3},\\ x_{2,6}x_{1,6}x_{1,3}x_{1,7}x_{1,2},\ x_{2,7}x_{1,7}x_{1,4}x_{1,5}x_{1,6},\ (x_{1,8}; x_{1,4}, x_{1,5}, x_{1,6}, x_{1,7})\}\\ \text{and} \qquad \{x_{2,1}x_{1,1}x_{1,2}x_{1,3}x_{1,4},\ x_{2,3}x_{1,3}x_{1,1}x_{1,7}x_{1,5},\ x_{2,4}x_{1,4}x_{1,1}x_{1,5}x_{1,3},\\ x_{1,2}x_{1,6}x_{1,1}x_{1,8}x_{2,8},\ x_{2,5}x_{1,5}x_{1,2}x_{1,8}x_{1,3},\ x_{2,6}x_{1,6}x_{1,3}x_{1,7}x_{1,2},\\ x_{1,5}x_{1,8}x_{1,6}x_{1,7}x_{2,7},\ x_{1,4}x_{1,8}x_{1,7}x_{1,4}x_{1,5},\ x_{2,2}x_{1,2}x_{1,4}x_{1,6}x_{1,5}\}.\end{gathered}$$

Hence by Remark 1.1, $K_3 \square K_8$ has a $(4; p, q)$-decomposition. $\square$

Lemma 2.13. *There exists a $(4; p, q)$-decomposition of $K_5 \square K_8$.*

Proof: Let $V(K_5 \square K_8) = \{x_{i,j} \colon 1 \le i \le 5,\ 1 \le j \le 8\}$. We can write $K_5 \square K_8 = (K_5 \square K_8 \backslash E(K_3 \square K_8)) \oplus (K_3 \square K_8)$. First we decompose $(K_5 \square K_8) \backslash E(K_3 \square K_8)$ into $\{0P_5, 28S_5\}$ as follows:

$$\begin{gathered}
\{(x_{1,1}; x_{3,1}, \boldsymbol{x_{4,1}}, \boldsymbol{x_{5,1}}, x_{1,2}),\ (x_{2,1}; x_{3,1}, x_{4,1}, \boldsymbol{x_{5,1}}, \boldsymbol{x_{2,8}})\},\\
\{(x_{1,2}; x_{3,2}, \boldsymbol{x_{4,2}}, \boldsymbol{x_{5,2}}, x_{1,3}),\ (x_{2,2}; x_{3,2}, x_{4,2}, \boldsymbol{x_{5,2}}, \boldsymbol{x_{2,3}})\},\\
\{(x_{1,3}; x_{3,3}, \boldsymbol{x_{4,3}}, \boldsymbol{x_{5,3}}, x_{1,4}),\ (x_{2,3}; x_{3,3}, x_{4,3}, \boldsymbol{x_{5,3}}, \boldsymbol{x_{2,4}})\},\\
\{(x_{1,4}; x_{3,4}, \boldsymbol{x_{4,4}}, \boldsymbol{x_{5,4}}, x_{1,5}),\ (x_{2,4}; x_{3,4}, x_{4,4}, \boldsymbol{x_{5,4}}, \boldsymbol{x_{2,5}})\},\\
\{(x_{1,5}; x_{3,5}, \boldsymbol{x_{4,5}}, \boldsymbol{x_{5,5}}, x_{1,6}),\ (x_{2,5}; x_{3,5}, x_{4,5}, \boldsymbol{x_{5,5}}, \boldsymbol{x_{2,7}})\},\\
\{(x_{1,6}; x_{3,6}, \boldsymbol{x_{4,6}}, \boldsymbol{x_{5,6}}, x_{1,7}),\ (x_{2,6}; x_{3,6}, x_{4,6}, \boldsymbol{x_{5,6}}, \boldsymbol{x_{2,1}})\},\\
\{(x_{1,7}; x_{3,7}, \boldsymbol{x_{4,7}}, \boldsymbol{x_{5,7}}, x_{1,8}),\ (x_{2,7}; x_{3,7}, x_{4,7}, \boldsymbol{x_{5,7}}, \boldsymbol{x_{2,6}})\},\\
\{(x_{1,8}; x_{3,8}, \boldsymbol{x_{4,8}}, \boldsymbol{x_{5,8}}, x_{1,1}),\ (x_{2,8}; x_{3,8}, x_{4,8}, \boldsymbol{x_{5,8}}, \boldsymbol{x_{2,2}})\},\\
\{(x_{1,7}; x_{1,2}, \boldsymbol{x_{1,3}}, \boldsymbol{x_{1,4}}, x_{1,5}),\ (x_{1,8}; x_{1,2}, x_{1,3}, \boldsymbol{x_{1,4}}, \boldsymbol{x_{1,5}})\},\\
\{(x_{1,2}; x_{1,5}, \boldsymbol{x_{1,4}}, \boldsymbol{x_{1,6}}, x_{2,2}),\ (x_{1,3}; x_{1,1}, x_{1,5}, \boldsymbol{x_{1,6}}, \boldsymbol{x_{2,3}})\},\\
\{(x_{1,1}; x_{1,4}, \boldsymbol{x_{1,5}}, \boldsymbol{x_{1,7}}, x_{2,1}),\ (x_{2,7}; x_{2,1}, x_{2,4}, \boldsymbol{x_{1,7}}, \boldsymbol{x_{2,8}})\},\\
\{(x_{1,6}; x_{1,1}, \boldsymbol{x_{1,4}}, \boldsymbol{x_{1,8}}, x_{2,6}),\ (x_{2,8}; x_{2,3}, x_{2,6}, \boldsymbol{x_{1,8}}, \boldsymbol{x_{2,4}})\},\\
\{(x_{2,4}; x_{2,1}, \boldsymbol{x_{2,2}}, \boldsymbol{x_{2,6}}, x_{1,4}),\ (x_{2,5}; x_{2,1}, x_{2,8}, \boldsymbol{x_{2,6}}, \boldsymbol{x_{1,5}})\},\\
\{(x_{2,2}; x_{2,1}, \boldsymbol{x_{2,5}}, \boldsymbol{x_{2,6}}, x_{2,7}),\ (x_{2,3}; x_{2,1}, x_{2,5}, \boldsymbol{x_{2,6}}, \boldsymbol{x_{2,7}})\}.
\end{gathered}$$

By Remark 1.2, we obtain a required even number of paths and stars from the paired stars given above. To obtain an odd number of paths consider the last $4S_5$ and decompose it into either $\{1P_5, 3S_5\}$ or $\{3P_5, 1S_5\}$ as follows:

$$\begin{gathered}
\{x_{1,4}x_{2,4}x_{2,2}x_{2,7}x_{2,3},\ (x_{2,1}; x_{2,4}, x_{2,2}, x_{2,3}, x_{2,5}),\\
(x_{2,6}; x_{2,2}, x_{2,3}, x_{2,4}, x_{2,5}),\ (x_{2,5}; x_{2,2}, x_{2,3}, x_{2,8}, x_{1,5})\}\\
\text{or} \qquad \{x_{1,4}x_{2,4}x_{2,2}x_{2,7}x_{2,3},\ x_{2,3}x_{2,6}x_{2,2}x_{2,1}x_{2,4},\\
x_{2,3}x_{2,1}x_{2,5}x_{2,6}x_{2,4},\ (x_{2,5}; x_{2,2}, x_{2,3}, x_{2,8}, x_{1,5})\}.
\end{gathered}$$

The remaining choices for odd number of paths can be obtained from the remaining paired stars (see Remark 1.2). Also, by Lemma 2.12, $K_3 \square K_8$ has a $(4; p, q)$-decomposition. Hence by Remark 1.1, $K_5 \square K_8$ has a $(4; p, q)$-decomposition. $\square$

Lemma 2.14. *There exists a $(4; p, q)$-decomposition of $K_7 \square K_8$.*

PROOF: Let $V(K_7 \square K_8) = \{x_{i,j} \colon 1 \le i \le 7,\ 1 \le j \le 8\}$. We can write $K_7 \square K_8 = (K_7 \square K_8 \backslash E(K_2 \square K_8)) \oplus (K_2 \square K_8)$ and $(K_7 \square K_8) \backslash E(K_2 \square K_8) = 8(K_7 \backslash E(K_2)) \oplus 5K_8$. By Lemma 2.3 and Example 1.1, $K_2 \square K_8$ ($\cong K_8 \square K_2$) and K_8 have a $(4; p, q)$-decomposition. So, it is enough to prove that $K_7 \backslash E(K_2)$ has a $(4; p, q)$-decomposition Let $V(K_7) = \{x_i \colon 1 \le i \le 7\}$. Now, $K_7 \backslash E(K_2)$ has a $(4; p, q)$-decomposition as follows:

1. For $p = 0$, $q = 5$, the required stars are $(x_1; x_4, x_5, x_6, x_7)$, $(x_2; x_1, x_5, x_6, x_7)$, $(x_3; x_1, x_2, x_6, x_7)$, $(x_4; x_2, x_3, x_6, x_7)$, $(x_5; x_3, x_4, x_6, x_7)$.
2. For $p = 1$, $q = 4$, the required paths and stars are $x_6x_1x_7x_5x_2$, $(x_2; x_1, x_4, x_6, x_7)$, $(x_3; x_1, x_2, x_6, x_7)$, $(x_4; x_1, x_3, x_6, x_7)$, $(x_5; x_3, x_4, x_6, x_1)$.
3. For $p = 2$, $q = 3$, the required paths and stars are $x_1x_4x_7x_5x_2$, $x_3x_4x_6x_1x_7$, $(x_2; x_1, x_4, x_6, x_7)$, $(x_3; x_1, x_2, x_6, x_7)$, $(x_5; x_3, x_4, x_6, x_1)$.

4. For $p=3$, $q=2$, the required paths and stars are $x_6x_1x_7x_5x_2$, $x_3x_5x_4x_2x_6$, $x_6x_5x_1x_2x_7$, $(x_3; x_1, x_2, x_6, x_7)$, $(x_4; x_1, x_3, x_6, x_7)$.

5. For $p=4$, $q=1$, the required paths and stars are $x_1x_4x_7x_5x_2$, $x_3x_4x_6x_1x_7$, $x_3x_5x_4x_2x_6$, $x_6x_5x_1x_2x_7$, $(x_3; x_1, x_2, x_6, x_7)$.

6. For $p=5$, $q=0$, the required paths are $x_2x_3x_1x_4x_7$, $x_6x_3x_7x_5x_2$, $x_3x_4x_6x_1x_7$, $x_3x_5x_4x_2x_6$, $x_6x_5x_1x_2x_7$. □

Lemma 2.15. *There exists a $(4;p,q)$-decomposition of $K_n \backslash E(K_i)$, when $n \equiv i \pmod 8$, $i \in \{3,5,7\}$.*

Proof: Let $n \equiv i \pmod 8$ and $n = 8k+i$, where k is a positive integer and $i \in \{3,5,7\}$. The graph $K_n \backslash E(K_i)$ can be viewed as edge-disjoint union of K_{8k} and $K_{8k,i}$. By Theorems 1.2 to 1.4, both the graphs K_{8k} and $K_{8k,i}$ have a $(4;p,q)$-decomposition. Hence by Remark 1.1, the graph $K_n \backslash E(K_i)$ has a $(4;p,q)$-decomposition. □

Theorem 2.1. *$K_m \Box K_n$ has a $(4;p,q)$-decomposition if and only if $mn(m+n-2) \equiv 0 \pmod 8$.*

Proof: *Necessity.* Since $K_m \Box K_n$ is $(m+n-2)$-regular and has mn vertices, $K_m \Box K_n$ has $mn(m+n-2)/2$ edges. Now, assume that $K_m \Box K_n$ has a $(4;p,q)$-decomposition. Then the number of edges in the graph must be divisible by 4, i.e., $8|mn(m+n-2)$ and hence $mn(m+n-2) \equiv 0 \pmod 8$, this condition is satisfied precisely when one of the following holds: (i) $m,n \equiv 0 \pmod 2$, (ii) $m,n \equiv 1 \pmod 8$, (iii) $m,n \equiv 5 \pmod 8$, (iv) $m \equiv 3 \pmod 8$, $n \equiv 7 \pmod 8$, (v) $m \equiv 0 \pmod 8$, $n \equiv 1 \pmod 2$.

Sufficiency. We construct the required decomposition in five cases.

Case 1. Let $m,n \equiv 0 \pmod 2$. We construct the required decomposition in three subcases separately.

(a) Let $m,n \equiv 0 \pmod 4$. Let $m=4k$ and $n=4l$, $k,l \in \mathbb{Z}^+$. We can write $K_m \Box K_n = kl(K_4 \Box K_4) \oplus 2kl(l+k-2)K_{4,4}$. By Lemma 2.7 and Theorem 1.1, $K_4 \Box K_4$ and $K_{4,4}$ each have a $(4;p,q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(4;p,q)$-decomposition.

(b) Let $m \equiv 0 \pmod 4$, $n \equiv 2 \pmod 4$. When $n=2$, by Lemmas 2.1, 2.3 and 2.5, $K_m \Box K_2$ has a $(4;p,q)$-decomposition for $m=4,8,12$. If $m>12$, and $m \equiv 0 \pmod 8$, let $m=8k$, $k>1$, be an integer. Then $K_m \Box K_2 = k(K_8 \Box K_2) \oplus k(k-1)K_{8,8}$. By Lemma 2.3 and Theorem 1.2, $K_8 \Box K_2$ and $K_{8,8}$ each have a $(4;p,q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(4;p,q)$-decomposition. If $m \equiv 4 \pmod 8$, let $m=8k+12$, $k \in \mathbb{Z}^+$. Then $K_m \Box K_2 = (K_{8k} \Box K_2) \oplus (K_{12} \Box K_2) \oplus 2K_{8k,12}$. By Lemma 2.5 and Theorem 1.2, $K_{12} \Box K_2$ and $K_{8k,12}$ each have a $(4;p,q)$-decomposition. Also, we proved that $K_{8k} \Box K_2$

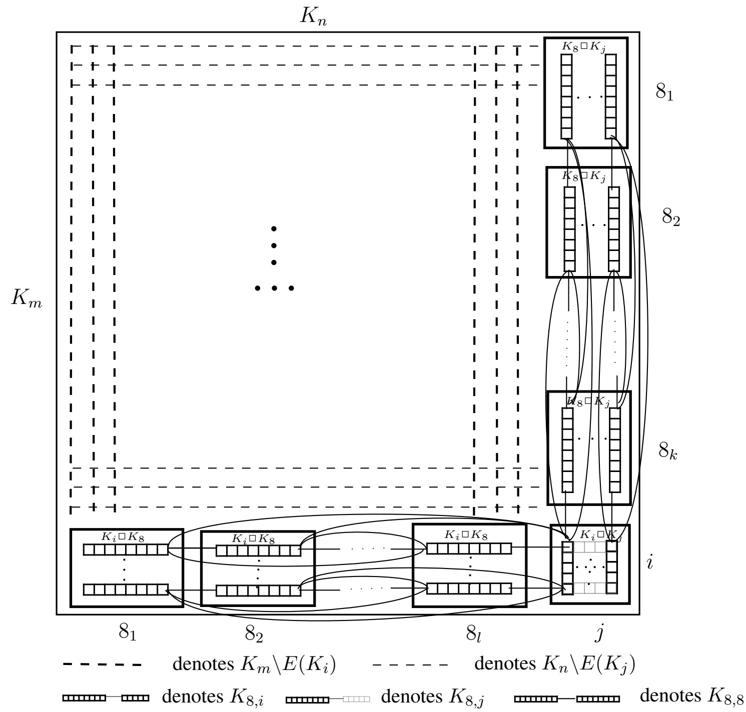

FIGURE 1. $K_m\Box K_n$.

has a $(4;p,q)$-decomposition in this case. Hence by Remark 1.1, $K_m\Box K_n$ has a $(4;p,q)$-decomposition.

When $n=6$, let $m=4k$, $k\in\mathbb{Z}^+$. Then $K_m\Box K_n = k(K_4\Box K_6)\oplus 3k(k-1)K_{4,4}$. By Lemma 2.8 and Theorem 1.1, $K_4\Box K_6$ and $K_{4,4}$ each have a $(4;p,q)$-decomposition. Hence by Remark 1.1, $K_m\Box K_n$ has a $(4;p,q)$-decomposition.

When $n>6$, let $m=4k$ and $n=4l+2$, $k,l\in\mathbb{Z}^+$. Then $K_m\Box K_n = (K_{4k}\Box K_{4(l-1)})\oplus(K_{4k}\Box K_6)\oplus 4kK_{4(l-1),6}$. By Case 1 (a), $K_{4k}\Box K_{4(l-1)}$ has a $(4;p,q)$-decomposition. Also, we proved that $K_{4k}\Box K_6$ has a $(4;p,q)$-decomposition in this case. Hence by Remark 1.1, $K_m\Box K_n$ has a $(4;p,q)$-decomposition.

(c) Let $m,n\equiv 2 \pmod 4$. When $n=2$, clearly there is no $(4;p,q)$-decomposition for $K_2\Box K_2$ and hence $m>2$. By Lemmas 2.2, 2.4 and 2.6, $K_6\Box K_2$, $K_{10}\Box K_2$ and $K_{14}\Box K_2$ each have a $(4;p,q)$-decomposition.

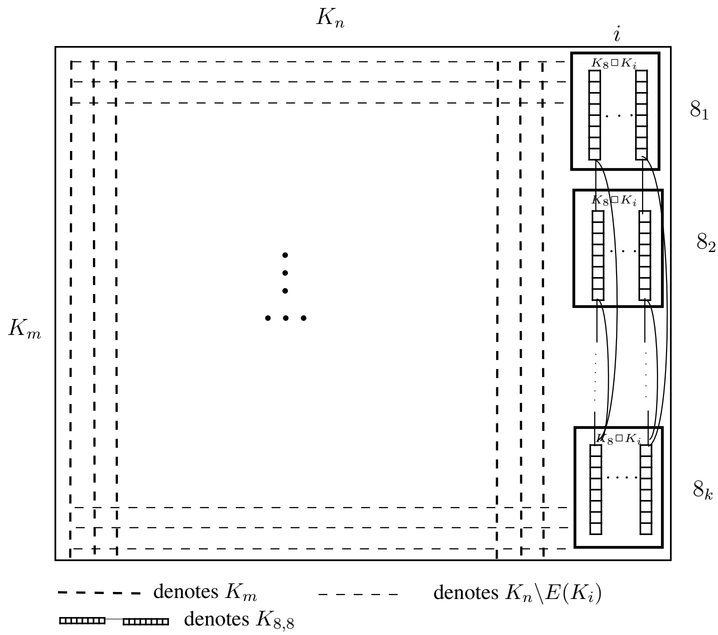

FIGURE 2. $K_m \square K_n$.

For $m > 14$, let $m = 4k+2$, $k > 3$, be an integer. Then $K_m \square K_2 = (K_{4(k-2)} \square K_2) \oplus (K_{10} \square K_2) \oplus K_{4(k-2),10}$. By Lemma 2.4, Case 1 (b) and Theorem 1.2, $K_{10} \square K_2$, $K_{4(k-2)} \square K_2$ and $K_{4(k-2),10}$ each have a $(4; p, q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

When $n = 6$, since $K_2 \square K_6$ ($\cong K_6 \square K_2$) and $K_6 \square K_6$ (by Lemmas 2.2, 2.9) each have a $(4; p, q)$-decomposition, $m > 6$. Let $m = 4k+2$, $k > 1$, be an integer, then $K_m \square K_6 = (K_{4(k-1)} \square K_6) \oplus (K_6 \square K_6) \oplus 6K_{4(k-1),6}$. By Lemma 2.9, Case 1 (b) and Theorems 1.1 and 1.2, $K_6 \square K_6$, $K_{4(k-1)} \square K_6$ and $K_{4(k-1),6}$ each have a $(4; p, q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

When $m, n > 6$, let $m = 4k+2$ and $n = 4l+2$, $k, l > 1$ are integers. We can write $K_m \square K_n = (K_{4k+2} \square K_{4(l-1)}) \oplus (K_{4k+2} \square K_6) \oplus (4k+2)K_{4(l-1),6} = (K_{4k+2} \square K_{4(l-1)}) \oplus (k-1)(K_4 \square K_6) \oplus (K_6 \square K_6) \oplus 3(k-1)(k-2)K_{4,4} \oplus 6(k-1)K_{4,6} \oplus (4k+2)K_{4(l-1),6}$. By Lemmas 2.8 and 2.9 and Theorems 1.1 and 1.2, $K_4 \square K_6$, $K_6 \square K_6$, $K_{4,6}$, $K_{4(l-1),6}$ and $K_{4,4}$ each have a $(4; p, q)$-decomposition. Also by Case 1 (b), $K_{4k+2} \square K_{4(l-1)}$ has a $(4; p, q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

Case 2. Let $m, n \equiv 1 \pmod 8$. We can write $K_m \square K_n = nK_m \oplus mK_n$. By Theorem 1.4, K_m and K_n each have a $(4; p, q)$-decomposition whenever $m, n \geq 16$. Hence by Example 1.2 and Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

Case 3. Let $m, n \equiv 5 \pmod 8$. Let $m = 8k+5$ and $n = 8l+5$, $k, l \geq 0$, be integers. We can write $K_m \square K_n = nK_m \oplus mK_n = 8l(K_m \backslash E(K_5)) \oplus 8k(K_n \backslash E(K_5)) \oplus k(K_8 \square K_5) \oplus l(K_5 \square K_8) \oplus \frac{5}{2}(k(k-1) + l(l-1))K_{8,8} \oplus (K_5 \square K_5) \oplus 5(k+l)K_{8,5}$ (see Figure 1 with $i = j = 5$). By Theorem 1.2 and Lemmas 2.10, 2.13 and 2.15, $K_{8,8}$, $K_{8,5}$, $K_m \backslash E(K_5)$, $K_n \backslash E(K_5)$, $K_5 \square K_8$ and $K_5 \square K_5$ each have a $(4; p, q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

Case 4. Let $m \equiv 3 \pmod 8$, $n \equiv 7 \pmod 8$. Let $m = 8k+3$, $n = 8l+7$, $k, l \geq 0$, are integers. We can write $K_m \square K_n = nK_m \oplus mK_n = 8k(K_n \backslash E(K_7)) \oplus 8l(K_m \backslash E(K_3)) \oplus l(K_3 \square K_8) \oplus k(K_7 \square K_8) \oplus \big((3l(l-1) + 7k(k-1))/2\big)K_{8,8} \oplus (K_3 \square K_7) \oplus 7kK_{8,3} \oplus 3lK_{8,7}$ (refer Figure 1 with $i = 3$, $j = 7$). By Lemmas 2.11, 2.12 and 2.14 and Theorems 1.2 and 1.3, $K_3 \square K_8$, $K_7 \square K_8$, $K_3 \square K_7$, $K_{8,3}$, $K_{8,7}$ and $K_{8,8}$ each have a $(4; p, q)$-decomposition. Also by Lemma 2.15, $K_m \backslash E(K_3)$ and $K_n \backslash E(K_7)$ each have a $(4; p, q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

Case 5. Let $m \equiv 0 \pmod 8$, $n \equiv 1 \pmod 2$. If $n \equiv 1 \pmod 8$, then K_m and K_n each have a $(4; p, q)$-decomposition, by Theorem 1.4 and Examples 1.1 and 1.2. Hence by Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition.

When $n \equiv i \pmod 8$ with $i = 3, 5, 7$, let $m = 8k$, $k \in \mathbb{Z}^+$. We can write $K_m \square K_n = nK_m \oplus mK_n = (n-i)K_m \oplus k(K_8 \square K_i) \oplus i(k(k-1)/2)K_{8,8} \oplus m(K_n \backslash E(K_i))$, $i \in \{3, 5, 7\}$ (see Figure 2). By Lemmas 2.12 to 2.15, Theorem 1.2 and Remark 1.1, $K_m \square K_n$ has a $(4; p, q)$-decomposition. $\square$

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