Comment.Math.Univ.Carolin. 62,3 (2021) 291–296

On the distribution of the roots of polynomial $z^k - z^{k-1} - \cdots - z - 1$

Carlos A. Gómez, Florian Luca

Abstract. We consider the polynomial $f_k(z) = z^k - z^{k-1} - \cdots - z - 1$ for $k \geq 2$ which arises as the characteristic polynomial of the k-generalized Fibonacci sequence. In this short paper, we give estimates for the absolute values of the roots of $f_k(z)$ which lie inside the unit disk.

Keywords: polynomial root distribution

Classification: 12F10, 11B39

1. Introduction

For an integer $k \geq 2$, the polynomial $f_k(z) = z^k - z^{k-1} - \cdots - z - 1$ arises as the characteristic polynomial of the linear recurrence sequence $(u_n)_{n\geq 0}$ of recurrence $u_{n+k} = u_{n+k-1} + \cdots + u_n$. The classic study of the linear recurrence sequences, see [2], is based on knowledge of the roots of their characteristic polynomial. While studying the roots of $f_k(z)$, it is common to work with the polynomial:

$$g_k(z) = (z-1)f_k(z) = z^{k+1} - 2z^k + 1. \tag{1}$$

Except for the extra root at $z = 1$, $g_k(z)$ has the same roots as $f_k(x)$.

By Descartes' rule of signs, the polynomial $f_k(z)$ has exactly one positive real root, say $z = \alpha_1$. Since $f_k(1) = 1 - k$ and $f_k(2) = 1$, it follows that $\alpha_1 \in (1, 2)$. In fact, it is known that $2(1 - 2^{-k}) < \alpha_1 < 2$, see [4, Lemma 2.3] or [9, Lemma 3.6]. Moreover, given that $\alpha_1^{k+1} - 2\alpha_1^k + 1 = 0$, we obtain $\alpha_1 = 2 - \alpha_1^{-k} < 2 - 2^{-k}$. So, $2(1 - 2^{-k}) < \alpha_1 < 2(1 - 2^{-(k+1)})$ for all $k \geq 2$. Thus α_1 approaches 2 as k tends to infinity. E. P. Miles in [6] showed that the roots of $f_k(z)$ are distinct and the remaining $k - 1$ roots of $f_k(z)$ different from α_1 lie inside the unit disk. He showed this by reducing the equation $f_k(z) = 0$ to a form where Rouché's theorem could be applied. This fact was reproved by M. D. Miller in [7] by an elementary argument. In particular, α_1 is a Pisot number and $f_k(z)$

DOI 10.14712/1213-7243.2021.025

C. A. Gómez was supported in part by Project 71280 (Universidad del Valle). F. Luca was supported in part by the Number Theory Focus Area Grant of CoEMaSS at Wits (South Africa).

is an irreducible polynomial over $\mathbb{Q}[z]$. Properties of the roots of the similar but more general polynomial $f_{k,c}(z) = z^k - z^{k-1} - \cdots - z - c$ for positive values of the parameter c have been proved in [5].

Recently, we gave the following estimate on the ratio of roots of $f_k(z)$, see Lemma 2.2 in [3].

Theorem 1. *If α and β are roots of $f_k(z)$ with $|\alpha| > |\beta|$, then*

$$\frac{|\alpha|}{|\beta|} > 1 + 8^{-k^4}.$$

This result was used in [3] as one of the main ingredients in the study of the zero-multiplicity of a particular linear recurrence sequence with characteristic polynomial $f_k(z)$.

Here, we give upper and lower bounds on the absolute values of the roots of $f_k(z)$ which lie inside the unit disk.

Theorem 2. *Let z_0 be a root of $f_k(z)$ with $\varrho = |z_0| < 1$. Then*

$$1 - \frac{\log 3}{k} < \varrho < 1 - \frac{1}{2^8 k^3}.$$

2. Proof of Theorem 2

We put $z_0 = \varrho e^{i\theta}$ with $0 < \varrho < 1$ and $\theta \in (0, 2\pi)$. By (1), we obtain that $1 \leq \varrho^k(2 + \varrho) < 3\varrho^k$, or

$$\varrho > 3^{-1/k} = e^{-\log 3/k} \geq 1 - \frac{\log 3}{k}.$$

Now, again by (1), we get

$$z_0^k(2 - z_0) = 1 \qquad \text{and} \qquad \overline{z_0}^k(2 - \overline{z_0}) = 1.$$

After multiplying the above identities, we obtain $\varrho^{2k}(2 - z_0)(2 - \overline{z_0}) = 1$. Then

$$\varrho^{-2k} = (2 - z_0)(2 - \overline{z_0}) = 4 - 4\mathrm{Re}(z_0) + \varrho^2,$$

which leads to

$$\varrho^2(\varrho^{-2k-2} - 1) = 4(1 - \mathrm{Re}(z_0)). \tag{2}$$

We assume that

$$1 - \varrho < \frac{c_0}{k^3} \qquad \text{with} \quad c_0 = 2^{-8}. \tag{3}$$

Then, $1 < \varrho^{-1} \leq (1 - (c_0/k^3))^{-1} < 1 + (2c_0/k^3)$ for all $k \geq 2$. Hence,

$$\varrho^{-(2k+2)} < \left(1 + \frac{2c_0}{k^3}\right)^{2k+2} < \exp\left(\frac{2c_0(2k+2)}{k^3}\right) < 1 + \frac{4c_0(2k+2)}{k^3}.$$

In the above inequalities, we have used that $e^x \geq 1+x$ for all real x and $e^x \leq 1+2x$ for all real x such that $|x| < 1/2$, as well as the fact that $2c_0(2k+2)/k^3 < 1/2$ for all $k \geq 2$. So, we obtain $\varrho^2\left(\varrho^{-2k-2} - 1\right) < 8\varrho^2 c_0(k+1)/k^3$ and by (2), we get

$$1 - \mathrm{Re}(z_0) < \frac{2\varrho^2 c_0(k+1)}{k^3}. \tag{4}$$

On the other hand,

$$\mathrm{Im}^2(z_0) = \varrho^2 - \mathrm{Re}^2(z_0) < \varrho^2 - \left(1 - \frac{2\varrho^2 c_0(k+1)}{k^3}\right)^2 < \frac{4\varrho^2 c_0(k+1)}{k^3}. \tag{5}$$

We now write $z_0 = 1 + z_1$. Hence, $z_1 = (\mathrm{Re}(z_0) - 1) + \mathrm{i}\mathrm{Im}(z_0)$. Thus, by (4) and (5), we get

$$\begin{aligned}
|z_1| &= ((\mathrm{Re}(z_0) - 1)^2 + \mathrm{Im}^2(z_0))^{1/2} \\
&< \sqrt{2}\left(\max\left\{\left(\frac{2\varrho^2 c_0(k+1)}{k^3}\right)^2, \frac{4\varrho^2 c_0(k+1)}{k^3}\right\}\right)^{1/2} \\
&< 2\sqrt{2}\varrho c_0^{1/2}\left(\frac{k+1}{k^3}\right)^{1/2} \\
&< \frac{4\varrho c_0^{1/2}}{k}.
\end{aligned}$$

Hence, we have proved that

$$z_0 = 1 + z_1, \qquad \text{where } |z_1| < \frac{4\varrho c_0^{1/2}}{k}. \tag{6}$$

We now analyze the polynomial function of complex value $(1+z)^\lambda$ for $\lambda = 1, 2, \ldots, k$ in $z = z_1$, according to the binomial theorem. We put $\eta := ((1 + z_1)^\lambda - 1 - \lambda z_1)/z_1^2$, so

$$\begin{aligned}
|\eta| &= \left|\sum_{j=2}^{\lambda} \binom{\lambda}{j} z_1^{j-2}\right| \leq \sum_{j=2}^{\lambda} \binom{\lambda}{j} |z_1|^{j-2} \\
&< \lambda^2 \sum_{j=2}^{\lambda} \binom{\lambda}{j-2} |z_1|^{j-2} = \lambda^2(1 + |z_1|)^{\lambda-2} \\
&\leq \lambda^2(1 + 2(\lambda - 2)|z_1|).
\end{aligned}$$

Here we have used that $\binom{\lambda}{j} < \lambda^2\binom{\lambda}{j-2}$ for $j = 2, \dots, k$, in addition to

$$(1 + |z_1|)^{\lambda-2} < \mathrm{e}^{(\lambda-2)|z_1|} < 1 + 2(\lambda - 2)|z_1|,$$

which holds because $(\lambda - 2)|z_1| < k|z_1| < 1/2$ by (6). Since $\varrho < 1$, it then follows that

$$|\eta| < \lambda^2(1 + 8c_0^{1/2}) \qquad \text{for} \quad 1 \le \lambda \le k.$$

Hence, for each $\lambda = 1, 2, \dots, k$

$$z_0^{\lambda} = (1 + z_1)^{\lambda} = 1 + \lambda z_1 + \delta_{\lambda}, \qquad \text{where } |\delta_{\lambda}| < 16c_0(1 + 8c_0^{1/2}). \tag{7}$$

Thus, we obtain

$$\begin{aligned} 0 = f_k(z_0) &= z_0^k - z_0^{k-1} - \dots - z_0^2 - z_0 - 1 \\ &= (1 + kz_1 + \delta_k) - (1 + (k-1)z_1 + \delta_{k-1}) - \dots - (1 + z_1 + \delta_1) - 1 \\ &= 1 - k - \frac{k(k-3)}{2} z_1 + \delta_k - \sum_{\lambda=1}^{k-1} \delta_{\lambda}. \end{aligned}$$

Now, by (2) we have that

$$k - 1 < \frac{k(k-3)}{2}|z_1| + \sum_{\lambda=1}^{k} |\delta_{\lambda}| \le (2c_0^{1/2} + 16c_0(1 + 8c_0^{1/2}))k < \frac{k}{3},$$

which is not possible. Thus, our assumption (3) is false. Hence, $\varrho \le 1 - c_0/k^3$. The fact that the inequality is strict follows because ϱ is an algebraic integer, while $1 - 1/(2^8k^3)$ is not. This completes the proof of Theorem 2. $\square$

3. An open problem

In 1950, P. Erdős and P. Turán in [1] investigated the angular distribution of zeros of complex polynomials $f(z) \in \mathbb{C}[z]$. Let I be an arc on the unit circle and let $N(I, f(z))$ be the number of zeros α's with $\alpha/|\alpha| = \mathrm{e}^{\mathrm{i}\theta}$ lying on this arc. A natural way to estimate the equidistribution of the roots of the polynomial $f(z)$ is the discrepancy, defined as

$$D(f(z)) = \max_I \left| N(I, f(z)) - \frac{|I|}{2\pi} N \right|.$$

This measures the maximum difference between the actual count of the number of arguments of roots found in a given arc, and the number that would be expected if all the angles were uniformly distributed.

The version of the Erdős–Turán inequality was recently given by K. Soundararajan in [8]. Assume $f(z)$ is monic and put

$$h(f(z)) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta, \qquad \text{where } \log^+(x) = \max\{\log x, 0\}.$$

Theorem 3. *For any monic polynomial $f(z)$ of degree $k > 1$,*

$$D(f(z)) \le \frac{8}{\pi}\sqrt{kh(f(z))}.$$

Remark. For the case of $f(z) = g_k(z) = z^{k+1} - 2z^k + 1$ it follows that $|f(e^{i\theta})| \le 4$, so $h(g_k(z)) \le \log 4$. In particular, if we take the largest subinterval I of $[0, 2\pi)$ free of any $\theta = \arg(\alpha/|\alpha|)$ for root α of $g_k(z)$ then $|I| < 16(\log 4)^{1/2}k^{-1/2}$. Numerically, since $16(\log 4)^{1/2} = 18.838\ldots$, it follows that by taking short intervals I_h of radius $18.9k^{1/2}$ around each angle $2\pi h/k$, $h = 1, 2, \ldots, k-1$, one will always find at least one $\theta \in I_h$.

Assume that

$$\alpha_1, \quad \widetilde{\alpha}_1, \quad \alpha_2, \quad \overline{\alpha_2}, \quad \ldots \quad \alpha_l, \quad \overline{\alpha_l}, \qquad \text{with } l = \left\lfloor \frac{k-1}{2} \right\rfloor,$$

are the roots of $f_k(z)$, where $\widetilde{\alpha}_1 \in (-1, 0)$ appears only when k is even. For $1 \le j \le l$ we write $\alpha_j = \varrho_j e^{i\theta_j}$, with $\theta_1 = 0$ and $\theta_j \in (0, 2\pi)$. As a consequence of the above remark for each $h \in \{0, \ldots, k-1\}$ there is $j \in \{1, \ldots, k\}$ such that

$$\left|\theta_j - \frac{2\pi h}{k}\right| < \frac{8(\log 4)^{1/2}}{k^{1/2}}.$$

Open problem. Prove that inequality

$$\left|\theta_j - \frac{2j\pi}{k}\right| < \frac{1}{k}$$

holds for all $k \ge 3$.

This problem is motivated by the computational verification made for each $k \in [3, 1000]$.

Acknowledgement. We thank the referee for a careful reading of the manuscript and for several suggestions which improved the presentation of our paper. F. Luca worked on this paper while visiting the Max Planck Institute for Mathematics in Bonn, Germany, from September 2019 to February 2020. He thanks this Institute for their hospitality and support.

References

[1] Erdős P., Turán P., *On the distribution of roots of polynomials*, Ann. of Math. (2) **51** (1950), 105–119.

[2] Everest G., van der Poorten A., Shparlinski I., Ward T., *Recurrence Sequences*, Mathematical Surveys and Monographs, 104, American Mathematical Society, Providence, 2003.

[3] Gómez C. A., Luca F., *On the zero-multiplicity unitary of a fifth-order linear recurrences*, International Journal of Number Theory **15** (2018), no. 3, 585–595.

[4] Hua L. K., Wang Y., *Applications of Number Theory to Numerical Analysis*, Springer, Berlin, 1981.

[5] Marques D., Trojovský P., *On characteristic polynomial of higher order generalized Jacobsthal numbers*, Adv. Difference Equ. **2019** (2019), Paper No. 392, 9 pages.

[6] Miles E. P., Jr., *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly **67** (1960), 745–752.

[7] Miller M. D., *Mathematical notes: On generalized Fibonacci numbers*, Amer. Math. Monthly **78** (1971), no. 10, 1108–1109.

[8] Soundararajan K., *Equidistribution of zeros of polynomials*, Amer. Math. Monthly **126** (2019), no. 3, 226–236.

[9] Wolfram D. A., *Solving generalized Fibonacci recurrences*, Fibonacci Quart. **36** (1998), no. 2, 129–145.

C. A. Gómez:
Departamento de Matemáticas, Universidad del Valle, Calle 13 No. 100-00, Cali, Colombia

E-mail: carlos.a.gomez@correounivalle.edu.co

F. Luca:
School of Mathematics, University of the Witwatersrand, 1 Jan Smuts Ave, Braamfontein, Johannesburg 2000, South Africa;
and:
Research Group in Algebraic Structures and Related Topics,
King Abdulaziz University, P. O. Box 80200, Jeddah 21598, Saudi Arabia;
and:
Centro de Ciencias Matemáticas, UNAM,
Antigua Carretera a Pátzcuaro # 8701, Residencial San José de la Huerta, 58089, Morelia, Mexico

E-mail: Florian.Luca@wits.ac.za

(*Received* January 23, 2020*, revised* June 25, 2020)