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Bihyperbolic numbers of the Fibonacci type and their idempotent representation

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Abstract. In this paper we introduce bihyperbolic numbers of the Fibonacci type. We present some of their properties using matrix generators and idempotent representations.

Keywords: hyperbolic number; bihyperbolic number; Fibonacci number; Pell number; Jacobsthal number

Classification: 11B37, 11B39

1. Introduction

Let $\mathbb{H}$ be the set of hyperbolic numbers $x+y\mathbf{h}$, where $x, y \in \mathbb{R}$, with the unipotent element $\mathbf{h}$ such that $\mathbf{h} \neq \pm 1$ and $\mathbf{h}^2 = 1$. Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1,\ j_2,\ j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1, \quad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1. \tag{1}$$

If $\zeta = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3$ and $\omega = y_0 + j_1 y_1 + j_2 y_2 + j_3 y_3$ are any two bihyperbolic numbers then the equality, the addition, the subtraction and the multiplication by scalar are defined in the natural way.
Equality: $\zeta = \omega$ only if $x_0 = y_0$, $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$;
addition: $\zeta + \omega = (x_0 + y_0) + j_1(x_1 + y_1) + j_2(x_2 + y_2) + j_3(x_3 + y_3)$;
subtraction: $\zeta - \omega = (x_0 - y_0) + j_1(x_1 - y_1) + j_2(x_2 - y_2) + j_3(x_3 - y_3)$;
multiplication by scalar $s \in \mathbb{R}$: $s\zeta = sx_0 + j_1 sx_1 + j_2 sx_2 + j_3 sx_3$.
Multiplication of bihyperbolic numbers can be made analogously as multiplications of algebraic expressions using the rules (1).

The addition and multiplication on $\mathbb{H}_2$ are commutative and associative. Moreover, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring.

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Properties of hyperbolic numbers we can find among others in [10], [11]. For the algebraic properties of bihyperbolic numbers, see [2], [9].

Let $n \geq 0$ be an integer. The nth Fibonacci number F_n is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There are many numbers given by the second order linear recurrence relations. The nth Pell number P_n is defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with the initial conditions $P_0 = 0$, $P_1 = 1$. The nth Jacobsthal number J_n is defined by $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ and $J_0 = 0$, $J_1 = 1$.

The Pell numbers and Jacobsthal numbers are well-known numbers in the number theory. They belong to the wide class of numbers of the Fibonacci type. These numbers have applications also in the theory of hypercomplex numbers, see [15], in particular complex numbers, see [6], quaternions, see [4], [5], [6], [7], [12], [13], bicomplex numbers, see [1], [8], hybrid numbers, see [14], [16], [17] etc.

In this paper we introduce bihyperbolic numbers of the Fibonacci type.

Let $n \geq 0$ be an integer. The nth bihyperbolic Fibonacci number BhF_n, the nth bihyperbolic Pell number BhP_n and the nth bihyperbolic Jacobsthal number BhJ_n are defined as

$$\begin{aligned} BhF_n &= F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3}, \\ BhP_n &= P_n + j_1 P_{n+1} + j_2 P_{n+2} + j_3 P_{n+3}, \\ BhJ_n &= J_n + j_1 J_{n+1} + j_2 J_{n+2} + j_3 J_{n+3}, \end{aligned}$$

respectively.

Using well-known identities for numbers of the Fibonacci type, we can get identities for bihyperbolic numbers of the Fibonacci type. In this paper we will focus on using matrix representations (matrix generators) and idempotent representation of bihyperbolic numbers to prove some properties of bihyperbolic numbers of the Fibonacci type. Some combinatorial properties of bihyperbolic numbers of the Fibonacci type we can find in [3].

2. Matrix generators for bihyperbolic numbers of the Fibonacci type

Matrix methods are very important for sequences defined recursively. Matrix generators are useful tools for obtaining identities and algebraic representation. In this paper we introduce the matrix generator for bihyperbolic Fibonacci numbers, bihyperbolic Pell numbers and bihyperbolic Jacobsthal numbers.

Let $BF(n) = \begin{bmatrix} BhF_{n+1} & BhF_n \\ BhF_n & BhF_{n-1} \end{bmatrix}$ be a matrix with entries being bihyperbolic Fibonacci numbers.

Theorem 1. *Let $n \geq 1$ be an integer. Then*

$$\begin{bmatrix} BhF_{n+1} & BhF_n \\ BhF_n & BhF_{n-1} \end{bmatrix} = \begin{bmatrix} BhF_2 & BhF_1 \\ BhF_1 & BhF_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1}. \tag{2}$$

Proof: If $n = 1$ then by simple calculations the result immediately follows. Assume that the equality is true for all integers $1, 2, \ldots, n$. We shall prove that the equality is true for integer $n+1$. Using the induction hypothesis we have

$$\begin{aligned} &\begin{bmatrix} BhF_2 & BhF_1 \\ BhF_1 & BhF_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} BhF_{n+1} & BhF_n \\ BhF_n & BhF_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} BhF_{n+1} + BhF_n & BhF_{n+1} \\ BhF_n + BhF_{n-1} & BhF_n \end{bmatrix}. \end{aligned}$$

From the definition of bihyperbolic Fibonacci numbers and Fibonacci numbers we have

$$\begin{aligned} &BhF_{n+1} + BhF_n \\ &\quad = F_{n+1} + j_1 F_{n+2} + j_2 F_{n+3} + j_3 F_{n+4} + F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3} \\ &\quad = F_{n+1} + F_n + j_1(F_{n+2} + F_{n+1}) + j_2(F_{n+3} + F_{n+2}) + j_3(F_{n+4} + F_{n+3}) \\ &\quad = F_{n+2} + j_1 F_{n+3} + j_2 F_{n+4} + j_3 F_{n+5} = BhF_{n+2}. \end{aligned}$$

In the same way we can prove that $BhF_n + BhF_{n-1} = BhF_{n+1}$, which ends the proof. □

Matrix generator for bihyperbolic Fibonacci numbers immediately gives the Cassini formula for these numbers.

Corollary 2. *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} BhF_{n+1} \cdot BhF_{n-1} - BhF_n^2 &= \left(BhF_2 \cdot BhF_0 - BhF_1^2\right) \cdot (-1)^{n-1} \\ &= (2j_1 + 3j_3) \cdot (-1)^n. \end{aligned}$$

Proof: Multiplication of bihyperbolic numbers is commutative, so we can use determinant properties to obtain the first equality. Calculating the determinants in (2) and putting $BhF_0 = 0 + j_1 + j_2 + 2j_3$, $BhF_1 = 1 + j_1 + 2j_2 + 3j_3$, $BhF_2 = 1 + 2j_1 + 3j_2 + 5j_3$, we have

$$BhF_2 \cdot BhF_0 - BhF_1^2 = -2j_1 - 3j_3$$

which ends the proof. □

In the same way we can obtain the matrix generator and the Cassini formula for bihyperbolic Pell numbers and bihyperbolic Jacobsthal numbers.

Let $BP(n) = \begin{bmatrix} BhP_{n+1} & BhP_n \\ BhP_n & BhP_{n-1} \end{bmatrix}$ be a matrix with entries being bihyperbolic Pell numbers.

Theorem 3. *Let $n \geq 1$ be an integer. Then*

$$\begin{bmatrix} BhP_{n+1} & BhP_n \\ BhP_n & BhP_{n-1} \end{bmatrix} = \begin{bmatrix} BhP_2 & BhP_1 \\ BhP_1 & BhP_0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

Corollary 4. *Let $n \geq 1$ be an integer. Then*

$$BhP_{n+1} \cdot BhP_{n-1} - BhP_n^2 = (-4j_1 - 12j_3) \cdot (-1)^{n-1}.$$

Let $BJ(n) = \begin{bmatrix} BhJ_{n+1} & BhJ_n \\ BhJ_n & BhJ_{n-1} \end{bmatrix}$ be a matrix with entries being bihyperbolic Jacobsthal numbers.

Theorem 5. *Let $n \geq 1$ be an integer. Then*

$$\begin{bmatrix} BhJ_{n+1} & BhJ_n \\ BhJ_n & BhJ_{n-1} \end{bmatrix} = \begin{bmatrix} BhJ_2 & BhJ_1 \\ BhJ_1 & BhJ_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

Corollary 6. *Let $n \geq 1$ be an integer. Then*

$$BhJ_{n+1} \cdot BhJ_{n-1} - BhJ_n^2 = (5 - 5j_1 + 5j_2 - 5j_3) \cdot (-2)^{n-1}.$$

3. Idempotent representation of bihyperbolic numbers of the Fibonacci type

Let consider bihyperbolic numbers

$$i_1 = \frac{1 + j_1 + j_2 + j_3}{4}, \qquad i_2 = \frac{1 - j_1 + j_2 - j_3}{4},$$
$$i_3 = \frac{1 + j_1 - j_2 - j_3}{4}, \qquad i_4 = \frac{1 - j_1 - j_2 + j_3}{4}.$$

Then i_1, i_2, i_3 and i_4 satisfy the following identities

$$\begin{aligned} i_1 + i_2 + i_3 + i_4 &= 1, \\ i_s^2 &= i_s \qquad \text{for } \ s = 1, 2, 3, 4, \\ i_s i_k &= 0 \qquad \text{for } \ s, k = 1, 2, 3, 4, \ \text{ and } \ s \neq k. \end{aligned}$$

For any bihyperbolic number $\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3$ putting

$$z_1 = x_0 + x_1 + x_2 + x_3, \qquad z_2 = x_0 - x_1 + x_2 - x_3,$$
$$z_3 = x_0 + x_1 - x_2 - x_3, \qquad z_4 = x_0 - x_1 - x_2 + x_3$$

we have

$$\zeta = z_1i_1 + z_2i_2 + z_3i_3 + z_4i_4. \tag{3}$$

The formula (3) is called the idempotent representation of the bihyperbolic number ζ. The addition and multiplication of bihyperbolic numbers can be realized "term-by-term" in their idempotent representations. More precisely, if $\zeta = z_1i_1 + z_2i_2 + z_3i_3 + z_4i_4$ and $\omega = w_1i_1 + w_2i_2 + w_3i_3 + w_4i_4$ are two bihyperbolic numbers, then

$$\begin{aligned} \zeta + \omega &= (z_1 + w_1)i_1 + (z_2 + w_2)i_2 + (z_3 + w_3)i_3 + (z_4 + w_4)i_4, \\ \zeta \cdot \omega &= (z_1 \cdot w_1)i_1 + (z_2 \cdot w_2)i_2 + (z_3 \cdot w_3)i_3 + (z_4 \cdot w_4)i_4. \end{aligned} \tag{4}$$

We will write bihyperbolic numbers of the Fibonacci type using their idempotent representations. For nth bihyperbolic Fibonacci number BhF_n we have

$$\begin{aligned} BhF_n &= F_n + j_1F_{n+1} + j_2F_{n+2} + j_3F_{n+3} \\ &= (F_n + F_{n+1} + F_{n+2} + F_{n+3})i_1 + (F_n - F_{n+1} + F_{n+2} - F_{n+3})i_2 \\ &\quad + (F_n + F_{n+1} - F_{n+2} - F_{n+3})i_3 + (F_n - F_{n+1} - F_{n+2} + F_{n+3})i_4. \end{aligned}$$

Using the recurrence relations for Fibonacci numbers, $F_{n+2} = F_{n+1} + F_n$ and $F_{n+3} = F_{n+2} + F_{n+1}$, we obtain

$$\begin{aligned} F_n + F_{n+1} + F_{n+2} + F_{n+3} &= 3F_n + 4F_{n+1}, \\ F_n - F_{n+1} + F_{n+2} - F_{n+3} &= F_n - 2F_{n+1}, \\ F_n + F_{n+1} - F_{n+2} - F_{n+3} &= -F_n - 2F_{n+1}, \\ F_n - F_{n+1} - F_{n+2} + F_{n+3} &= F_n. \end{aligned}$$

Thus the idempotent representation of the nth bihyperbolic Fibonacci number BhF_n has the form

$$BhF_n = (3F_n + 4F_{n+1})i_1 + (F_n - 2F_{n+1})i_2 + (-F_n - 2F_{n+1})i_3 + F_ni_4. \tag{5}$$

In the same way, using recurrence relations for Pell and Jacobsthal numbers we obtain the idempotent representation of the nth bihyperbolic Pell number BhP_n

$$BhP_n = (4P_n + 8P_{n+1})i_1 - 4P_{n+1}i_2 + (-2P_n - 6P_{n+1})i_3 + (2P_n + 2P_{n+1})i_4 \tag{6}$$

and the idempotent representation of the nth bihyperbolic Jacobsthal number BhJ_n

(7) $BhJ_n = (5J_n + 5J_{n+1})i_1 + (J_n - 3J_{n+1})i_2 + (-3J_n - 3J_{n+1})i_3 + (J_n + J_{n+1})i_4.$

Theorem 7 (Cassini identity for bihyperbolic Fibonacci numbers)**.** *Let $n \geq 1$ be an integer. Then*

$$BhF_{n+1} \cdot BhF_{n-1} - BhF_n^2 = (5i_1 - 5i_2 - i_3 + i_4) \cdot (-1)^n. \tag{8}$$

Proof: Using (4) and (5) we have

$$\begin{aligned}
&BhF_{n+1} \cdot BhF_{n-1} - BhF_n^2\\
&\quad= \left[(3F_{n+1} + 4F_{n+2})(3F_{n-1} + 4F_n) - (3F_n + 4F_{n+1})^2\right] i_1\\
&\qquad+ \left[(F_{n+1} - 2F_{n+2})(F_{n-1} - 2F_n) - (F_n - 2F_{n+1})^2\right] i_2\\
&\qquad+ \left[(-F_{n+1} - 2F_{n+2})(-F_{n-1} - 2F_n) - (-F_n - 2F_{n+1})^2\right] i_3\\
&\qquad+ \left[F_{n+1}F_{n-1} - F_n^2\right] i_4\\
&\quad= [9(F_{n+1}F_{n-1} - F_n^2) + 16(F_{n+2}F_n - F_{n+1}^2)\\
&\qquad+ 12(F_{n+2}F_{n-1} - F_nF_{n+1})]i_1 + [(F_{n+1}F_{n-1} - F_n^2) + 4(F_{n+2}F_n - F_{n+1}^2)\\
&\qquad- 2(F_{n+2}F_{n-1} - F_nF_{n+1})]i_2 + [(F_{n+1}F_{n-1} - F_n^2) + 4(F_{n+2}F_n - F_{n+1}^2)\\
&\qquad+ 2(F_{n+2}F_{n-1} - F_nF_{n+1})]i_3 + [F_{n+1}F_{n-1} - F_n^2]i_4.
\end{aligned}$$

Using idempotent representations of bihyperbolic Fibonacci numbers, Cassini identity for Fibonacci numbers $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and Vajda's identity for Fibonacci numbers $F_{n+2}F_{n-1} - F_nF_{n+1} = (-1)^n$ we obtain that

$$\begin{aligned}
BhF_{n+1} \cdot BhF_{n-1} - BhF_n^2 &= [9(-1)^n + 16(-1)^{n+1} + 12(-1)^n]i_1\\
&\quad+ [(-1)^n + 4(-1)^{n+1} - 2(-1)^n]i_2\\
&\quad+ [(-1)^n + 4(-1)^{n+1} + 2(-1)^n]i_3 + (-1)^n i_4
\end{aligned}$$

and (8) is proved. □

In the same way, using Cassini identity for Pell numbers $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$, Vajda's type identity for Pell numbers $P_{n+2}P_{n-1} - P_nP_{n+1} = 2(-1)^n$, Cassini identity for Jacobsthal numbers $J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}$ and Vajda's type identity for Jacobsthal numbers $J_{n+2}J_{n-1} - J_nJ_{n+1} = (-1)^n 2^{n-1}$ we obtain next theorems.

Theorem 8 (Cassini identity for bihyperbolic Pell numbers)**.** *Let $n \geq 1$ be an integer. Then*

$$BhP_{n+1} \cdot BhP_{n-1} - BhP_n^2 = (16i_1 - 16i_2 - 8i_3 + 8i_4) \cdot (-1)^n.$$

Theorem 9 (Cassini identity for bihyperbolic Jacobsthal numbers)**.** *Let $n \geq 1$ be an integer. Then*

$$BhJ_{n+1} \cdot BhJ_{n-1} - BhJ_n^2 = 20i_2 \cdot (-2)^{n-1}.$$

Note that Cassini identities obtained in Theorems 7, 8, 9 are idempotent versions of Cassini identities obtained in Corollaries 2, 4, 6.

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