The (dis)connectedness of products of Hausdorff spaces in the box topology

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Abstract. In this paper the following two propositions are proved:

(a) If X_α, $\alpha \in A$, is an infinite system of connected spaces such that infinitely many of them are nondegenerated completely Hausdorff topological spaces then the box product $\square_{\alpha\in A} X_\alpha$ can be decomposed into continuum many disjoint nonempty open subsets, in particular, it is disconnected.

(b) If X_α, $\alpha \in A$, is an infinite system of Brown Hausdorff topological spaces then the box product $\square_{\alpha\in A} X_\alpha$ is also Brown Hausdorff, and hence, it is connected.

A space is Brown if for every pair of its open nonempty subsets there exists a point common to their closures. There are many examples of countable Brown Hausdorff spaces in literature.

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1. Introduction

All considered spaces are nondegenerated.

It is well known (cf. [15]) that the topological product $\prod_{\alpha\in A} X_\alpha$ of an indexed family $\{X_\alpha\}_{\alpha\in A}$ of connected spaces is connected. Let us endow the product with the box topology and denote the new space as $\square_{\alpha\in A} X_\alpha$.

Theorem 1.1 ([11]). *Let X_α, $\alpha \in A$, be an infinite system of connected regular T_1-spaces. Then the space $\square_{\alpha\in \mathcal{A}} X_\alpha$ is disconnected.*

This result motivated

Question 1.2 ([3]). Let X_α, $\alpha \in A$, be an infinite system of connected spaces. Under what conditions on the system is the space $\square_{\alpha\in A} X_\alpha$ (dis)connected?

In [3] the authors suggested two independent sufficient conditions on topological spaces X_α, $\alpha \in A$, which imply disconnectedness of the product $\square_{\alpha\in A} X_\alpha$.

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In view of Theorem 1.1 these conditions are meaningful for connected spaces with axioms lower than T_3.

Thus a condition called a *decreasing open filtration property,* in short, *DOF-property,* is valid for T_1-spaces with at least one regular non-isolated point (and not only for such spaces). Other condition, called an *increasing open filtration property,* shortly, *IOF-property,* holds for example for the Khalimsky line $\mathbb{K}$, see [6], a topological space $(\mathbb{Z}, \tau)$, where $\mathbb{Z}$ is the set of all integers and τ is the topology on $\mathbb{Z}$ generated by the base $\mathcal{B} = \{\{2k+1\}, \{2k-1, 2k, 2k+1\} \colon k \in \mathbb{Z}\}$. It is easy to see that $\mathbb{K}$ is a connected T_0-space (but not T_1). Let us note, see [12], that the Khalimsky line $\mathbb{K}$ and the spaces $\mathbb{K}^n$, $n \geq 2$, are basic objects of digital topology.

In [4] the authors observed that if each space X_α, $\alpha \in A$, is *hyperconnected* (i.e. every pair of nonempty open subsets of such a space has nonempty intersection) then the product $\square_{\alpha \in A} X_\alpha$ is also hyperconnected, in particular, connected. Let us note that hyperconnected spaces are not Hausdorff, and there are examples of hyperconnected T_1-spaces (some of them can even be found in [4]).

In this article I consider Question 1.2 in the realm of Hausdorff spaces.

In particular, I will show the following:

(a) If X_α, $\alpha \in A$, is an infinite system of connected spaces such that infinitely many of them are completely Hausdorff topological spaces then the product $\square_{\alpha \in A} X_\alpha$ can be decomposed into continuum many disjoint nonempty open subsets, in particular, it is disconnected.

(b) If X_α, $\alpha \in A$, is an infinite system of Brown Hausdorff topological spaces then the product $\square_{\alpha \in A} X_\alpha$ is also Brown Hausdorff, and hence, it is connected.

2. Auxiliary notions and facts

Recall that a topological space X is called *completely Hausdorff* or *functionally Hausdorff* if for any distinct points $x, y \in X$ there exists a continuous function $f \colon X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

It is easy to see that every connected completely Hausdorff space has cardinality greater than or equal to $\mathfrak{c}$.

Example 2.1. Let X be the topological space $(\mathbb{R}, \tau)$, where $\mathbb{R}$ is the set of reals, and τ is the topology on $\mathbb{R}$ generated by the family $\{(a,b) \setminus M \colon a < b, M \subset \mathbb{R}, |M| \leq \aleph_0\}$. Let us note that X is a connected homogeneous completely Hausdorff topological space without regular points. (A point p of a topological space T is called a *regular point of* T if for any closed subset F of T such that

$p \notin F$ there are two disjoint open subsets U and V of T such that $p \in U$ and $F \subseteq V$.) Observe that the identity mapping from X to $\mathbb{R}$ is continuous.

A topological space X is called a *Urysohn space* if for any distinct points $x, y \in X$ there are open neighborhoods U and V of x and y, respectively, such that U and V have disjoint closures.

It is evident that each completely Hausdorff space is Urysohn, and there are different examples of countable connected Urysohn spaces (which cannot be completely Hausdorff because of their cardinality), see for examples [7], [8], [9], [14], [17], [19]. The examples from [17], [19] have at least one regular point.

A topological space X is called a *Brown space* if every pair of open nonempty subsets of X has nonempty intersection of their closures (cf. [1]).

It is evident that every hyperconnected space is Brown, and every Brown space is connected. There are different examples of connected countable Hausdorff Brown spaces (which can neither have regular points nor be Urysohn by its definition), see for examples [2], [5], [10], [13], [16].

Recall [3] that a space X possesses

(1) a *DOF-property,* if there exists a countable sequence $\{O_i\}_{i=1,2,\dots}$ of open subsets of X such that $\mathrm{Cl}_X O_{i+1} \subsetneqq O_i$ for each $i = 1, 2, \dots$ and $\bigcap_{i\in\mathbb{N}} O_i \neq \emptyset$;

(2) an *IOF-property,* if there exists a countable sequence $\{O_i\}_{i=1,2,\dots}$ of open subsets of X such that $\emptyset \neq O_1 \subset \mathrm{Cl}_X O_i \subsetneqq O_{i+1}$ for each $i = 1, 2 \dots$ and $\bigcup_{i\in\mathbb{N}} O_i = X$.

Proposition 2.2 ([3]). *Let X_α, $\alpha \in A$, be an infinite system of connected spaces such that infinitely many of them possess the DOF-property (or IOF-property). Then the product $\square_{\alpha\in A} X_\alpha$ can be decomposed into continuum many disjoint nonempty open subsets.*

3. Disconnectedness of products of completely Hausdorff spaces with the box topology

Proposition 3.1. *Let $f\colon X \to Y$ be a continuous mapping of a topological space X onto a topological space Y with the DOF-property. Then X possesses the DOF-property.*

PROOF: Let $\{O_i\}_{i\in\mathbb{N}}$ be a countable sequence of open subsets of Y such that $\mathrm{Cl}_Y O_{i+1} \subsetneqq O_i$ for each $i = 1, 2, \dots$ and $\bigcap_{i=1,2\dots} O_i \neq \emptyset$. Put $V_i = f^{-1}O_i$ for each $i \geq 1$. Note that $\{V_i\}_{i\in\mathbb{N}}$ is a countable sequence of open subsets of X such that $\bigcap_{i\in\mathbb{N}} V_i = f^{-1}\big(\bigcap_{i\in\mathbb{N}} O_i\big) \neq \emptyset$, $\mathrm{Cl}_X V_{i+1} \subset f^{-1}\mathrm{Cl}_Y O_{i+1} \subset V_i$ and $\mathrm{Cl}_X V_{i+1} \neq V_i$

(since $V_i \setminus \mathrm{Cl}_X V_{i+1} \supseteq f^{-1}O_i \setminus f^{-1}\mathrm{Cl}_Y O_{i+1} = f^{-1}(O_i \setminus \mathrm{Cl}_Y O_{i+1}) \neq \emptyset$) for each $i = 1, 2, \dots$ Hence X possesses the DOF-property. $\square$

Corollary 3.2. *Every connected completely Hausdorff topological space X possesses DOF-property.*

PROOF: Consider two distinct points $x, y \in X$ and a continuous mapping $f\colon X \to \mathbb{R}$ of X to the real line such that $f(x) \neq f(y)$. Let us note that the image $Y = f(X)$ of X is a nondegenerated connected subset of the real line. Since Y is a T_1 space with at least one regular non-isolated point, the space Y possesses the DOF-property. By Proposition 3.1 the space X also possesses the DOF-property. $\square$

Remark 3.3. Every infinite completely Hausdorff space can be continuously mapped onto a subspace of the real line with at least one non-isolated point. This implies that every such a space has the DOF-property.

Theorem 3.4. *Let X_α, $\alpha \in A$, be an infinite system of connected spaces such that infinitely many of them are completely Hausdorff topological spaces. Then the product $\square_{\alpha \in A} X_\alpha$ can be decomposed into continuum many disjoint nonempty open subsets.*

PROOF: Apply Corollary 3.2 and Proposition 2.2 (the case of DOF-property). $\square$

We get immediately

Corollary 3.5. *The product $X^{\aleph_0}$, where X is the space from Example* 2.1, *with the box topology can be decomposed into continuum many disjoint nonempty open subsets.*

4. Connectedness of products of Brown Hausdorff spaces with the box topology

Proposition 4.1. *Let X_α, $\alpha \in A$, be any system of Brown spaces. Then the product $\square_{\alpha \in \mathcal{A}} X_\alpha$ is also Brown.*

PROOF: Let U and V be two nonempty open subsets of $\square_{\alpha \in \mathcal{A}} X_\alpha$. Consider a point $x \in U$ and a point $y \in V$. Let $B = \prod_{\alpha \in \mathcal{A}} B_\alpha$ and $C = \prod_{\alpha \in \mathcal{A}} C_\alpha$ be basic open neighborhoods of x and y, respectively, such that $B \subset U$ and $C \subset V$. Note that for each $\alpha \in A$ the space X_α is Brown. So $\mathrm{Cl}_{X_\alpha} B_\alpha \cap \mathrm{Cl}_{X_\alpha} C_\alpha \neq \emptyset$. Then put $X = \square_{\alpha \in \mathcal{A}} X_\alpha$ and observe that $\mathrm{Cl}_X B \cap \mathrm{Cl}_X C = \left(\prod_{\alpha \in \mathcal{A}} \mathrm{Cl}_{X_\alpha} B_\alpha\right) \cap \left(\prod_{\alpha \in \mathcal{A}} \mathrm{Cl}_{X_\alpha} C_\alpha\right) = \prod_{\alpha \in \mathcal{A}}((\mathrm{Cl}_{X_\alpha} B_\alpha) \cap (\mathrm{Cl}_{X_\alpha} C_\alpha)) \neq \emptyset$. Hence, the space $\square_{\alpha \in \mathcal{A}} X_\alpha$ is Brown. $\square$

Corollary 4.2 ([1]). *Let X_α, $\alpha \in A$, be an infinite system of Brown spaces. Then the topological product $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is Brown.*

Proof: Recall, see [1], that a continuous image of a Brown space is also Brown. Since the identity mapping $\mathrm{id}\colon \square_{\alpha\in\mathcal{A}} X_\alpha \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ is a continuous bijection, it follows from Proposition 4.1 that the space $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is Brown. $\square$

Theorem 4.3. *Let X_α, $\alpha \in A$, be an infinite system of Brown Hausdorff topological spaces. Then the product $\square_{\alpha\in A} X_\alpha$ is Brown Hausdorff.*

Proof: Apply Proposition 4.1 and the known fact that the product of Hausdorff spaces with the box topology is also Hausdorff. $\square$

We get immediately

Corollary 4.4. *Any product X^α, where X is the Bing space, see* [2], *the Golomb space, see* [5], *the Kirch space, see* [10], *the Lawrence space, see* [13], *or the Ritter space, see* [16], *with the box topology is connected. (Recall that all these spaces X are Hausdorff and Brown.)*

5. Box products of countable connected Urysohn spaces

Remark 5.1. Since the Roy space R, see [17], and the Vought space V, see [19], two countable connected Urysohn spaces, have regular non-isolated points, the products $R^{\aleph_0}$ and $V^{\aleph_0}$ with the box topology can be decomposed into continuum many disjoint nonempty open subsets by Proposition 2.2 (the case of DOF-property).

The following proposition is evident.

Proposition 5.2. *Each countable connected homogeneous Hausdorff space has no regular points.*

In [18] (cf. [8]) it was suggested a way to produce countable connected Urysohn homogeneous spaces, and hence (by Proposition 5.2) without regular points.

Example 5.3. Let X be any connected countable Urysohn space, see [7], [8], [9], [14], [17], [19]. For each positive integer i consider a copy X_i of X with two distinct points p_i, q_i in X_i. In the topological sum $\bigoplus_{i\in\mathbb{N}} X_i$ identify points q_i and p_{i+1} for every $i \geq 1$. Denote the quotient space by X_ω and the quotient mapping by pr. Note that the space X_ω is countable connected and Urysohn. Moreover, if X does not have regular points then neither does X_ω.

Proposition 5.4. *The space X_ω possesses the IOF-property.*

Proof: Let U_i and V_i be open neighborhoods of p_i and q_i in X_i with disjoint closures. Put $O_1 = pr(X_1 \setminus \mathrm{Cl}_{X_1} V_1)$. Then for each positive integer $i \geq 2$ put

$$O_i = pr((X_1 \oplus \cdots \oplus X_i) \setminus \mathrm{Cl}_{X_i} V_i).$$

Note that all O_i are open subsets of X_ω such that $\emptyset \neq O_1 \subset \mathrm{Cl}_X O_i \subsetneqq O_{i+1}$ for each $i = 1, 2 \ldots$ and $\bigcup_{i \in \mathbb{N}} O_i = X_\omega$. Indeed, $pr(p_1) \in O_1$, hence $O_1 \neq \emptyset$. Then for each integer $i \geq 2$ we have

$$\mathrm{Cl}_{X_\omega} O_i \subseteq pr((X_1 \oplus \cdots \oplus X_i) \setminus V_i) \subseteq O_{i+1}.$$

Moreover, $pr(q_i) \in O_{i+1} \setminus \mathrm{Cl}_{X_\omega} O_i$ and $\bigcup_{i \in \mathbb{N}} O_i \supseteq \bigcup_{i \in \mathbb{N}} pr(X_1 \oplus \cdots \oplus X_i) = X_\omega$. Hence, the space X_ω possesses the IOF-property. □

Example 5.5. Let us extend the space X_ω by a point p as follows. In the union $X_\omega \cup \{p\}$ a base at the point p consists of the sets $\{p\} \cup \left(X_\omega \setminus \bigcup_{i \leq k} prX_i\right)$, $k = 1, 2, \ldots$ For the points of the part X_ω we take any base from the space X_ω. Denote the new space by X_ω^+. It is easy to see that the space X_ω^+ is countable connected Urysohn, and the point p is a regular point of X_ω^+.

Corollary 5.6. *The product $(X_\omega)^{\aleph_0}$ ($(X_\omega^+)^{\aleph_0}$, respectively) with the box topology can be decomposed into continuum many disjoint nonempty open subsets.*

Proof: Apply Proposition 5.4 and Proposition 2.2 (for both cases). □

In the end one can ask

Question 5.7. Are there countable connected Urysohn spaces X_i, $i = 1, 2, \ldots$, such that the product $\square_{i=1,2,\ldots} X_i$ is connected?

In connection to the question above it would be interesting to precise what examples from [7], [8], [9], [14] do not have regular points.

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