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Limited p-converging operators and relation with some geometric properties of Banach spaces

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Abstract. By using the concepts of limited p-converging operators between two Banach spaces X and Y, L_p-sets and L_p-limited sets in Banach spaces, we obtain some characterizations of these concepts relative to some well-known geometric properties of Banach spaces, such as $*$-Dunford–Pettis property of order p and Pelczyński's property of order p, $1 \le p < \infty$.

Keywords: Gelfand–Phillips property; Schur property; p-Schur property; weakly p-compact set; reciprocal Dunford–Pettis property of order p

Classification: 47L05, 46B25

1. Introduction

Suppose that X is a Banach space and $1 \le p \le \infty$. The space of all weakly p-summable sequences in X is defined by

$$l_p^{\mathrm{weak}}(X) := \{(x_n)\colon (x_n, x^*) \in l_p,\ \forall\, x^* \in X^*\}.$$

This is a Banach space with norm

$$\|(x_n)\|_p^{\mathrm{weak}} = \sup\biggl\{\biggl(\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle|^p\biggr)^{1/p}:\ \|x^*\| \le 1\biggr\}.$$

Note that for $p = \infty$, $l_\infty^{\mathrm{weak}}(X) = l_\infty(X)$ is the Banach space of all (weakly) bounded sequences in X with supremum norm, see [10, page 33]. Moreover, by $c_0^{\mathrm{weak}}(X)$ we represent the closed subspace of $l_\infty(X)$ containing all weakly null sequences in X.

An operator T between two Banach spaces X and Y is said to be p-converging if it transfers weakly p-summable sequences into norm null sequences. The class of all p-converging operators from X into Y is denoted by $C_p(X,Y)$. Also T is

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called p-summing if there is a constant $c \geq 0$ such that for all choices of $(x_k)_{k=1}^n$ in X we have

$$\left(\sum_{k=1}^{n} \|Tx_k\|^p\right)^{1/p} \leq c \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x^*\rangle|^p\right)^{1/p} : \|x^*\| \leq 1\right\}.$$

The set of all p-summing operators from X into Y is denoted by $\Pi_p(X,Y)$.

For each $1 \leq p < \infty$ a sequence (x_n) in a Banach space X is said to be weakly p-convergent to an $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable, i.e., $(x_n - x) \in l_p^{\mathrm{weak}}(X)$. The weakly ∞-convergent sequences are simply the weakly convergent sequences. Also, a bounded set K in a Banach space is said to be relatively weakly p-compact, $1 \leq p \leq \infty$, if every sequence in K has a weakly p-convergent subsequence, see [3]. If the limit point of each weakly p-convergent subsequence is in K, then K is weakly p-compact set. Moreover, according to [4], we say that a Banach space $X \in \mathcal{W}_p$ if the closed unit ball B_X of X is a weakly p-compact set. A bounded operator T from X into Y is called weakly p-compact, $1 \leq p \leq \infty$, if $T(B_X)$ is relatively weakly p-compact. The space of all weakly p-compact operators from X into Y is denoted by $W_p(X,Y)$; while the space of all bounded operators and weakly compact operators from X into Y are denoted by $L(X,Y)$ and $W(X,Y)$, respectively. Weakly ∞-compact operators are precisely those $T \in L(X,Y)$ for which $T(B_X)$ is relatively weakly compact, that is, $W_\infty(X,Y) = W(X,Y)$.

A Banach space X has the Dunford–Pettis (DP) property, if every weakly compact operator T from X into arbitrary Banach space Y is a Dunford–Pettis operator, that is, T carries weakly convergent sequences into norm convergent ones. Moreover, if $1 \leq p \leq \infty$, the Banach space X has the Dunford–Pettis property of order p (DP_p) if for each Banach space Y, every weakly compact operator $T\colon X \to Y$ is p-converging; in other words $W(X,Y) \subseteq C_p(X,Y)$, see [3]. By definition, ∞-converging operators are equal to Dunford–Pettis ones. So the Dunford–Pettis property of order ∞ is the same as DP property. Every Banach space with DP property, such as the sequence spaces c_0 and l_1, have the DP_p property, see [3].

Also the Banach space X has the Schur property if every weakly null sequence in X converges in norm. The simplest Banach space with the Schur property is l_1. A Banach space X has the p-Schur property, $1 \leq p \leq \infty$, if every weakly p-compact subset of X is compact. In other words, if $1 \leq p < \infty$, X has the p-Schur property if and only if every sequence $(x_n) \in l_p^{\mathrm{weak}}(X)$ is a norm null sequence, for example, l_p has the 1-Schur property. Moreover, X has the ∞-Schur property if and only if every sequence in $c_0^{\mathrm{weak}}(X)$ is norm null. So,

∞-Schur property coincides with the Schur property. Also one note that every Schur space has the p-Schur property for all $p \geq 1$, see [6].

A subset K of a Banach space X is called limited (or Dunford–Pettis (DP)), if for each weak* null (weak null, respectively) sequence (x_n^*) in X^*,

$$\lim_{n\to\infty} \sup_{x\in K} |\langle x, x_n^*\rangle| = 0.$$

In particular, a sequence $(x_n) \subset X$ is limited if and only if $\langle x_n, x_n^*\rangle \to 0$ for all weak*-null sequences (x_n^*) in X^*.

In general, every relatively compact subset of X is limited and so is Dunford–Pettis. If every limited subset of X is relatively compact, then X has the Gelfand–Phillips (GP) property. For example the classical Banach spaces c_0 and l_1 have the GP property and every Schur space and spaces containing no copy of l_1, such as reflexive spaces have the same property, see [2]. The reader can find some useful and additional properties of limited and DP sets and Banach spaces with the Schur and GP properties in [1], [11], [12], [15], [19], [20], [22], [24].

In this note, using the concepts of limited p-converging operators between Banach spaces and L_p-limited subsets in dual of Banach spaces, we obtain some characterizations of the DP_p^* property of X. We shall also obtain some necessary and sufficient conditions for Pelczyński's property (V) of order p which has been introduced and studied in [18]. In particular, we will present a new class of Banach spaces with Pelczyński's property (V) of order p. More precisely, we will prove that if $X \in \mathcal{W}_p$ and Y is a Banach space with Pelczyński's property (V) of order p such that $L(X, Y^*) = \Pi_p(X, Y^*)$, then $X \otimes_\pi Y$ has Pelczyński's property (V) of order p.

2. Main results

An operator $T \in L(X, Y)$ is called limited completely continuous if it carries limited and weakly null sequences in X to norm null ones in Y. The class of all limited completely continuous operators from X into Y is denoted by $L_{\mathrm{cc}}(X, Y)$, see [23]. Also, an operator $T \in L(X, Y)$ is limited p-converging if it transfers limited and weakly p-summable sequences into norm null sequences, see [14]. We denote the space of all limited p-converging operators from X into Y by $C_{\mathrm{l}p}(X, Y)$.

It is clear that every weakly p-compact operator is weakly compact. On the other hand by [23, Corollary 2.5] every weakly compact operator is limited completely continuous. Also limited completely continuous operators are limited p-converging. Therefore we have

$$W_p(X, Y) \subseteq W(X, Y) \subseteq L_{\mathrm{cc}}(X, Y) \subseteq C_{\mathrm{l}p}(X, Y).$$

Theorem 2.1. *The following statements for any bounded operator $T\colon X \to Y$ are equivalent.*

1. $T \in C_{\mathrm{l}p}(X,Y)$.
2. *Operator T transfers limited weakly p-compact sets into relatively norm compact ones.*
3. *If $S\colon Z \to X$ is limited weakly p-compact operator, i.e., $S(B_Z)$ is limited and weakly p-compact, then TS is compact.*
4. *If $S\colon l_1 \to X$ is limited weakly p-compact, then TS is compact.*

Proof: (1) $\Rightarrow$(2) Let $A \subset X$ be limited weakly p-compact and (Tx_n) is a sequence in $T(A)$. Since A is weakly p-compact, we conclude that there is a subsequence (x_{n_k}) of (x_n) and $x_0 \in X$ such that $(x_{n_k} - x_0) \in l_p^{\mathrm{weak}}(X)$. By assumption, $\|Tx_{n_k} - Tx_0\| \to 0$ which implies that $T(A)$ is relatively compact.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Assume that (x_n) is limited weakly p-summable. We shall prove that $\|Tx_n\| \to 0$. Define

$$S\colon l_1 \to X, \qquad S(\alpha_1, \alpha_2, \dots) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

First, note that S is well defined, since (x_n) is weakly p-summable. We claim that S is limited weakly p-compact.

Since (x_n) is limited and

$$S(B_{l_1}) = \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n \colon \sum_{n=1}^{\infty} |\alpha_n| \leq 1 \Big\},$$

it follows that S is a limited operator. Assume that $q > 1$ such that $1/p + 1/q = 1$. It is easy to see that the set

$$\Big\{ \sum_{n=1}^{\infty} \alpha_n x_n \colon \sum_{n=1}^{\infty} |\alpha_n|^q \leq 1 \Big\}$$

is the continuous image by the natural operator associated to $(\alpha_n) \in B_{l_q}$ and so is weakly p-compact, see e.g. [10]. On the other hand, it is clear that

$$\Big\{ \sum_{n=1}^{\infty} \alpha_n x_n \colon \sum_{n=1}^{\infty} |\alpha_n| \leq 1 \Big\} \subseteq \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n \colon \sum_{n=1}^{\infty} |\alpha_n|^q \leq 1 \Big\}.$$

It implies that $S(B_{l_1})$ is relatively weakly p-compact. Then by (4) the operator TS is compact. If (e_n) is the standard basis for l_1, then each subsequence (e_{n_k}) of (e_n), has a new subsequence, which is denoted again by (e_{n_k}), such that

$(Tx_{n_k}) = (TSe_{n_k})$ is norm convergent. Since the sequence (Tx_n) is weakly null it follows that $\|Tx_n\| \to 0$. $\square$

A Banach space X is said to have the DP*-property of order p, for $1 \le p \le \infty$, if all weakly p-compact sets in X are limited. In short, we say that X has the DP$_p^*$ property, see [13]. It is clear that every p-converging operator is limited p-converging, but the converse in general is false. For example, let T be the identity operator on c_0. By [6, Corollary 2.8] c_0 does not have the p-Schur property. Then T is not p-converging while $T \in C_{\mathrm{l}p}(c_0)$, since c_0 has the GP property.

In the following, we give a characterization of this converse assertion, with respect to the DP$_p^*$ property of Banach spaces.

Theorem 2.2 ([13])**.** *Let $1 \le p \le \infty$. The Banach space X has the* DP$_p^*$ *property if and only if $\langle x_n, x_n^* \rangle \to 0$ as $n \to \infty$ for all $(x_n) \in l_p^{\mathrm{weak}}(X)$ and all* weak* *null sequence (x_n^*) in X^*.*

Theorem 2.3. *The Banach space X has the* DP$_p^*$ *property if and only if $C_p(X,Y) = C_{\mathrm{l}p}(X,Y)$ for every Banach space Y.*

Proof: Let $T \in C_{\mathrm{l}p}(X,Y)$ and $(x_n) \in l_p^{\mathrm{weak}}(X)$. Theorem 2.2 implies that (x_n) is limited and so $\|Tx_n\| \to 0$. Hence $T \in C_p(X,Y)$.

Conversely, if X does not have the DP$_p^*$ property, then there are $(x_n) \in l_p^{\mathrm{weak}}(X)$ and a weak*-null sequence (x_n^*) in X^* and $\varepsilon > 0$ such that $|\langle x_n, x_n^* \rangle| > \varepsilon$ for all integer n. Define $T\colon X \to c_0$ by $Tx = (\langle x, x_n^* \rangle)$ and let A be a limited subset of X. Then $T(A)$ is also limited in c_0. Since c_0 has the GP property, $T(A)$ is relatively compact. Theorem 2.1 shows that $T \in C_{\mathrm{l}p}(X, c_0)$. Moreover, $\|Tx_n\| \ge |\langle x_n, x_n^* \rangle| \ge \varepsilon$. Therefore $T \notin C_p(X, c_0)$, which completes the proof. $\square$

Recall that according to [17], a bounded subset K of a Banach space X is p-limited if for every $(x_n^*) \in l_p^{\mathrm{weak}}(X^*)$ there exists $(\alpha_n) \in l_p$ such that $|\langle x, x_n^* \rangle| \le \alpha_n$ for all $x \in K$ and all $n \in \mathbb{N}$. Equivalently, K is p-limited if

$$\lim_n \sup_{x \in K} |\langle x, x_n^* \rangle| = 0$$

for every $(x_n^*) \in l_p^{\mathrm{weak}}(X^*)$.

It is clear that every limited set and every Dunford–Pettis set are p-limited. We refer to [9] for more information about p-limited subsets of Banach spaces.

Theorem 2.4. *Let X^* has the* DP$_p^*$ *property. If $T\colon X \to Y$ and $T(B_X)$ is not p-limited, then T fixes a copy of l_1.*

Proof: By assumptions, there exist $\varepsilon > 0$, $(y_k^*) \in l_p^{\mathrm{weak}}(Y^*)$ and a sequence $(x_k) \subset B_X$ such that $|\langle Tx_k, y_k^* \rangle| \ge \varepsilon$ for all integers k. We claim that (Tx_n) does not have a weakly Cauchy subsequence. Otherwise, by passing to subsequence, we can assume that the sequence (Tx_n) is weakly Cauchy. For each $m \in \mathbb{N}$, $\lim_{n\to\infty}\langle Tx_m, y_n^*\rangle = 0$. Therefore there is an $n_m \in \mathbb{N}$ such that $|\langle Tx_m, y_{n_m}^*\rangle| < \varepsilon/2$. We also have

$$|\langle Tx_{n_m} - Tx_m, y_{n_m}^*\rangle| \geq |\langle Tx_{n_m}, y_{n_m}^*\rangle| - |\langle Tx_m, y_{n_m}^*\rangle| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

for all $m \in \mathbb{N}$. Since the sequence $(x_{n_m} - x_m)_{m\in\mathbb{N}}$ is weakly null and $(y_{n_m}^* \circ T) \in l_p^{\text{weak}}(X^*)$, it follows from the DP_p^* property of X^* that

$$\lim_{m\to\infty}\langle Tx_{n_m} - Tx_m, y_{n_m}^*\rangle = 0,$$

which is a contradiction. Hence (x_n) has no weakly Cauchy subsequence, since the image of a weakly Cauchy sequence is weakly Cauchy. Therefore the Rosenthal's l_1-theorem implies the existence of a subsequence of (x_n) and a subsequece of (Tx_n) which is equivalent to the usual l_1 basis. Therefore a copy of l_1 in Y is fixed by T. □

Let us recall that according to [18] a bounded subset K of X^* is said to be p-(V) set if

$$\lim_n \sup_{x^*\in K} |\langle x_n, x^*\rangle| = 0$$

for all $(x_n) \in l_p^{\text{weak}}(X)$. The authors in [18] have used this notion to define Pelczyński's property (V) of order p as a p-version of Pelczyński's property (V). Also, a bounded subset K of X^* is called an L-set, if each weakly null sequence (x_n) in X tends to 0 uniformly on K, see [12]. It is clear that ∞-(V) sets are L-sets. According to this point of view in this article we choose the name L_p-sets instead of the p-(V) subsets of X^*.

Obviously, a sequence $(x_n^*) \in X^*$ is a L_p-set if and only if $\lim_{n\to\infty}\langle x_n, x_n^*\rangle = 0$ for all $(x_n) \in l_p^{\text{weak}}(X)$.

In the following, we introduce the notion of L_p-limited subsets of the dual space X^*.

Definition 2.5. Let $1 \leq p \leq \infty$. A subset K of a dual space X^* of X is L_p-limited set if

$$\lim_n \sup_{x^*\in K} |\langle x_n, x^*\rangle| = 0$$

for every limited sequence $(x_n) \in l_p^{\text{weak}}(X)$.

For example, the Schur property of l_1 implies that the closed unit ball of $l_\infty = l_1^*$ is an L_p-set and so L_p-limited set. The closed unit ball of $c_0^* = l_1$ shows that L_p-limited sets are not L_p-sets, in general. In fact c_0 has the GP property and so every limited weakly null sequence in c_0 is norm null, hence the closed unit ball of c_0^* is an L_p-limited set. But c_0 fail to have the p-Schur property. Then this closed unit ball is not an L_p-set. The reader is referred to [8] for more information about the relationships between L_p-sets and L_p-limited sets.

Proposition 2.6. *A Banach space X has the p-Schur property if and only if every bounded subset of X^* is an L_p-set. In particular, the closed unit ball of each l_p space is an L_1-set.*

Proof: If X has the p-Schur property and $(x_n) \in l_p^{\mathrm{weak}}(X)$, then

$$\sup\{|\langle x_n, x^*\rangle| \colon x^* \in B_{X^*}\} = \|x_n\| \to 0.$$

Thus B_{X^*} is an L_p-set. So, every bounded subset of X^* is an L_p-set. The converse is proven in a similar way.

It is clear that, for every Banach space X, every p-limited subset of X^* is an L_p-set and the closed convex hull of an L_p-limited set is also L_p-limited. Furthermore, every L_p-limited set in X^* is bounded. In fact, if $K \subseteq X^*$ is an L_p-limited set which is unbounded, then there are (x_n^*) in K and (y_n) in B_X such that $|\langle y_n, x_n^*\rangle| > n^2$ for all n. Let $x_n = y_n/n^2$. Then

$$\sum_{n=1}^{\infty} \|x_n\|^p = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \|y_n\|^p < \infty.$$

Hence (x_n) is a limited sequence in $l_p^{\mathrm{weak}}(X)$. Therefore

$$0 = \lim_{n\to\infty} \sup_{x_n^* \in K} |\langle x_n, x_n^*\rangle| \geq \lim_{n\to\infty} |\langle x_n, x_n^*\rangle| = \lim_{n\to\infty} \frac{1}{n^2} |\langle y_n, x_n^*\rangle| > 1.$$

This is a contradiction. □

Theorem 2.7. *The Banach space X has the* DP_p^* *property if and only if every L_p-limited subset of X^* is L_p-set.*

Proof: It is clear that, for an operator $T \colon X \to Y$, $T \in C_{\mathrm{lp}}(X, Y)$ if and only if $T^*(B_{Y^*})$ is an L_p-limited set. Also, $T \in C_p(X, Y)$ if and only if $T^*(B_{Y^*})$ is an L_p-set. Now, assume that every L_p-limited subset of X^* is L_p-set and $T\colon X \to Y$ is a limited p-converging operator. Then $T^*(B_{Y^*})$ is an L_p-limited set. By assumption $T^*(B_{Y^*})$ is an L_p-set. Hence T is p-converging. Therefore Theorem 2.3 completes the proof. The converse follows easily from Theorem 2.2. □

In [16] A. Grothendieck introduced the reciprocal Dunford–Pettis (RDP) property: a Banach space X has the RDP property if for every Banach space Y, every completely continuous operator $T \colon X \to Y$ is weakly compact. Recall that Banach space X has Pelczyński property (V) if for every Banach space Y, every unconditionally converging operator $T \in L(X, Y)$, (i.e. any operator mapping weakly unconditionally converging series into unconditionally converging ones) is weakly compact.

The concept of Pelczyński property (V) of order p has been introduced in [18]. In fact, a Banach space X has the Pelczyński property (V) of order p (property p-(V)) if $C_p(X,Y) \subseteq W(X,Y)$ for every Banach space Y.

Note that property 1-(V) is equivalent to Pelczyński property (V) and ∞-(V) is equivalent to the RDP property. Also, since every completely continuous operator is p-converging, then every Banach space which has property p-(V) for some $1 \leq p \leq \infty$ has the RDP property. Then we have the following well-known result; every Banach space X with Pelczyński (V) property has the RDP property.

Moreover, every reflexive Banach space has property p-(V) and if X is non reflexive with the p-Schur property, then X does not have property p-(V); indeed, the identity operator $i\colon X \to X$ is p-converging, but it is not weakly compact.

Theorem 2.8 ([18, Theorem 2.4])**.** *A Banach space X has property p-(V) if and only if every L_p-set in X^* is relatively weakly compact.*

Theorem 2.9. *If a Banach space $X \in \mathcal{W}_p$, then every L_p-set in X^* is relatively compact.*

Proof: Suppose that $X \in \mathcal{W}_p$ and $K \subseteq X^*$ is an L_p-set. Then K is bounded. Without loss of generality, we may assume that K is weak* closed and so is weak* compact. Define

$$T\colon X \to C(K), \quad \langle Tx, x^* \rangle = \langle x, x^* \rangle, \qquad x \in X,\ x^* \in K.$$

Clearly, T is bounded. Indeed,

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\| \leq 1} \Big(\sup_{x^* \in K} |\langle x, x^* \rangle| \Big) = \sup_{x^* \in K} \|x^*\|.$$

On the other hand, T is p-converging, because if $(x_n) \in l_p^{\mathrm{weak}}(X)$, then

$$\|Tx_n\| = \sup_{x^* \in K} |\langle Tx_n, x^* \rangle| = \sup_{x^* \in K} |\langle x_n, x^* \rangle| \to 0.$$

Therefore T is compact and so $T^*\colon C(K)^* \to X^*$ is compact. For $x^* \in K$ define $\delta_{x^*} \in C(K)^*$ by

$$\delta_{x^*}(f) = f(x^*), \qquad f \in C(K).$$

Hence for all $x \in X$ we have

$$\langle x, T^*(\delta_{x^*}) \rangle = \langle Tx, \delta_{x^*} \rangle = \langle Tx, x^* \rangle = \langle x, x^* \rangle.$$

Then $T^*(\delta_{x^*}) = x^*$. Moreover,

$$K = \{T^*\delta_{x^*} : x^* \in K\} = T^*\{\delta_{x^*} : x^* \in K\} \subseteq T^*(B_{C(K)^*}).$$

Since T^* is compact, we conclude that K is relatively compact. $\square$

As a corollary, every Banach space $X \in \mathcal{W}_p$ has property p-(V). But the converse is not true in general. For example, the Hilbert space l_2 has property 1-(V), but it is not weakly 1-compact, see [6, page 132].

The following characterization of spaces having DP_p property has an essential role to achieve our next results.

Theorem 2.10 ([3, Proposition 3.2])**.** *For a given Banach space X and $1 \leq p \leq \infty$ the following are equivalent:*

(1) *Space X has the DP_p property.*
(2) *If $(x_n) \in l_p^{\mathrm{weak}}(X)$ and $(x_n^*) \in c_0^{\mathrm{weak}}(X^*)$, then $\langle x_n, x_n^* \rangle \to 0$.*

Corollary 2.11. *If X has the DP_p property and $Y \in \mathcal{W}_p$, then $L(X, Y^*) = C_p(X, Y^*)$.*

Proof: Assume that $T \in L(X, Y^*)$ and $(x_n) \in l_p^{\mathrm{weak}}(X)$. Let $(y_n) \in l_p^{\mathrm{weak}}(Y)$. Since $(T^*(y_n))$ is weakly null, then $\langle Tx_n, y_n \rangle = \langle x_n, T^*y_n \rangle \to 0$ by Theorem 2.10. It follows that (Tx_n) is an L_p-set. Therefore Theorem 2.9 implies that (Tx_n) is relatively compact, and so $T \in C_p(X, Y)$. $\square$

Corollary 2.12. *If a Banach space X has the DP_p property and $Y^* \in \mathcal{W}_p$, then $L(X, Y) = C_p(X, Y)$.*

Proof: Let $T \in L(X, Y)$ and let $(x_n) \in l_p^{\mathrm{weak}}(X)$. Then by previous corollary, (Tx_n) is an L_p-set in Y^{**}. Hence an appeal to Theorem 2.9 shows that this sequence is relatively compact in Y^{**} and so in Y. $\square$

Note that if a Banach space $X \in \mathcal{W}_p$ and for some Banach space Y, $T \in C_p(X, Y)$, then for each sequence (x_n) in B_X, there is a subsequence (x_{n_k}) weakly p-convergent to some $x \in B_X$, and so $\|Tx_{n_k} - Tx\| \to 0$ as $k \to \infty$. Therefore T is compact. This will be used in the proof of the following theorem.

Theorem 2.13. *For Banach spaces X and Y such that $X, Y^* \in \mathcal{W}_p$ the following assertions are equivalent*

(1) *For each $T \in L(X, Y^{**})$ and each sequence $(x_n) \in l_p^{\mathrm{weak}}(X)$, (Tx_n) is an L_p-set.*
(2) *Every $T \in L(X, Y^{**})$ is compact.*
(3) *Every $T \in L(Y^*, X^*)$ is compact.*

Proof: (1) $\Rightarrow$ (2) Let $T \in L(X, Y^{**})$ and $(x_n) \in l_p^{\mathrm{weak}}(X)$. Then (Tx_n) is an L_p-set in Y^{**}. Since $Y^* \in \mathcal{W}_p$, by Theorem 2.9, (Tx_n) is a relatively compact set. Therefore $\|Tx_n\| \to 0$. Hence $T \in C_p(X, Y^{**})$ and we are done since $X \in \mathcal{W}_p$.

(2) $\Rightarrow$ (3) If $T \in L(Y^*, X^*)$, then $T^*|_X \in L(X, Y^{**})$ is compact. Therefore $T = (T^*|_X)^*|_{Y^*} \colon Y^* \to X^*$ is compact.

(3) $\Rightarrow$ (1) Let $T \in L(X, Y^{**})$ and $(x_n) \in l_p^{\mathrm{weak}}(X)$ such that (Tx_n) is not an L_p-set. So there are $\varepsilon > 0$ and $(y_n^*) \in l_p^{\mathrm{weak}}(Y^*)$ such that (by passing to a subsequence, if necessary)

$$|\langle Tx_n, y_n^* \rangle| > \varepsilon, \qquad \forall\, n \in \mathbb{N}.$$

Hence,

$$|\langle T^*|_{Y^*}(y_n^*), x_n \rangle| > \varepsilon, \qquad \forall\, n \in \mathbb{N}.$$

Since $T^*|_{Y^*}$ is compact, there is a subsequence $(y_{n_k}^*)_k$ such that $(T^*|_{Y^*}(y_{n_k}^*))$ is norm null and we have a contradiction. $\square$

Theorem 2.14. *Let X be a Banach space and $X \in \mathcal{W}_p$ and let Y be a Banach space with property p-(V). If $L(X, Y^*) = \Pi_p(X, Y^*)$, then $X \otimes_\pi Y$ has property p-(V).*

Proof: Let H be an L_p-subset of $(X \otimes_\pi Y)^* = L(X, Y^*)$ and (h_n) be a sequence in H. If $(x_n) \in l_p^{\mathrm{weak}}(X)$ we claim that $\|h_n(x_n)\|_{Y^*} \to 0$. If this were false, there would exist $\varepsilon > 0$, (h_{n_k}), (x_{n_k}) and $(y_k) \subseteq B_Y$ such that

$$|\langle h_{n_k}(x_{n_k}), y_k \rangle| > \varepsilon$$

for all $k \in \mathbb{N}$. On the other hand, for every $T \in (X \otimes_\pi Y)^* = L(X, Y^*)$,

$$\sum_{k=1}^{\infty} |T(x_{n_k} \otimes y_k)|^p = \sum_{k=1}^{\infty} |\langle Tx_{n_k}, y_k \rangle|^p \leq \sum_{k=1}^{\infty} \|Tx_{n_k}\|^p < \infty,$$

since T is p-summing and $(x_{n_k}) \in l_p^{\mathrm{weak}}(X)$. Hence $(x_{n_k} \otimes y_k) \in l_p^{\mathrm{weak}}(X \otimes_\pi Y)$ and so by assumption on H, $\langle h_{n_k}(x_{n_k}), y_n \rangle \to 0$ which is a contradiction. Similarly we can prove that if $(y_n) \in l_p^{\mathrm{weak}}(Y)$, then $\|h_n^*(y_n)\|_{X^*} \to 0$.

If $y^{**} \in Y^{**}$, then the sequence $(h_n^*(y^{**})) \subseteq X^*$ is an L_p-set. Because, If $(x_n) \in l_p^{\mathrm{weak}}(X)$, then

$$|\langle h_n^*(y^{**}), x_n \rangle| = |\langle h_n(x_n), y^{**} \rangle| \leq \|y^{**}\| \|h_n(x_n)\|_{Y^*} \to 0.$$

Hence Theorem 2.9 implies that $(h_n^*(y^{**}))$ is a relatively compact set. By passing to a subsequence, we may assume that this sequence is weakly convergent to some x^*. Similarly, we can prove that for all $x^{**} \in X^{**}$, the sequence $(h_n^{**}(x^{**}))$ is an L_p-set and so is a relatively weakly compact subset of Y^{***}, by virtue of Theorem 2.8. But $h_n \colon X \to Y^*$ is compact for all $n \in \mathbb{N}$; so $(h_n^{**}(x^{**})) \subseteq Y^*$.

Now consider two arbitrary subsequences $(h_{n_k}^{**}(x^{**}))$ and $(h_{n_p}^{**}(x^{**}))$ which are weakly convergent to z_1 and z_2, respectively. It is easy to see that $z_1 = z_2$. Indeed, if $y^{**} \in Y^{**}$, then we have

$$\begin{aligned}\langle z_1, y^{**}\rangle &= \lim_k \langle h_{n_k}^{**}(x^{**}), y^{**}\rangle = \lim_k \langle x^{**}, h_{n_k}^{*}(y^{**})\rangle \\ &= \lim_n \langle x^{**}, h_n^{*}(y^{**})\rangle = \lim_p \langle x^{**}, h_{n_p}^{*}(y^{**})\rangle \\ &= \lim_p \langle h_{n_p}^{**}(x^{**}), y^{**}\rangle = \langle z_2, y^{**}\rangle.\end{aligned}$$

Hence there is $h_0(x^{**}) \in Y^*$ such that $h_0(x^{**}) = w - \lim_n h_n^{**}(x^{**})$. Now we claim that h_0 is w^*-w^* continuous. In fact, we show that h_0 is w^*-w^* continuous from X^{**} into Y^*. Let (x_α^{**}) be a w^*-null net in X^{**} and $y^{**} \in Y^{**}$. Since

$$\langle h_0(x_\alpha^{**}), y^{**}\rangle = \lim_n \langle h_n^{**}(x_\alpha^{**}), y^{**}\rangle = \lim_n \langle x_\alpha^{**}, h_n^{*}(y^{**})\rangle = \langle x_\alpha^{**}, x^*\rangle,$$

we observe that $\lim_\alpha \langle h_0(x_\alpha^{**}), y^{**}\rangle = 0$ and h_0 is w^*-w^* continuous. Now consider $h \in L(X, Y^*) = \Pi_p(X, Y^*)$ defined by $h = h_0|_X$. If $x^{**} \in X^{**}$, then there is a net $(x_\alpha) \subset X$ which is w^*-converging to x^{**}. So we obtain

$$h^{**}(x^{**}) = w^* - \lim_\alpha h^{**}(x_\alpha) = w^* - \lim_\alpha h(x_\alpha) = w^* - \lim_\alpha h_0(x_\alpha) = h_0(x^{**}).$$

Therefore $h^{**} = h_0$. By the construction of h_0 we thus have $\lim_n \langle h_n^{**}(x^{**}), y^{**}\rangle = \langle h^{**}(x^{**}), y^{**}\rangle$ for all $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$. Corollary 4.1.5 of [21] implies that $h_n \stackrel{w}{\to} h$ in $L(X, Y^*)$. Therefore H is relatively weakly compact. □

Recall that a Banach space X has the p-Gelfand–Phillips (p-GP) property if every limited weakly p-compact subset of X is relatively compact, see [13]. It should be noted that this notion has been called "limited p-Schur property" in [7]. More precisely, X has the p-GP property if and only if every limited sequence $(x_n) \in l_p^{\mathrm{weak}}(X)$ is norm null. It is easy to see that every Banach space with the p-Schur property and every Banach space with GP property is p-GP for all $1 \le p \le \infty$. Moreover, X has the GP property if and only if every limited weakly null sequence in X is norm null, see e.g., [11]. Therefore the ∞-GP property is equivalent to the GP property.

If X is a p-GP space with the DP_p^* property, then X has the p-Schur property. Indeed, if $(x_n) \in l_p^{\mathrm{weak}}(X)$, then by the DP_p^* property of X, we conclude that $\langle x_n, x_n^*\rangle \to 0$ for all w^*-null sequence $(x_n^*) \subset X^*$. Therefore (x_n) is limited and so $\|x_n\| \to 0$. Furthermore, if $X \in \mathcal{W}_p$ has the p-GP property, then X has the GP property.

By a similar argument of Proposition 2.6, it is evident that a Banach space X has the p-GP property if and only if every bounded subset of X^* is an L_p-limited set. Since l_1 has the p-Schur property for all $1 \le p \le \infty$ so B_{l_1} is an L_p-limited set which is not weakly compact. Also, l_2 has the 1-Schur property. It follows that B_{l_2} is an L_1-limited set, while we know that it is not weakly 1-compact, see [6, page 132].

Theorem 2.15. *For a Banach space X, the following are equivalent.*

(1) *Every L_p-limited set in X^* is weakly compact.*
(2) *For each Banach space Y, $C_{\mathrm{l}p}(X,Y) = W(X,Y)$.*
(3) *$C_{\mathrm{l}p}(X,l_\infty) = W(X,l_\infty)$.*

Proof: (1) $\Rightarrow$ (2) If $T \in C_{\mathrm{l}p}(X,Y)$, then $T^*(B_{Y^*})$ is an L_p-limited set in X^*. So by hypothesis, it is weakly compact and so T^* is a weakly compact operator. Therefore $T \in W(X,Y)$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) If (1) does not hold, then there is an L_p-limited subset A of X^* which is not weakly compact. So there is a sequence $(x_n^*) \subset A$ with no weakly p-convergent subsequence. Now let $T\colon X \to l_\infty$ be defined by

$$Tx = (\langle x, x_n^*\rangle), \qquad x \in X.$$

As (x_n^*) is L_p-limited set, for every limited sequence $(x_m) \in l_p^{\mathrm{weak}}(X)$ we have

$$\|Tx_m\| = \sup_n |\langle x_m, x_n^*\rangle| \to 0$$

as $m \to \infty$. Thus $T \in C_{\mathrm{l}p}(X,l_\infty)$. Clearly $T^*(e_n^*) = x_n^*$ for all $n \in \mathbb{N}$. Hence T^* is not weakly p-compact. So $T \notin W(X,l_\infty)$. □

It is clear that the class $C_{\mathrm{l}p}(X,Y)$ is a closed linear subspace of $L(X,Y)$ which has the ideal property. In sequel, we prove that the operator ideal $C_{\mathrm{l}p}$ of all limited p-converging operators between Banach spaces, by meaning of [5], is injective but it is not surjective.

Theorem 2.16. *The operator ideal $C_{\mathrm{l}p}$ is injective but not surjective.*

Proof: Suppose that $T \in L(X,Y)$ and $J\colon Y \to Z$ is an isometric embedding, such that JT is limited p-converging. If $(x_n) \in l_p^{\mathrm{weak}}(X)$ is limited, then $\|JTx_n\| \to 0$ and so $\|Tx_n\| \to 0$ as $n \to 0$. Therefore T belongs to $C_{\mathrm{l}p}$. Hence $C_{\mathrm{l}p}$ is injective.

Now assume that X is a Banach space without the p-GP property. Then the identity operator $i\colon X \to X$ is not limited p-converging. On the other hand, one define $\Phi\colon l_1(B_X) \to X$ via

$$\Phi(\varphi) = \sum_{x \in B_X} \varphi(x)x, \qquad \varphi \in l_1(B_X).$$

It is easy to see that Φ is a surjective operator. Thus the Schur property and so the p-GP property of $l_1(B_X)$ imply that the operator $\Phi = i\Phi$ belongs to $C_{\mathrm{l}p}$, while the identity operator i does not. Hence $C_{\mathrm{l}p}$ is not surjective. $\square$

Theorem 2.17. *The Banach space X has the p-GP property if and only if $L(X,Y) = C_{\mathrm{l}p}(X,Y)$ for every Banach space Y.*

Proof: Suppose that X has the p-GP property. If $T \in L(X,Y)$ and $(x_n) \in l_p^{\mathrm{weak}}(X)$ is a limited sequence, then $\|x_n\| \to 0$. Hence $\|Tx_n\| \to 0$.

Conversely, if $Y = X$, then the identity operator on X belongs to $C_{\mathrm{l}p}$. Therefore X has the limited p-Schur property. $\square$

Similarly, we can prove that the Banach space X has the p-GP property if and only if $L(Y,X) = C_{\mathrm{l}p}(Y,X)$ for every Banach space Y.

Theorem 2.18. *The Banach space X has the DP_p^* property if and only if $L(X,Y) = C_p(X,Y)$ for every p-GP Banach space Y.*

Proof: Assume that X has the DP_p^* property and Y is a p-GP space. Consider limited sequence $(x_n) \in l_p^{\mathrm{weak}}(X)$. Then for every operator $T \in L(X,Y)$, $(Tx_n) \in l_p^{\mathrm{weak}}(Y)$ is a limited sequence. So $\|T(x_n)\| \to 0$ and by Theorem 2.3 $T \in C_p(X,Y)$.

Conversely suppose that $Y = c_0$, $(x_n) \in l_p^{\mathrm{weak}}(X)$ and (x_n^*) is a weak* null sequence in X^*. Define $T\colon X \to c_0$ by $Tx = (\langle x, x_n^*\rangle)$. Then by assumption, $\|Tx_n\| \to 0$. Therefore

$$|\langle x_n, x_n^*\rangle| \le \sup_k |\langle x_n, x_k^*\rangle| = \|Tx_n\| \to 0$$

as $n \to \infty$. By Theorem 2.2, X has the DP_p^* property. $\square$

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