A study of universal elements in classes of bases of topological spaces

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Abstract. The universality problem focuses on finding universal spaces in classes of topological spaces. Moreover, in "Universal spaces and mappings" by S. D. Iliadis (2005), an important method of constructing such universal elements in classes of spaces is introduced and explained in details. Simultaneously, in "A topological dimension greater than or equal to the classical covering dimension" by D. N. Georgiou, A. C. Megaritis and F. Sereti (2017), new topological dimension is introduced and studied, which is called quasi covering dimension and is denoted by $\dim_q$. In this paper, we define the base dimension-like function of the type $\dim_q$, denoted by $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$, and study the property of universality for this function. Especially, based on the method of "Universal spaces and mappings" by S. D. Iliadis (2005), we prove that in classes of bases which are determined by $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$ there exist universal elements.

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1. Introduction

The small inductive dimension (ind), the large inductive dimension (Ind) and the covering dimension (dim) are three topological dimensions which have been studied in details, see for example [9], [10], [1], [11].

Recently, a new dimension for topological spaces, called quasi covering dimension ($\dim_q$) was introduced, proving that it is always greater than or equal to dim and many properties of this dimension have been studied, see [5]. Also, this dimension has been studied in the view of matrix theory, giving algorithms which compute this dimension for finite spaces, see for example [2], [3], [4].

Moreover, the notion of universality for spaces attracts the interest of finding universal spaces in various classes of spaces, see [8], [6], [7]. A space T is said to be *universal* in a class $\mathbb{P}$ of spaces if: (a) $T \in \mathbb{P}$ and (b) for every $X \in \mathbb{P}$ there

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exists an embedding of X into T. A space T, which satisfies the condition (b) only, is said to be *containing* for the class $\mathbb{P}$.

For example, for a fixed infinite cardinal τ, in the class of all T_0-spaces of weight less than or equal to τ, in the class of all regular T_0-spaces of weight less than or equal to τ and in the class of all completely regular T_0-spaces of weight less than or equal to τ there exist universal elements, see [7].

However, in [8] the author gives a different approach to the universality problem. Given an indexed collection $\mathbf{S}$ of T_0-spaces of weight less than or equal to a given infinite cardinal τ, the author constructs by a standard manner some containing T_0-spaces for $\mathbf{S}$ of weight less than or equal to τ. This method is proved to be a significant tool for the study of the universality problem, see [8], [6], [7].

In addition, the universality problem is also investigated for classes of bases of topological spaces. This study leads to the definition of base dimension-like functions of the types ind, Ind and dim, see [8], verifying the universality property in classes of bases which are characterized by these dimensions.

In this paper, based on the topological dimension quasi covering dimension, $\dim_q$, we define the base dimension-like function of the type $\dim_q$ and we prove the existence of universal elements in classes of bases that are determined by this function. This result is based on the method of constructing containing spaces as it is presenting in [8].

2. Preliminaries – a method of constructing containing spaces

In this section, we recall basic definitions, notations and the method of constructing containing spaces that are needed for this paper and for more details we refer to [8].

We agree that throughout the paper we denote by τ a fixed infinite cardinal, by $\mathcal{F}$ the set of all finite subsets of τ and we assume that all spaces are T_0-spaces of weight less than or equal to τ. We shall use the symbol "$\equiv$" in a relation, meaning that one or both sides of the relation are new notations.

Moreover, we shall use indexed sets. An *indexed set* is a mapping F of a set Λ into a set Y. Hence, F is a subset of $\Lambda \times Y$ and its elements have the form (λ, y_λ), where $\lambda \in \Lambda$ and $y_\lambda = F(\lambda)$. However, for the simplicity of notations each element (λ, y_λ) of F is identified with the element y_λ of Y and the indexed set F is denoted by $\{y_\lambda : \lambda \in \Lambda\}$.

An indexed set $F\colon \Lambda \to Y$ is said to be an *indication* of Y if $F(\Lambda) = Y$ and the set Λ is called the *indexing set* of F. Usually, in order to emphasize the use of the set Λ, F is called a Λ-indexed set.

A family B of open subsets of a space X is said to be a *base* for X if any open subset of X is a union of elements of B.

Any τ-indexed base $M = \{U_\delta \colon \delta \in \tau\}$ of X is called a *mark* of X and a space X is called *marked* if a mark of X is chosen.

Let $\mathbf{S}$ be an indexed collection of spaces. An $\mathbf{S}$-indexed collection

$$\mathbf{B} \equiv \{B^X \colon X \in \mathbf{S}\},$$

where B^X is a base for X of cardinality less than or equal to τ, is called a *co-base* for $\mathbf{S}$.

An $\mathbf{S}$-indexed collection $\mathbf{M} \equiv \{M^X \colon X \in \mathbf{S}\}$ of τ-indexed families is said to be a *co-indication* of an $\mathbf{S}$-indexed set $\mathbf{B} \equiv \{B^X \colon X \in \mathbf{S}\}$ of families if M^X is an indication of B^X for every $X \in \mathbf{S}$.

Also, an $\mathbf{S}$-indexed collection

$$\mathbf{M} \equiv \{\{U_\delta^X \colon \delta \in \tau\} \colon X \in \mathbf{S}\}, \tag{2.1}$$

where $\{U_\delta^X \colon \delta \in \tau\}$ is a τ-indexed base for X, is called a *co-mark* of $\mathbf{S}$. The co-mark $\mathbf{M}$ of $\mathbf{S}$ is said to be a *co-extension* of a co-mark

$$\mathbf{M}^+ \equiv \{\{V_\delta^X \colon \delta \in \tau\} \colon X \in \mathbf{S}\}$$

of $\mathbf{S}$ if there exists a one-to-one mapping θ of τ into itself such that for every $X \in \mathbf{S}$ and for every $\delta \in \tau$, $V_\delta^X = U_{\theta(\delta)}^X$. The corresponding mapping θ is called an *indicial mapping from* $\mathbf{M}^+$ *to* $\mathbf{M}$.

We shall deal with $\mathcal{F}$-indexed families of equivalence relations on $\mathbf{S}$. For such a family $\mathrm{R} \equiv \{\sim^s \colon s \in \mathcal{F}\}$ we shall write

$$\mathrm{C}(\mathrm{R}) = \bigcup\{\mathrm{C}(\sim^s) \colon s \in \mathcal{F}\},$$

where $\mathrm{C}(\sim^s)$ denotes the set of all equivalence classes of the relation "$\sim^s$". We will, frequently, refer to the minimal ring of subsets of $\mathbf{S}$ containing $\mathrm{C}(\mathrm{R})$, which is denoted by $\mathrm{C}^\diamondsuit(\mathrm{R})$.

Let $\mathrm{R}_1 \equiv \{\sim_1^s \colon s \in \mathcal{F}\}$ and $\mathrm{R}_0 \equiv \{\sim_0^s \colon s \in \mathcal{F}\}$ be two $\mathcal{F}$-indexed families of equivalence relations on $\mathbf{S}$. It is said that R_1 is a *final refinement* of R_0 if for every $s \in \mathcal{F}$, there exists $t \in \mathcal{F}$ such that $\sim_1^t \subset \sim_0^s$.

An $\mathcal{F}$-indexed family $\mathrm{R} \equiv \{\sim^s \colon s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{S}$ is said to be *admissible* if the following conditions are satisfied:

(a) $\sim^\emptyset = \mathbf{S} \times \mathbf{S}$;
(b) for every $s \in \mathcal{F}$ the number of $\sim^s$-equivalence classes is finite; and
(c) $\sim^s \subset \sim^t$, if $t \subset s$.

For every $s \in \mathcal{F} \setminus \{\emptyset\}$, on the class of all marked spaces an equivalence relation, denoted by $\sim_m^s$, is defined as follows, see [8]: Two marked spaces X and Y are $\sim_m^s$*-equivalent* if there exists an isomorphism i of the algebra of subsets of X

generated by the set $\{U^X_\delta : \delta \in s\}$ onto the algebra of subsets of Y generated by the set $\{U^Y_\delta : \delta \in s\}$ such that $i(U^X_\delta) = U^Y_\delta$ for every $\delta \in s$. This relation is called *s-standard* and the isomorphism i is called *natural*.

We consider the co-mark (2.1) of $\mathbf{S}$. For every $s \in \mathcal{F} \setminus \{\emptyset\}$, the s-standard equivalence relation "$\sim^s_m$" defines on $\mathbf{S}$ an equivalence relation, denoted by "$\sim^s_{\mathbf{M}}$", as follows, see [8]:
Two elements X and Y of $\mathbf{S}$ are *$\sim^s_{\mathbf{M}}$-equivalent* if and only if the marked spaces X and Y are $\sim^s_m$-equivalent (that is, $\sim^s_{\mathbf{M}} = \sim^s_m \cap (\mathbf{S}\times\mathbf{S})$). We also set $\sim^\emptyset_{\mathbf{M}} = \mathbf{S}\times\mathbf{S}$.

We denote by

$$\mathrm{R}_{\mathbf{M}} \equiv \{\sim^s_{\mathbf{M}} : s \in \mathcal{F}\}$$

the indexed family of equivalence relations "$\sim^s_{\mathbf{M}}$" on $\mathbf{S}$, which is called **M**-*standard*. An admissible family R of equivalence relations on $\mathbf{S}$ is said to be **M**-*admissible* if R is a final refinement of $\mathrm{R}_{\mathbf{M}}$.

Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an **M**-admissible family of equivalence relations on $\mathbf{S}$. On the set of all pairs (x, X), where $X \in \mathbf{S}$ and $x \in X$, we consider an equivalence relation, denoted by "$\sim^{\mathbf{M}}_{\mathrm{R}}$", as follows: $(x, X) \sim^{\mathbf{M}}_{\mathrm{R}} (y, Y)$ if and only if $X \sim^s Y$ for every $s \in \mathcal{F}$, and either $x \in U^X_\delta$ and $y \in U^Y_\delta$ or $x \notin U^X_\delta$ and $y \notin U^Y_\delta$ for every $\delta \in \tau$.

The set of all equivalence classes of the relation "$\sim^{\mathbf{M}}_{\mathrm{R}}$" is denoted by $\mathrm{T}(\mathbf{M}, \mathrm{R})$ or simply by T, that is $\mathrm{T} = \mathrm{C}(\sim^{\mathbf{M}}_{\mathrm{R}})$. (It is assumed that $\mathrm{T}(\mathbf{M}, \mathrm{R}) = \emptyset$, if all elements of $\mathbf{S}$ are empty.) See [8].

For every $\mathbf{H} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ the set of all $\mathbf{a} \in \mathrm{T}(\mathbf{M}, \mathrm{R})$ for which there exists an element $(x, X) \in \mathbf{a}$ such that $X \in \mathbf{H}$ is denoted by $\mathrm{T}(\mathbf{M}, \mathrm{R}, \mathbf{H}) \equiv \mathrm{T}(\mathbf{H})$. For every $\delta \in \tau$ and $\mathbf{H} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ we denote by $U^{\mathrm{T}}_\delta(\mathbf{H})$ the set of all $\mathbf{a} \in \mathrm{T}(\mathbf{M}, \mathrm{R})$ for which there exists an element $(x, X) \in \mathbf{a}$ such that $X \in \mathbf{H}$ and $x \in U^X_\delta$.

We denote by $\mathrm{B}^{\mathrm{T}}_*$ the set of all sets of the form $U^{\mathrm{T}}_\delta(\mathbf{H})$, where $\delta \in \tau$, $\mathbf{H} \in \mathrm{C}(\sim^t)$ and $\sim^t \subset \sim^{\{\delta\}}_{\mathbf{M}}$. The set $\mathrm{B}^{\mathrm{T}}_*$ is a base for a topology on the set T, see Lemma 1.2.6 of [8]. The set T equipped with the topology for which the set $\mathrm{B}^{\mathrm{T}}_*$ is a base will be called *containing space for the indexed collection* $\mathbf{S}$ *corresponding to the co-mark* $\mathbf{M}$ *and the family* R. Since $|\mathrm{B}^{\mathrm{T}}_*| \leqslant \tau$, the weight of T is $\leqslant \tau$.

The subspaces of T of the form $\mathrm{T}(\mathbf{L})$, where $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$, are said to be *primary subspaces* of T.

For every subset κ of τ and $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ we set

(1) $\mathrm{B}^{\mathrm{T}}_\kappa \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) : \delta \in \kappa \text{ and } \mathbf{H} \in \mathrm{C}(\mathrm{R})\}$;
(2) $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa} \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) : \delta \in \kappa \text{ and } \mathbf{H} \in \mathrm{C}^\diamondsuit(\mathrm{R})\}$;
(3) $\mathrm{B}^{\mathbf{L}}_\kappa \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) \in \mathrm{B}^{\mathrm{T}}_\kappa : \mathbf{H} \subset \mathbf{L}\}$;
(4) $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa} \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) \in \mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa} : \mathbf{H} \subset \mathbf{L}\}$.

If for every $X \in \mathbf{S}$ the family $\{U^X_\delta : \delta \in \kappa\}$ is a base for X, then the families $\mathrm{B}^{\mathrm{T}}_\kappa$ and $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa}$ are bases for the space T and the families $\mathrm{B}^{\mathbf{L}}_\kappa$ and $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa}$ are bases for the space $\mathrm{T}(\mathbf{L})$, see Corollary 1.2.8 of [8]. The families $\mathrm{B}^{\mathrm{T}}_{\kappa}$ and $\mathrm{B}^{\mathbf{L}}_{\kappa}$ (or $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa}$ and $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa}$) are called *$\kappa$-standard bases* (*$(\diamondsuit,\kappa)$-standard bases*, respectively) for the corresponding spaces T and $\mathrm{T}(\mathbf{L})$.

For every element X of $\mathbf{S}$ there exists a *natural embedding* e^X_{T} of X into the space $\mathrm{T}(\mathbf{M},\mathrm{R})$ defined as follows: for every $x \in X$, $e^X_{\mathrm{T}}(x) = \mathbf{a}$, where $\mathbf{a}$ is the element of $\mathrm{T}(\mathbf{M},\mathrm{R})$ containing the pair (x, X), see Proposition 1.2.10 of [8].

We suppose that for every $X \in \mathbf{S}$ a subset Q^X of X is given. The $\mathbf{S}$-indexed set

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\} \tag{2.2}$$

is called a *restriction* of $\mathbf{S}$. Especially, a restriction $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ of $\mathbf{S}$ is called *open* (or *closed*) if for every $X \in \mathbf{S}$, Q^X is an open (a closed, respectively) subset of X.

We consider the restriction (2.2) of $\mathbf{S}$. The *trace on* $\mathbf{Q}$ *of the co-mark* $\mathbf{M}$ *of* $\mathbf{S}$ is the co-mark

$$\mathbf{M}|_{\mathbf{Q}} \equiv \{\{U^X_\delta \cap Q^X : \delta \in \tau\} : Q^X \in \mathbf{Q}\}$$

of $\mathbf{Q}$. The *trace on* $\mathbf{Q}$ *of an equivalence relation "$\sim$" on* $\mathbf{S}$ is the equivalence relation on $\mathbf{Q}$ denoted by $\sim|_{\mathbf{Q}}$ and defined as follows: $Q^X \sim|_{\mathbf{Q}} Q^Y$ if and only if $X \sim Y$.

Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an $\mathcal{F}$-indexed family of equivalence relations on $\mathbf{S}$. The *trace on* $\mathbf{Q}$ *of the family* R is the $\mathcal{F}$-indexed family

$$\mathrm{R}|_{\mathbf{Q}} \equiv \{\sim^s|_{\mathbf{Q}} : s \in \mathcal{F}\}$$

of equivalence relations on $\mathbf{Q}$. The *trace on* $\mathbf{Q}$ *of an element* $\mathbf{H}$ *of* $\mathrm{C}^{\diamondsuit}(\mathrm{R})$ is the element

$$\mathbf{H}|_{\mathbf{Q}} \equiv \{Q^X \in \mathbf{Q} : X \in \mathbf{H}\}$$

of $\mathrm{C}^{\diamondsuit}(\mathrm{R}|_{\mathbf{Q}})$.

An $\mathbf{M}$-admissible family R of equivalence relations on $\mathbf{S}$ is said to be $(\mathbf{M},\mathbf{Q})$-*admissible* if $\mathrm{R}|_{\mathbf{Q}}$ is an $\mathbf{M}|_{\mathbf{Q}}$-admissible family of equivalence relations on $\mathbf{Q}$.

If R is an $(\mathbf{M},\mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$, then we can consider the containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}},\mathrm{R}|_{\mathbf{Q}})$ for the indexed collection $\mathbf{Q}$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{Q}}$ and the $\mathbf{M}|_{\mathbf{Q}}$-admissible family $\mathrm{R}|_{\mathbf{Q}}$. The containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}},\mathrm{R}|_{\mathbf{Q}})$ is denoted briefly by $\mathrm{T}|_{\mathbf{Q}}$.

There exists a natural embedding of $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}},\mathrm{R}|_{\mathbf{Q}})$ into $\mathrm{T}(\mathbf{M},\mathrm{R})$. So we can consider the containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}},\mathrm{R}|_{\mathbf{Q}})$ as a subspace of the space $\mathrm{T}(\mathbf{M},\mathrm{R})$. The subsets of this form will be called *specific subsets* of $\mathrm{T}(\mathbf{M},\mathrm{R})$.

Also, if $\mathbf{L}$ is an element of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$ and $\mathbf{E} = \mathbf{L}|_{\mathbf{Q}}$, consequently the subspace $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}},\mathrm{R}|_{\mathbf{Q}},\mathbf{L}|_{\mathbf{Q}})$ of $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}},\mathrm{R}|_{\mathbf{Q}})$ is denoted by $\mathrm{T}(\mathbf{E}) \equiv \mathrm{T}(\mathbf{L}|_{\mathbf{Q}})$.

Let R be an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. A restriction $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ is said to be an $(\mathbf{M}, \mathrm{R})$-*complete restriction* if the family R is $(\mathbf{M}, \mathbf{Q})$-admissible and the subset $\mathrm{T}|_{\mathbf{Q}}$ of T satisfies the following condition: for every point $\mathbf{a}$ of $\mathrm{T}|_{\mathbf{Q}}$ and for every element (x, X) of $\mathbf{a}$ we have $x \in Q^X$.

In our consideration, by a *class of subsets* we mean a class $\mathbb{F}$ consisting of pairs (Q, X), where Q is a subset of a space X. Let $\mathbb{F}$ be a class of subsets. A restriction $\mathbf{Q}$ of $\mathbf{S}$ is said to be an $\mathbb{F}$-*restriction* if $(Q^X, X) \in \mathbb{F}$ for every $X \in \mathbf{S}$. Also, a restriction $\mathbf{Q}$ of $\mathbf{S}$ is said to be *complete* if there exists a co-mark $\mathbf{M}$ of $\mathbf{S}$ and an $(\mathbf{M}, \mathbf{Q})$-admissible family R of equivalence relations on $\mathbf{S}$ such that $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction. A class $\mathbb{F}$ of subsets is said to be *complete* if for every indexed collection $\mathbf{S}$ of spaces any $\mathbb{F}$-restriction is complete.

An element (Q^T, T) of a class $\mathbb{F}$ of subsets is said to be *universal* in $\mathbb{F}$ if for every $(Q^Z, Z) \in \mathbb{F}$ there exists an embedding h of Z into T such that $Q^Z \subseteq h^{-1}(Q^T)$.

Definition 2.1 ([8])**.** A nonempty class $\mathbb{F}$ of subsets is said to be *saturated* if for every indexed collection $\mathbf{S}$ of spaces and for every $\mathbb{F}$-restriction $\mathbf{Q}$ of $\mathbf{S}$, there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$, there exists an $(\mathbf{M}, \mathbf{Q})$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and elements $\mathbf{L}$ and $\mathbf{H}$ of $\mathrm{C}^\diamondsuit(\mathrm{R})$ for which $\mathbf{L} \subset \mathbf{H}$, we have $(\mathrm{T}(\mathbf{H}|_{\mathbf{Q}}), \mathrm{T}(\mathbf{L})) \in \mathbb{F}$.

The co-mark $\mathbf{M}^+$ is called an *initial co-mark corresponding to the* $\mathbb{F}$*-restriction* $\mathbf{Q}$ (or *corresponding to the restriction* $\mathbf{Q}$ *and the class* $\mathbb{F}$). Also the family R^+ is called an *initial family corresponding to the co-mark* $\mathbf{M}$ *and the* $\mathbb{F}$*-restriction* $\mathbf{Q}$ (or *corresponding to the co-mark* $\mathbf{M}$*, the restriction* $\mathbf{Q}$ *and the class* $\mathbb{F}$).

By a *class of bases* we mean a class consisting of pairs (B, X), where B is a base for a space X such that $\emptyset, X \in B$. Let $\mathbb{D}$ be a class of bases. We say that a base B for a space X is a $\mathbb{D}$-*base* if $(B, X) \in \mathbb{D}$. A co-base $\mathbf{B} \equiv \{\mathrm{B}^X : X \in \mathbf{S}\}$ for a collection $\mathbf{S}$ of spaces is said to be a $\mathbb{D}$-*co-base* if for every $X \in \mathbf{S}$, B^X is a $\mathbb{D}$-*base* for X.

An element (B^T, T) of a class $\mathbb{D}$ of bases is said to be *universal* in $\mathbb{D}$ if for every $(B^X, X) \in \mathbb{D}$ there exists an embedding h of X into T such that $B^X = \{h^{-1}(V) : V \in B^T\}$.

Definition 2.2 ([8])**.** A class $\mathbb{D}$ of bases is said to be *saturated* if for every indexed collection $\mathbf{S}$ of spaces, for every $\mathbb{D}$-co-base $\mathbf{B}$ for $\mathbf{S}$ and for every co-indication $\mathbf{N}$ of $\mathbf{B}$, there exists a co-extension $\mathbf{M}^+$ of $\mathbf{N}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$, there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every element $\mathbf{L} \in \mathrm{C}^{\diamond}(\mathrm{R})$, we have $(\mathrm{B}^{\mathbf{L}}_{\diamond,\theta(\tau)}, \mathrm{T}(\mathbf{L})) \in \mathbb{D}$, where θ is an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

The co-mark $\mathbf{M}^+$ is called an *initial co-mark corresponding to the co-indication* $\mathbf{N}$ *of* $\mathbf{B}$ *and the class* $\mathbb{D}$. Also the family R^+ is called an *initial family corresponding to the co-mark* $\mathbf{M}$*, the co-indication* $\mathbf{N}$ *of* $\mathbf{B}$*, and the class* $\mathbb{D}$.

Facts. In [8] the following results are proved:

(1) The class of all bases is saturated.
(2) The nonempty intersection of not more than τ saturated classes of bases is also a saturated class of bases.
(3) In any saturated class of bases there exist universal elements.
(4) The class of all subsets is saturated.
(5) The nonempty intersection of not more than τ saturated classes of subsets is also a saturated class of subsets.
(6) In any saturated class of subsets there exist universal elements.

In [5] the notion of a new topological dimension, called quasi covering dimension and denoted by $\dim_q$, is introduced, introducing firstly the meaning of the quasi cover.

Let X be a topological space. A *cover* of X is a nonempty set of subsets of X, whose union is X. If B is a base for X, then a cover c of X is said to be B*-cover* if all elements of c are elements of B.

A *quasi cover* of X is a nonempty set of subsets of X, whose union is a dense element of X. Two quasi covers c_1 and c_2 are said to be *similar* (in short, $c_1 \sim c_2$) if their unions are the same dense subset of X.

A family r of subsets of X is said to be a *refinement* of a family c of subsets of X (in short, $r \prec c$) if each element of r is contained in an element of c.

If $\mathbb{F}$ is a class of subsets, then a cover c of a space X is said to be $\mathbb{F}$*-cover* if $(V, X) \in \mathbb{F}$ for every $V \in c$. An $\mathbb{F}$-cover of a space X, which is a refinement of a cover c of X, is called $\mathbb{F}$*-refinement* of c.

Moreover, it is said that $\mathbb{F}$ satisfies the *finite union condition* if the conditions $(F_i, X) \in \mathbb{F}$, $i \in j \in \omega$, imply that $\left(\bigcup\{F_i \colon i \in j\}, X\right) \in \mathbb{F}$. It is said that $\mathbb{F}$ satisfies the *empty subset condition* if the condition $(Q, X) \in \mathbb{F}$ implies that $(\emptyset, X) \in \mathbb{F}$.

We also recall the notion of the order. The *order* of a family r of a space X is:

(a) The integer -1 if r consists of the empty set only (and therefore, $X = \emptyset$).
(b) An integer $n \in \omega$ if the intersection of any $n+2$ distinct elements of r is empty and there exists $n+1$ distinct elements of r, whose intersection is not empty.
(c) The symbol ∞ if for every $n \in \omega$, there exist n distinct elements of r, whose intersection is not empty.

Definition 2.3 ([5]). The *quasi covering dimension* ($\dim_q$) of a space X is defined as follows:

(1) $\dim_q(X) \leqslant n$, where $n \in \{-1, 0, 1, \ldots\}$, if and only if for every finite open quasi cover c of X, there exists a finite open quasi cover r of X, such that $r \prec c$, $r \sim c$ and the order of r is less than or equal to n.

(2) $\dim_q(X) = n$, where $n \in \{0, 1, 2, \ldots\}$, if and only if $\dim_q(X) \leqslant n$ and $\dim_q(X) \nleqslant n-1$.

If there exists no integer n for which $\dim_q(X) \leqslant n$, then we put $\dim_q(X) = \infty$.

3. Base dimension-like function of the type $\dim_q$

In this section we define the base dimension-like function of the type $\dim_q$ and based on the method of constructing containing spaces as it is presenting in Section 2, we prove the universality property for this dimension.

Let B be a base for a space X. A quasi cover c of X is said to be B-*quasi cover* if all elements of c are elements of B.

Let $\mathbb{F}$ be a class of subsets. A quasi cover c of a space X is said to be $\mathbb{F}$-*quasi cover* if $(V, X) \in \mathbb{F}$ for every $V \in c$. An $\mathbb{F}$-quasi cover of a space X, which is refinement of a quasi cover c of a space X and is similar to c, is called $\mathbb{F}$-*quasi refinement.*

Definition 3.1. For every class $\mathbb{F}$ of subsets we denote by $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$ the (unique) function that has as domain the class of all bases and as range the set $\{-1, \infty\} \cup \omega$ satisfying the following condition:

$$\mathrm{b}\text{-}\dim_q^{\mathbb{F}}(B, X) \leqslant n, \qquad n \in \{-1\} \cup \omega,$$

if and only if for every finite B-quasi cover c of the space X, there exists an $\mathbb{F}$-quasi refinement of c, which has order less than or equal to n. If there exists no integer n for which $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}(B, X) \leqslant n$, then we put $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}(B, X) = \infty$.

Theorem 3.2. *For every $n \in \{-1\} \cup \omega$ the class $\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathbb{F}} \leqslant n)$ of all bases (B, X) with $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}(B, X) \leqslant n$ is saturated, provided that $\mathbb{F}$ is a complete and saturated class of subsets satisfying the finite union condition and the empty subset condition.*

Proof: Let $\mathbb{F}$ be a complete and saturated class of subsets satisfying the finite union condition and the empty subset condition and let n be an element of $\{-1\} \cup \omega$. We shall prove that the class

$$\mathbb{P}_n \equiv \mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathbb{F}} \leqslant n)$$

is saturated. Let $\mathbf{S}$ be an indexed collection of spaces, $\mathbf{B} \equiv \{B^X : X \in \mathbf{S}\}$ a $\mathbb{P}_n$-co-base for $\mathbf{S}$ and

$$\mathbf{N} \equiv \{\{V^X_\varepsilon : \varepsilon \in \tau\} : X \in \mathbf{S}\}$$

a co-indication of $\mathbf{B}$. Thus, for every $X \in \mathbf{S}$ we have $\mathrm{b}\text{-}\dim^{\mathbb{F}}_q(B^X, X) \leqslant n$. We prove that there exists a co-extension $\mathbf{M}^+$ of $\mathbf{N}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$, we have $(\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\theta(\tau)}, \mathrm{T}(\mathbf{L})) \in \mathbb{P}_n$, where θ is an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

Firstly, for every $q \in \mathcal{F} \setminus \{\emptyset\}$ and for every $\eta \in q$ we construct an $\mathbb{F}$-restriction of $\mathbf{S}$:

$$\mathbf{W}(q, \eta) \equiv \{W^X(q, \eta) : X \in \mathbf{S}\}.$$

Let $q = \{\eta_0, \ldots, \eta_k\}$ be an element of $\mathcal{F} \setminus \{\emptyset\}$. For every $X \in \mathbf{S}$ we consider the indexed set

$$\mathcal{V}^X(q) \equiv \{V^X_\eta : \eta \in q\}.$$

If it is not true that the set $\mathcal{V}^X(q)$ is a quasi cover of X, then for every $\eta \in q$ we set $W^X(q, \eta) = \emptyset$. We suppose that the set $\mathcal{V}^X(q)$ is a quasi cover (and therefore, a B^X-quasi cover) of X. Since $\mathrm{b}\text{-}\dim^{\mathbb{F}}_q(B^X, X) \leqslant n$, there exists an $\mathbb{F}$-quasi refinement r^X_q of $V^X(q)$, which has order less than or equal to n. We set

$$W^X(q, \eta_0) = \bigcup \{O \in r^X_q : O \subseteq V^X_{\eta_0}\}.$$

If there is no such elements $O \in r^X_q$, then it is supposed that $W^X(q, \eta_0) = \emptyset$. Also, we set

$$W^X(q, \eta_i) = \bigcup \{O \in r^X_q : O \subseteq V^X_{\eta_i} \text{ and } O \nsubseteq V^X_\eta \text{ for every } \eta \in \{\eta_0, \ldots, \eta_{i-1}\}\}$$

for every $i \in \{1, \ldots, k\}$. If there is no such elements $O \in r^X_q$, it is supposed that $W^X(q, \eta_i) = \emptyset$. Since r^X_q is an $\mathbb{F}$-quasi cover of order less than or equal to n and $\mathbb{F}$ satisfies the finite union condition and the empty subset condition, the set $W^X(q) \equiv \{W^X(q, \eta) : \eta \in q\}$ is also $\mathbb{F}$-quasi cover of X, similar to r^X_q $\bigl($since $\bigcup\{O : O \in r^X_q\} = \bigcup\{W^X(q, \eta) : \eta \in q\}\bigr)$ of order less than or equal to n. Moreover, we have $W^X(q, \eta) \subset V^X_\eta$ for every $\eta \in q$. We note that the quasi cover $W^X(q)$ has also the following property: if $\eta_0, \ldots, \eta_{n+1}$ are distinct elements of q such that $W^X(q, \eta_i) \neq \emptyset$ for every $i \in \{0, \ldots, n+1\}$, then $W^X(q, \eta_0), \ldots, W^X(q, \eta_{n+1})$ are distinct elements of $W^X(q)$ and therefore,

$$W^X(q, \eta_0) \cap \ldots \cap W^X(q, \eta_{n+1}) = \emptyset.$$

Let $\mathbf{M}^+$ be a co-mark of $\mathbf{S}$, which is a co-extension of $\mathbf{N}$. (We denote by θ^N an indicial mapping from $\mathbf{N}$ to $\mathbf{M}^+$.) Without loss of generality we can suppose that for every $q \in \mathcal{F} \setminus \{\emptyset\}$ and for every $\eta \in q$, $\mathbf{M}^+$ is an initial co-mark corresponding to the restriction $\mathbf{W}(q,\eta)$ of $\mathbf{S}$ and the class $\mathbb{F}$. Moreover, since $\mathbb{F}$ is a complete class, we can suppose that there exists a family R_0^+ of equivalence relations on $\mathbf{S}$ such that for every $q \in \mathcal{F} \setminus \{\emptyset\}$ and for every $\eta \in q$ we have:

(a) the family R_0^+ is $(\mathbf{M}^+, \mathbf{W}(q,\eta))$-admissible; and
(b) the restriction $W^X(q,\eta)$ of $\mathbf{S}$ is $(\mathbf{M}^+, \mathrm{R}_0^+)$-complete.

We show that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the co-indication $\mathbf{N}$ of $\mathbf{B}$ and the class $\mathbb{P}_n$. Indeed, let

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$. We denote by θ^+ an indicial mapping from $\mathbf{M}^+$ to $\mathbf{M}$. Then, the co-mark $\mathbf{M}$ is a co-extension of the co-mark $\mathbf{N}$ and $\theta \equiv \theta^+ \circ \theta^{\mathbf{N}}$ is an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$. (Therefore, $V_\varepsilon^X = U_{\theta(\varepsilon)}^X$ for every $\varepsilon \in \tau$ and $X \in \mathbf{S}$.) For every $q \in \mathcal{F} \setminus \{\emptyset\}$ and for every $\eta \in q$ we consider a family $\mathrm{R}_{q,\eta}^+$ of equivalence relations on $\mathbf{S}$, which is an initial family corresponding to the co-mark $\mathbf{M}$, the restriction $\mathbf{W}(q,\eta)$ of $\mathbf{S}$ and the class $\mathbb{F}$. We denote by R^+ a family of equivalence relations on $\mathbf{S}$, which is a final refinement of all families $\mathrm{R}_{q,\eta}^+$. In addition, we suppose that R^+ is a final refinement of the family R_0^+.

We prove that R^+ is an initial family of $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $\mathbb{P}_n$. We consider an arbitrary admissible family $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and prove that for every $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ we have $(\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\theta(\tau)}, \mathrm{T}(\mathbf{L})) \in \mathbb{P}_n$, where $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\theta(\tau)}$ is the $(\diamondsuit, \theta(\tau))$-standard base for the subspace $\mathrm{T}(\mathbf{L})$ of T.

Let $\mathbf{L} \in \mathrm{C}^\diamondsuit(\mathrm{R})$ and

$$c \equiv \{U_{\delta_0}^{\mathrm{T}}(\mathbf{L}_0), \ldots, U_{\delta_k}^{\mathrm{T}}(\mathbf{L}_k)\}$$

be a finite $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\theta(\tau)}$-quasi cover of $\mathrm{T}(\mathbf{L})$, where $\mathbf{L}_0 \subset \mathbf{L}, \ldots, \mathbf{L}_k \subset \mathbf{L}$ and where $\delta_0, \ldots, \delta_k \in \theta(\tau)$. Then, the union of the elements of c is an open and dense subset D of $\mathrm{T}(\mathbf{L})$, which is determined by the $(\diamondsuit, \theta(\tau))$-standard base $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\theta(\tau)}$ of $\mathrm{T}(\mathbf{L})$. We set $s = \{\delta_0, \ldots, \delta_k\}$ and $q = \theta^{-1}(s)$. For every $X \in \mathbf{L}$ we denote by q^X the set of all elements η of q such that for some $i \in \{0, \ldots, k\}$, $\theta(\eta) = \delta_i$ and $X \in \mathbf{L}_i$. It is easy to see that $q^X \neq \emptyset$ and that the set

$$V^X(q^X) = \{V_\eta^X : \eta \in q^X\}$$

is a B^X-quasi cover of X. Then the set

$$W^X(q^X) = \{W^X(q^X, \eta) \colon \eta \in q^X\}$$

is an $\mathbb{F}$-quasi cover of X, which is similar to $V^X(q^X)$.

Let t be an element of $\mathcal{F}$ such that if $\mathbf{K}$ is $\sim^t$-equivalence class and $\mathbf{K} \cap \mathbf{L}_i \neq \emptyset$ for some $i \in \{0, \dots, k\}$, then $\mathbf{K} \subset \mathbf{L}_i$. Obviously, if $X \sim^t Y$, then $q^X = q^Y$.

Let $\mathbf{K}$ be an element of $\mathrm{C}(\sim^t)$ such that $\mathbf{K} \subset \mathbf{L}$. We let $q^{\mathbf{K}} = q^X$, where $X \in \mathbf{K}$. By the above, $q^{\mathbf{K}}$ is independent of the element X of $\mathbf{K}$.

We consider the quasi cover

$$c(\mathbf{K}) \equiv \{U^{\mathrm{T}}_{\delta_i}(\mathbf{L}_i) \cap \mathrm{T}(\mathbf{K}) = U^{\mathrm{T}}_{\delta_i}(\mathbf{K}) \colon \delta_i \in \theta(q^{\mathbf{K}}), \mathbf{K} \subset \mathbf{L}_i\}$$

of $\mathrm{T}(\mathbf{K})$. We can see that the union of the elements of $c(\mathbf{K})$ is the open and dense subset $D \cap \mathrm{T}(\mathbf{K}) \equiv D_{\mathbf{K}}$ of $\mathrm{T}(\mathbf{K})$. We prove that

$$r(\mathbf{K}) \equiv \{\mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}}, \eta)} \cap \mathrm{T}(\mathbf{K}) = \mathrm{T}(\mathbf{K}|_{\mathbf{W}(q^{\mathbf{K}}, \eta)}) \colon \eta \in q^{\mathbf{K}}\}$$

is an $\mathbb{F}$-quasi refinement of $c(\mathbf{K})$ of order less than or equal to n.

Firstly, we note that since $\mathbb{F}$ is saturated, by the choice of the family R^+ for every $\eta \in q^{\mathbf{K}}$ we have

$$(\mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}}, \eta)} \cap \mathrm{T}(\mathbf{K}), \mathrm{T}) \in \mathbb{F},$$

that is $r(\mathbf{K})$ is an $\mathbb{F}$-quasi cover, similar to $c(\mathbf{K})$. Also, for every $X \in \mathbf{K}$ and for every $\eta \in q^{\mathbf{K}}$ we have

$$W^X(q^{\mathbf{K}}, \eta) \subset V^X_\eta$$

and, therefore,

$$e^X_{\mathrm{T}}(W^X(q^{\mathbf{K}}, \eta)) \subset e^X_{\mathrm{T}}(V^X_\eta),$$

where e^X_{T} is the natural embedding of X into T. By Corollary 1.3.6 of [8] we have

$$\begin{aligned}
\mathrm{T}(\mathbf{K}|_{\mathbf{W}(q^{\mathbf{K}}, \eta)}) &= \bigcup \{e^X_{\mathrm{T}}(W^X(q^{\mathbf{K}}, \eta)) \colon X \in \mathbf{K}\} \\
&\subset \bigcup \{e^X_{\mathrm{T}}(V^X_\eta) \colon X \in \mathbf{K}\} \\
&= \bigcup \{e^X_{\mathrm{T}}(U^X_{\delta_i}) \colon X \in \mathbf{K}\} = U^{\mathrm{T}}_{\delta_i}(\mathbf{K}),
\end{aligned}$$

where $i \in \{0, \dots, k\}$ such that $\delta_i = \theta(\eta)$ and $\mathbf{K} \subset \mathbf{L}_i$, which means that $r(\mathbf{K})$ is an $\mathbb{F}$-refinement of $c(\mathbf{K})$.

Now, we prove that $r(\mathbf{K})$ has order less than or equal to n. Indeed, in the opposite case, we suppose that there exist $n + 2$ distinct elements $\eta_0, \dots, \eta_{n+1}$ of $q^{\mathbf{K}}$ such that

$$\mathrm{T}(\mathbf{K}|_{\mathbf{W}(q^{\mathbf{K}}, \eta_0)}) \cap \ldots \cap \mathrm{T}(\mathbf{K}|_{\mathbf{W}(q^{\mathbf{K}}, \eta_{n+1})}) \neq \emptyset$$

or

$$(\mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta_0)} \cap \mathrm{T}(\mathbf{K})) \cap \ldots \cap (\mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta_{n+1})} \cap \mathrm{T}(\mathbf{K})) \neq \emptyset.$$

Let $\mathbf{a}$ be an element of the above intersection and let $(x, X) \in \mathbf{a}$. Then

$$\mathbf{a} \in \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta_0)} \cap \ldots \cap \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta_{n+1})}.$$

Since the restrictions $\mathbf{W}(q^{\mathbf{K}}, \eta_i)$, $i \in \{0, \ldots, n+1\}$, are $(\mathbf{M}^+, \mathrm{R}_0^+)$-complete, by Lemma 2.2.4 of [8] these restrictions are also $(\mathbf{M}, \mathrm{R})$-complete. Therefore,

$$x \in W^X(q^X, \eta_0) \cap \ldots \cap W^X(q^X, \eta_{n+1}).$$

Since $W^X(q^X, \eta_0), \ldots, W^X(q^X, \eta_{n+1})$ are distinct elements of $W^X(q^X)$, the above contradicts the fact that the non-indexed set $W^X(q)$ has order less than or equal to n.

We consider the set

$$r \equiv \bigcup \{r(\mathbf{K}) \colon \mathbf{K} \in \mathrm{C}(\sim^t) \ \text{ and } \ \mathbf{K} \subset \mathbf{L}\},$$

which is an $\mathbb{F}$-refinement of c and has order less than or equal to n. We complete our proof, stating that r is similar to c, that is $\bigcup_{V \in r} V = D$.

Indeed, let $\mathbf{a} \in \bigcup_{V \in r} V$. Then there exists $V \in r$ such that $\mathbf{a} \in V$. By consideration of the set r, there exists $\mathbf{K} \in \mathrm{C}(\sim^t)$ with $\mathbf{K} \subset \mathbf{L}$ such that $V \in r(\mathbf{K})$, that is there exists $\eta \in q^{\mathbf{K}}$ such that $V = \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K})$. Thus, $\mathbf{a} \in \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K})$ and since

$$\mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K}) \subseteq \bigcup_{\eta \in q^{\mathbf{K}}} \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K}),$$

we have that $\mathbf{a} \in \bigcup_{\eta \in q^{\mathbf{K}}} \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K})$. Finally, since $r(\mathbf{K})$ is an $\mathbb{F}$-quasi cover of $\mathrm{T}(\mathbf{K})$, which is similar to $c(\mathbf{K})$, we have $\bigcup_{\eta \in q^{\mathbf{K}}} \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K}) = D \cap \mathrm{T}(\mathbf{K})$ and therefore, $\mathbf{a} \in D$.

Conversely, let $\mathbf{a} \in D$. Since $\mathbf{a} \in \mathrm{T}(\mathbf{L})$, there exists $\mathbf{K} \in \mathrm{C}(\sim^t)$ with $\mathbf{K} \subset \mathbf{L}$ such that $\mathbf{a} \in \mathrm{T}(\mathbf{K})$. Then $\mathbf{a} \in D \cap \mathrm{T}(\mathbf{K})$. Since $r(\mathbf{K})$ is an $\mathbb{F}$-quasi cover of $\mathrm{T}(\mathbf{K})$, which is similar to $c(\mathbf{K})$, we have $\bigcup_{\eta \in q^{\mathbf{K}}} \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K}) = D \cap \mathrm{T}(\mathbf{K})$ and therefore, there exists $\eta \in q^{\mathbf{K}}$ such that $\mathbf{a} \in \mathrm{T}|_{\mathbf{W}(q^{\mathbf{K}},\eta)} \cap \mathrm{T}(\mathbf{K})$. Thus, $\mathbf{a} \in V$ for some $V \in r$, that is, $\mathbf{a} \in \bigcup_{V \in r} V$. □

Using the fact that in any saturated class of bases there exist universal elements, see Proposition 2.5.4 of [8], we get the following corollary.

Corollary 3.3. *For every $n \in \{-1\} \cup \omega$, in the class $\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathbb{F}} \leqslant n)$ there exist universal elements.*

Remark 3.4. If $\mathbb{F}$ is the class $\mathbb{P}(\mathrm{Op})$ (or the class $\mathbb{P}(\mathrm{Cl})$ or the class $\mathbb{P}(\mathrm{rCl})$) of subsets consisting of all pairs (Q, X), where Q is an open (closed or regular closed, respectively) subset of X, then $\mathbb{F}$ is a complete and saturated class of subsets, satisfying the finite union condition and the empty subset condition, see [8]. In each of these cases, the function $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$ is denoted by $\mathrm{b}\text{-}\dim_q^{\mathrm{Op}}$, $\mathrm{b}\text{-}\dim_q^{\mathrm{Cl}}$ and $\mathrm{b}\text{-}\dim_q^{\mathrm{rCl}}$, respectively.

Therefore, by Corollary 3.3, we get the following corollary, which verifies the existence of universal elements in different classes of bases.

Corollary 3.5. *For every $n \in \{-1\} \cup \omega$, in the classes*

(1) $\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathrm{Op}} \leqslant n)$,
(2) $\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathrm{Cl}} \leqslant n)$, *and*
(3) $\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathrm{rCl}} \leqslant n)$,

there exist universal elements.

In [8] several base dimension-like functions of types ind, Ind, and dim are defined and studied from the universality point of view. For each base dimension-like function df and for every $m \in \{-1\} \cup \omega$, we denote by $\mathbb{P}(df \leqslant m)$ the class of all bases (B, X) with $df(B, X) \leqslant m$.

Using the fact that the nonempty intersection of not more than τ many saturated class of bases is also a saturated class of bases we obtain the following results. We refer the reader to the book [8] where detail information about mentioned base dimension-like functions can be found, as well as results related to them.

Corollary 3.6. *Let df be one of the base dimension-like functions $\mathrm{b}_0\text{-}\mathrm{ind}_{\mathbb{L}}^{\mathbb{B}}$, $\mathrm{b}_1\text{-}\mathrm{ind}_{\mathbb{L}}^{\mathbb{B}}$, $\mathrm{b}^0\text{-}\mathrm{ind}_{\mathbb{L}}^{\mathbb{B}}$, $\mathrm{b}^1\text{-}\mathrm{ind}_{\mathbb{L}}^{\mathbb{B}}$. If $\mathbb{B}$ and $\mathbb{L}$ are saturated classes of bases, and $\mathbb{F}$ is a complete and saturated class of subsets satisfying the finite union condition and the empty subset condition, then the class*

$$\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathbb{F}} \leqslant n) \cap \mathbb{P}(df \leqslant m), \qquad n, m \in \{-1\} \cup \omega$$

is saturated provided that it is nonempty, and hence in this class there exist universal elements.

Corollary 3.7. *Let df be one of the base dimension-like functions $\mathrm{b}^{\vee}\text{-}\mathrm{Ind}_{\mathbb{B}}^{\mathbb{F}_1}$, $\mathrm{b}\text{-}\mathrm{Ind}_{\mathbb{B}}^{\mathbb{F}_1}$, $\mathrm{b}^{\diamondsuit}\text{-}\mathrm{Ind}_{\mathbb{B}}^{\mathbb{F}_1}$. If $\mathbb{B}$ and $\mathbb{F}_1$ are saturated classes of bases and subsets, respectively, satisfying the finite free union condition, and $\mathbb{F}_2$ is a complete and saturated class of subsets satisfying the finite union condition and the empty subset condition, then the class*

$$\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathbb{F}_2} \leqslant n) \cap \mathbb{P}(df \leqslant m), \qquad n, m \in \{-1\} \cup \omega$$

is saturated provided that it is nonempty, and hence in this class there exist universal elements.

Corollary 3.8. *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are complete and saturated classes of subsets satisfying the finite union condition and the empty subset condition, then the class*

$$\mathbb{P}(\mathrm{b}\text{-}\dim_q^{\mathbb{F}_2} \leqslant n) \cap \mathbb{P}(\mathrm{b}\text{-}\dim^{\mathbb{F}_1} \leqslant m), \qquad n, m \in \{-1\} \cup \omega$$

is saturated provided that it is nonempty, and hence in this class there exist universal elements.

We complete our study, presenting some results for the universality problem on classes of spaces, determined by the quasi covering dimension.

Definition 3.9 ([8])**.** Let $\mathbb{M}$ be a class of bases and df a base dimension-like function. We denote by $\mathbb{M}\text{-}df$ the dimension-like function with as domain the class of all spaces and as range the class $\{-1, \infty\} \cup \mathcal{O}$, where $\mathcal{O}$ is the set of all ordinals, such that for every space X, $\mathbb{M}\text{-}df(X)$ is the minimal element α of $\{-1, \infty\} \cup \mathcal{O}$ for which there exists an element (B, X) of $\mathbb{M}$ such that $df(B, X) \leqslant \alpha$. If there is no such element (B, X) of $\mathbb{M}$, then it is supposed that $\mathbb{M}\text{-}df(X) = \infty$.

Remark 3.10. Based on the above definition, for every class $\mathbb{M}$ of bases we can consider the space dimension-like function $\mathbb{M}\text{-}\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$ such that for every space X, $\mathbb{M}\text{-}\mathrm{b}\text{-}\dim_q^{\mathbb{F}}(X)$ is the minimal element α of $\{-1, \infty\} \cup \mathcal{O}$ for which there exists an element (B, X) of $\mathbb{M}$ with $\dim_q^{\mathbb{F}}(B, X) \leqslant \alpha$, and $\mathbb{M}\text{-}\mathrm{b}\text{-}\dim_q^{\mathbb{F}}(X) = \infty$, if there is no such element (B, X) of $\mathbb{M}$.

If $\mathbb{M}$ is the class of all bases and $\mathbb{F}$ is one of the classes $\mathbb{P}(\mathrm{Op})$, $\mathbb{P}(\mathrm{Cl})$, $\mathbb{P}(\mathrm{rCl})$ of Remark 3.4, then the function $\mathbb{M}\text{-}\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$ is denoted by $\mathrm{s}\text{-}\mathrm{b}\text{-}\dim_q^{\mathrm{Op}}$, $\mathrm{s}\text{-}\mathrm{b}\text{-}\dim_q^{\mathrm{Cl}}$ and $\mathrm{s}\text{-}\mathrm{b}\text{-}\dim_q^{\mathrm{rCl}}$, respectively.

Definition 3.11 ([8])**.** A class $\mathbb{P}$ of spaces is said to be *saturated* if for every indexed collection $\mathbf{S}$ of elements of $\mathbb{P}$, there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every element $\mathbf{L} \in \mathrm{C}^{\diamond}(\mathrm{R})$, the space $\mathrm{T}(\mathbf{L})$ belongs to $\mathbb{P}$.

The co-mark $\mathbf{M}^+$ is called an *initial co-mark of* $\mathbf{S}$ *corresponding to the class* $\mathbb{P}$ and the family R^+ is called an *initial family of* $\mathbf{S}$ *corresponding to the co-mark* $\mathbf{M}$ *and the class* $\mathbb{P}$.

Facts. In [8] the following results are proved:

(1) The class of all spaces is saturated.
(2) The nonempty intersection of not more than τ saturated classes of spaces is also a saturated class of spaces.
(3) In any saturated class of spaces there exist universal elements.

Therefore, since the base dimension-like function $\mathrm{b}\text{-}\dim_q^{\mathbb{F}}$ satisfies the saturation property (Theorem 3.2), and especially, the base dimension-like functions $\mathrm{b}\text{-}\dim_q^{\mathrm{Op}}$, $\mathrm{b}\text{-}\dim_q^{\mathrm{Cl}}$, $\mathrm{b}\text{-}\dim_q^{\mathrm{rCl}}$ satisfy the saturation property, by Proposition 3.3.15 of [8] and the fact that in any saturated class of spaces there exist universal elements, we have the following result.

Proposition 3.12. *For every $n \in \{-1\} \cup \omega$, the classes*

(1) $\mathbb{P}(\mathrm{s}\text{-}\mathrm{b}\text{-}\dim_q^{\mathrm{Op}} \leqslant n)$,
(2) $\mathbb{P}(\mathrm{s}\text{-}\mathrm{b}\text{-}\dim_q^{\mathrm{Cl}} \leqslant n)$, *and*
(3) $\mathbb{P}(\mathrm{s}\text{-}\mathrm{b}\text{-}\dim_q^{\mathrm{rCl}} \leqslant n)$

are saturated classes of spaces and therefore, in these classes there exist universal elements.

It is well known that in the class of all T_0-spaces of weight less than or equal to τ there exist universal elements, see Corollary 2.1.5 of [8]. This implies that there is a containing space for the class of all T_0-spaces of weight less than or equal to τ and $\dim_q \leqslant n$. So the following natural problem arises.

Open problem. Let τ be a fixed infinite cardinal and n a natural number. Does there exist a universal element in the class of all T_0-spaces X of weight less than or equal to τ and $\dim_q(X) \leqslant n$?

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