# On hereditary normality of $\omega^*$, Kunen points and character $\omega_1$

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*Abstract.* We show that $\omega^* \setminus \{p\}$ is not normal, if $p$ is a limit point of some countable subset of $\omega^*$, consisting of points of character $\omega_1$. Moreover, such a point $p$ is a Kunen point and a super Kunen point.

*Keywords:* non-normality point; butterfly point; Kunen point; super Kunen point

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## 1. Introduction

We investigate properties of Čech–Stone compactification $\beta\omega$ of the countable discrete space $\omega = \{0, 1, 2, \dots\}$. One of the most intriguing problems in this area was stated probably around 1960 by L. Gillman in [4] or in [3]: Is $\omega^* \setminus \{p\}$ non-normal for any point $p$ of the remainder $\omega^* = \beta\omega \setminus \omega$? If so, then $p$ is called a non-normality point of $\omega^*$.

Assuming that the continuum hypothesis of CH (axiom of choice) is valid, a positive answer was obtained independently of N. M. Warren in [8] and M. Rajagopalan in [6] in 1972. A. Bešlagić and E. K. van Douwen in [1] in 1990 weakened CH to Martin's axiom (MA).

But so far not much is known within ZFC (Zermelo–Fraenkel set theory with the axiom of choice). Thus $p \in \omega^*$ is called a Kunen point, if there exists a discrete subset $P$ of $\omega^*$ of cardinality $\omega_1$, that is no more than countable outside any open neighbourhood of $p$. Any Kunen point is a non-normality point of $\omega^*$ (E. van Douwen).

A. Szymański in [7] in 2012 proved the same, if $p$ is a non-isolated point of some closed subset of $\omega^*$ of countable $\pi$-weight.

Some other more technical results were obtained in [5].

A. Błaszczyk and A. Szymański in [2] stated in 1980, that $p$ is a non-normality point, if $p$ is a limit point of some countable discrete subset $P$ of $\omega^*$. Now we

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omit the discrete requirement, assuming instead that $P$ consists of points of the character $\omega_1$.

**Theorem 1.** *Let $P$ be a countable subset of $\omega^*$, consisting of points of the character $\omega_1$. Then $\omega^* \setminus \{p\}$ is not normal for any point $p$ of $\omega^*$, which is in the closure of $P$. Moreover, $p$ is a Kunen point and a super Kunen point.*

## 2. Preliminaries

In our article $|N| = \omega$ and $|R| = \mathcal{C}$. Each ordinal number $\alpha$ can be represented in a unique way in the form $\alpha = \beta + n$, where $\beta$ is a limit ordinal and $n \in \omega$. Then $\alpha$ is even if $n$ is even and odd otherwise.

By $[\,]$ we always denote closure operator in $\omega^*$, by $Oa$ – a clopen neighbourhood of $a$, i.e. closed and open in $\omega^*$ set, containing $a$. A set $A$ is strongly discrete if there is a cellular family $\{Oa\colon a \in A\}$. A family $\{O_\alpha a\}_{\alpha<\tau}$ is called a clopen local base in $a$, if each $Oa$ contains some $O_\alpha a$. The minimal cardinality of the local base is called the character in $a$ and denoted $\chi(a)$.

**Definition.** A point $p$ of $\omega^*$ is called a super Kunen point, if there is a strongly discrete subset $P$ of $\omega^*$ of cardinality $\omega_1$, that is no more than countable outside any neighbourhood $Op$.

Of course, any super Kunen point is a Kunen point.

## 3. Proofs

Let from now on $P = \{p_i\colon i < \omega\}$ be a countable subset of $\omega^*$, consisting of points of character $\omega_1$ and let $p$ be any point of $[P] \setminus P$. For every $i < \omega$ assume $\{O_{i\alpha}\colon \alpha < \omega_1\}$ to be a clopen local base of cardinality $\omega_1$ in $p_i$. For any clopen neighbourhood $O$ of $p$ we denote

$$\mathcal{K}(O) = \min\{\lambda < \omega_1\colon \forall\, i < \omega(p_i \in O \to \exists\, \alpha \le \lambda(O_{i\alpha} \subset O))\}.$$

We define a filter $\mathcal{F}$ on $\omega$ as follows:

$$\mathcal{F} = \{\{i \in \omega\colon p_i \in O\}\colon O \text{ is a clopen neighbourhood of } p\}.$$

Some of the following facts are simple and sometimes well-known.

**Lemma 1.** *Every nonempty $G_\delta$-subset of $\omega^*$ has nonempty interior in $\omega^*$.*

**Lemma 2.** *Every point $q$ of $\omega^*$ of character $\omega_1$ is a super Kunen point.*

Proof: Let $\{O_\alpha : \alpha < \omega_1\}$ be a local base in $q$. By the previous lemma for every $\alpha < \omega_1$ we can find a nonempty clopen set $U_\alpha$ so that $q \notin U_\alpha$ and

$$U_\alpha \subset \bigcap_{\beta<\alpha} O_\beta \setminus \bigcup_{\beta<\alpha} U_\beta.$$

For any points $x_\alpha \in U_\alpha$ the set $\{x_\alpha : \alpha < \omega_1\}$ witnesses that $q$ is a super Kunen point. □

**Lemma 3.** *The family* $\{\bigcap_{\alpha<\lambda_i} O_{i\alpha} : i < \omega\}$ *is cellular for some* $\lambda_i < \omega_1$.

Proof: We can choose every $\lambda_i$ so that the set $\bigcap_{\alpha<\lambda_i} O_{i\alpha}$ is disjoint from both $\bigcap_{\beta<\lambda_j} O_{j\beta}$ for every $j < i$ and $\{p_j : j > i\}$. □

**Lemma 4.** *The family* $\{U_{i\alpha} : i < \omega$ *and* $\alpha < \omega_1\}$ *is cellular for some nonempty clopen sets* $U_{i\alpha}$ *such that* $U_{i\alpha} \subset \bigcap_{\beta<\alpha} O_{i\beta}$.

Proof: By the previous lemma we can choose every $U_{i\alpha}$ so that

$$U_{i\alpha} \subset \bigcap \{O_{i\beta} : \beta < \max\{\alpha, \lambda_i\}\},$$

$p_i \notin U_{i\alpha}$ and $U_{i\beta} \cap U_{i\alpha} = \emptyset$ for every $\beta < \alpha$. □

**Lemma 5.** *For every* $\alpha < \omega_1$ *there is a point* $a_\alpha \in \omega^*$ *such that* $a_\alpha \notin [P]$ *and*

$$a_\alpha \in \bigcap_{F\in\mathcal{F}} \Big[\bigcup_{i\in F} U_{i\alpha}\Big].$$

Proof: For $U = \bigcup_{i<\omega} U_{i\alpha}$ assume $D \subset \omega^* \setminus [U]$ to be $\sigma$-compact. Then $X = \omega \cup U \cup D$ is $\sigma$-compact and, so, normal. Since $\omega \subset X$, then $[X]_{\beta\omega} = \beta X$ is a Čech–Stone compactification of $X$. Since $U$ and $D$ are closed in $X$, then $[U] \cap [D] = [U]_{\beta X} \cap [D]_{\beta X} = \emptyset$. Hence $[U]$ is a $P$-set.

In every $U_{i\alpha}$ we can find a cellular family of $C$-many nonempty clopen sets $\{V_{i\beta} : \beta < C\}$ and put $V_\beta = \bigcup_{i<\omega} V_{i\beta}$ for any $\beta < C$. Since $[U] = \beta U$ by the standard arguments, then $[V_\beta]$ are disjoint clopen subsets of $[U]$.

Thus $[V_{\beta_0}] \cap P = \emptyset$ for some $\beta_0 < C$, and by the first paragraph of this proof, this implies $[V_{\beta_0}] \cap [P] = \emptyset$. So we can choose $a_\alpha$ to be any point of $\bigcap_{F\in\mathcal{F}} \big[\bigcup_{i\in F} V_{i\beta_0}\big]$. □

From now on every point $a_\alpha$ satisfies the conditions of Lemma 5.

**Lemma 6.** *Let* $O$ *be any clopen neighbourhood of* $p$. *If* $\alpha > \mathcal{K}(O)$ *for some* $\alpha < \omega_1$, *then* $a_\alpha \in O$.

Proof: For $F = \{i \in \omega : p_i \in O\}$ we get $F \in \mathcal{F}$. For any $i \in F$ there is $\alpha_i \leq \mathcal{K}(O)$ such that $O_{i\alpha_i} \subset O$. Then $\alpha > \alpha_i$ implies $U_{i\alpha} \subset O_{i\alpha_i} \subset O$ by our construction and

$$a_\alpha \in \Big[\bigcup_{i\in F} U_{i\alpha}\Big] \subset \Big[\bigcup_{i\in F} O_{i\alpha_i}\Big] \subset O.$$

$\square$

**Lemma 7.** *The set* $\{a_\alpha\colon \alpha<\omega_1\}$ *is discrete. Hence* $p$ *is a Kunen point.*

PROOF: For any $\alpha<\omega_1$ let $O$ be any clopen neighbourhood of $p$ such that $a_\alpha\notin O$. Then $a_\beta\in O$ for every $\beta>\mathcal{K}(O)$ by the previous lemma. Since the sets

$$C=\bigcup\{U_{i\alpha}\colon i<\omega\} \qquad \text{and} \qquad D=\bigcup\{U_{i\beta}\colon\ i<\omega,\ \beta\neq\alpha \ \text{and}\ \beta\le\mathcal{K}(O)\}$$

are $\sigma$-compact, open and disjoint, $[C]\cap[D]=\emptyset$. Since $a_\alpha\in[C]$ and $a_\beta\in[D]$ if $\beta\neq\alpha$ and $\beta\le\mathcal{K}(O)$, then the open set $\omega^*\setminus\big(O\cup[D]\big)$ contains $a_\alpha$ and non of $a'_\beta s$ for $\beta\neq\alpha$. $\square$

It implies that $\omega^*\setminus\{p\}$ is not normal by E. van Douwen. We shall give now another proof. Denote $A=\{a_\alpha\colon \alpha<\omega_1 \ \text{even}\}$ and $B=\{a_\alpha\colon \alpha<\omega_1\ \text{odd}\}$.

**Lemma 8.** *Since* $p=[A]\cap[B]$, $p$ *is a butterfly-point.*

PROOF: By Lemma 6 we get $p\in[A]\cap[B]$. On the other hand, let $O$ be any clopen neighbourhood of $p$. Then

$$\begin{aligned}[A]\cap[B]\setminus O &\subset [\{a_\alpha\colon \alpha\le\mathcal{K}(O)\ \text{even}\}]\cap[\{a_\alpha\colon \alpha\le\mathcal{K}(O)\ \text{odd}\}]\\ &\subset \Big[\bigcup\{U_{i\alpha}\colon i<\omega\ \text{and}\ \alpha\le\mathcal{K}(O)\ \text{even}\}\Big]\\ &\quad\cap\Big[\bigcup\{U_{i\alpha}\colon i<\omega\ \text{and}\ \alpha\le\mathcal{K}(O)\ \text{odd}\}\Big]=\emptyset,\end{aligned}$$

because $\omega^*$ is an $F$-space. $\square$

**Lemma 9.** *The space* $\omega^*\setminus\{p\}$ *is not normal.*

PROOF: For any continuous map $f\colon \omega^*\setminus\{p\}\to[0,1]$ it is enough to show that $f(A)\cap f(B)\neq\emptyset$.

For every $i<\omega$ we choose $\alpha_i<\omega_1$ so that $p\notin O_{i\alpha_i}$ and put $W=\bigcup_{i\in\omega}O_{i\alpha_i}$. Since $Y=\omega\cup W$ is regular and $\sigma$-compact, it is normal. Since $W$ is closed in $Y$, the restriction $f/W$ has a continuous extension $g\colon Y\to[0,1]$. Since $\omega\subset Y\subset\beta\omega$, then $g$ has a continuous extension $\tilde g\colon \beta\omega\to[0,1]$. For its restriction $h=\tilde g/\omega^*$ onto $\omega^*$ we have $h^{-1}h(p)=\bigcap_{i\in\omega}O_i$ for some clopen $O_i\subset\omega^*$. If $\alpha>\sup_{i<\omega}\alpha_i$ for some $\alpha<\omega_1$, then $a_\alpha\in\big[\bigcup_{i<\omega}U_{i\alpha}\big]\subset\big[\bigcup_{i<\omega}O_{i\alpha_i}\big]$, i.e. $a_\alpha\in[W]\setminus\{p\}$. Since $f/W=h/W$, then $f(a_\alpha)=h(a_\alpha)$. If $\alpha>\sup_{i\in\omega}\mathcal{K}(O_i)$, then $a_\alpha\in\bigcap_{i\in\omega}O_i$ by Lemma 6 and, so, $h(a_\alpha)=h(p)$. But then $h(p)\in f(A)\cap f(B)$. $\square$

**Lemma 10.** *There is a strongly discrete subset of* $\{a_\alpha\colon \alpha<\omega_1\}$ *of cardinality* $\omega_1$.

Proof: We shall construct by induction on $\lambda < \omega_1$ both a countable set $A_\lambda \subset \{a_\alpha \colon \alpha < \omega_1\}$ and a cellular family of clopen neighbourhoods of its points $\mathcal{B}_\lambda = \{Oa \colon a \in A_\lambda\}$ so that $A_\lambda \subsetneq A_\gamma$ if $\lambda < \gamma < \omega_1$ and $Oa \cap P = \emptyset$ for any $Oa \in \mathcal{B}_\lambda$.

First we put $A_0 = \{a_0\}$ and choose $\mathcal{B}_0 = \{Oa_0\}$ so that $Oa_0 \cap P = \emptyset$.

If $A_\alpha$ and $\mathcal{B}_\alpha$ have been constructed for every $\alpha < \lambda$ for some limit ordinal $\lambda < \omega_1$, then we put $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ and $\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$.

Assume $A_\lambda$ and $B_\lambda$ have been constructed for some ordinal $\lambda < \omega_1$. Then $V_a = \omega^* \setminus Oa$ is a clopen neighbourhood of $P$ for each $Oa \in \mathcal{B}_\lambda$. Let $\alpha > \sup_{a\in A_\lambda} \mathcal{K}(V_a)$ for some $\alpha < \omega_1$. Then $a_\alpha \in \left[\bigcup_{i\in\omega} U_{i\alpha}\right]$. For any $i \in \omega$ and $a \in A_\lambda$ we have $U_{i\alpha} \subset O_{i\beta} \subset V_a$ for some $\beta \le \mathcal{K}(V_a)$, i.e. $U_{i\alpha} \cap O_a = \emptyset$. Hence $\left[\bigcup_{i\in\omega} U_{i\alpha}\right] \cap \left[\bigcup_{a\in A_\lambda} Oa\right] = \emptyset$, because $\omega^*$ is an $F$-space, and $a_\alpha \notin \left[\bigcup_{a\in A_\lambda} Oa\right]$. There is a clopen neighbourhood $Oa_\alpha$, which does not intersect neither $P$ nor $\bigcup_{a\in A_\lambda} Oa$. We put $A_{\lambda+1} = A_\lambda \cup \{a_\alpha\}$ and $\mathcal{B}_{\lambda+1} = \mathcal{B}_\lambda \cup \{Oa_\alpha\}$.

Finally, the set $\bigcup_{\alpha<\omega_1} A_\alpha$ is as required. □

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