Two remarks on the maximal-ideal space of $\mathbf{H}^\infty$

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Abstract. The topology of the maximal-ideal space of H^∞ is discussed.

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In this note I discuss two properties of the topology of $\mathbf{M}$, the maximal-ideal space of $\mathbf{H}^\infty$, which is the Banach algebra of analytic functions which are bounded in the unit disc $\mathbf{D} = \{z\colon |z| < 1\}$. The collection of homomorphisms from $\mathbf{H}^\infty$ to $\mathbf{C}$ is referred to as $\mathbf{M}$ (which technically is the collection of the kernels of these homomorphisms). Each $f \in \mathbf{H}^\infty$ is identified with the function on $\mathbf{M}$ which takes every homomorphism ϕ to $\phi(f)$. (I shall use "z" for the identity function or for a complex number, according to whatever may be convenient.) If $\lambda \in \mathbf{C}$ and $|\lambda| = 1$, a compact subset $\mathbf{M}_\lambda$ (called the fiber over λ) of $\mathbf{M}$ is defined to be the collection of homomorphisms ϕ for which $\phi(z) = \lambda$. A great deal is known about $\mathbf{M}$ and about the fibers $\mathbf{M}_\lambda$. See [1] and [2]. The disc $\mathbf{D}$ is identified with a subset of $\mathbf{M}$ by having each point z correspond to the homomorphism evaluation at z.

Two comments I wish to make are in regard to the fibers. In particular, $\mathbf{M}_1$ is known to be connected; the proof relies on a difficult theorem [1, page 88]. I shall present an easy proof by use of a different difficult theorem, the Corona theorem [2, pages 185, 315], which is that $\mathbf{D}$ is dense in $\mathbf{M}$. The second comment regards subsets of $\mathbf{D}$ which may or may not have $\mathbf{M}_1$ in their closures.

Proposition 1. *The fiber* $\mathbf{M_1}$ *is connected.*

Proof: The portion of the disc $\{z = re^{\mathrm{i}\theta}\colon\ R < r < 1 \text{ and } |\theta| < \delta\}$, where δ is a small positive number, has $\mathbf{M_1}$ in its closure $C(R, \delta)$. This is obvious from the fact that $\mathbf{D}$ is dense in $\mathbf{M}$. The set $C(R, \delta)$ is compact and connected, being the closure of a connected set in a compact space. Form $\bigcap_{R\to 1} C(R, \delta)$, the intersection of nested sets, to obtain $C(\delta)$, again compact and containing $\mathbf{M_1}$. Now take

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$\bigcap_{\delta\to 0} C(\delta)$, once again obtaining a compact and connected space. Obviously, this last contains $\mathbf{M_1}$ and nothing else. $\square$

Proposition 2. *Let U be a subset of $\mathbf{D}$. If the closure $\overline{U}$ of U in $\mathbf{C}$ does not contain an arc of the circle $\mathbf{T} = \{z\colon |z| = 1\}$ which has 1 in its (relative) interior, then the closure of U in $\mathbf{M}$ does not contain all of $\mathbf{M_1}$.*

Proof: Recall the pseudo-hyperbolic metric $\varrho(z,u) = |z-u|/|1-\bar{u}z|$ between two points of $\mathbf{D}$. Convergence of $\prod \varrho(z,u_n)$ at any point z of the disc (and therefore at all points of the disc) is the necessary and sufficient condition that there be a (convergent) Blaschke product with $\{u_n\colon n=1,2,\dots\}$ as its zeroes. If no u_n is 0 and they are all distinct, the standard Blaschke product $B(z)$ with $\{u_n\colon n=1,2,\dots\}$ as its zeroes is this:

$$B(z) = B(z,\{u_1,u_2,\dots\}) = \prod \Big[\frac{-\bar{u}_n}{u_n}\,\frac{z-u_n}{1-\bar{u}_n z}\Big].$$

(Note that $B(0) = \prod |u_n| > 0$.)

Observe that if $\delta > 0$, then for every θ, $\varrho(z, r\mathrm{e}^{\mathrm{i}\theta}) \to 1$ uniformly on $\{z\colon |z| \le 1,\ |z-\mathrm{e}^{\mathrm{i}\theta}| \ge \delta\}$ as $r \to 1^-$.

If the closure $\overline{U}$ of U in $\mathbf{C}$ omits small arcs $A_n = \{\mathrm{e}^{\mathrm{i}\theta}\colon \theta \in J_n\}$, where $J_n = [a_n, b_n]$ are pairwise disjoint intervals in $(0,1)$ and $b_n \to 0$ as $n \to \infty$, we shall see that the closure of U in $\mathbf{M}$ does not contain all of $\mathbf{M}_1$. (There is no loss of generality in using $(0,1)$ instead of $(-1,0)\cup(0,1)$. And if $(-\varepsilon,\varepsilon)$ is disjoint from $\overline{U}$, the proposition is obvious.) For each J_n let $K_n = (a_n', b_n')$, where a_n' and b_n' are any numbers for which $a_n < a_n' < b_n' < b_n$, and let r_n be positive numbers converging to 1 so that the open sets $C_n = \{r\mathrm{e}^{\mathrm{i}\theta}\colon r_n < r < 1, \theta \in K_n\}$ are nonempty, are contained in $\mathbf{D}$, and are disjoint from $\overline{U}$.

For any set V of points (not including 0) of $\mathbf{D}$ let $B(z,V)$ be the Blaschke product having V as its set of zeroes, as indicated above, if it converges. We shall choose a sequence V by induction. First select u_1 to be any point of C_1 so that $\inf\{\varrho(z,u_1)\colon z \in \overline{U}\} > 1/2$. Next pick $u_2 \in C_2$ so that $\inf\{\varrho(z,u_2)\colon z \in \overline{U}\cup\{u_1\}\} > 3/4$. And choose $u_3 \in C_3$ so that $\inf\{\varrho(z,u_3)\colon z \in \overline{U}\cup\{z_1,z_2\}\} > 7/8$. Etc. We eventually pick $u_n \in C_n$ so that $\inf\{\varrho(z,u_n)\colon z \in \overline{U} \cup \{u_1, u_2,\dots,u_{n-1}\}\} > 1-2^{-n}$. Etc. We obtain a sequence u_n which converges to 1.

The choices of u_n and the above inequalities show that

$$B(z) = B(z,\{u_1,u_2,\dots\})$$

converges and $|B(z)| \ge c > 0$ for some c and all $z \in \overline{U}$. This means that if a homomorphism Φ is in the cluster set of U in $\mathbf{M}$, then $|\Phi(B)| \ge c > 0$. However, the sequence u_n has a cluster set in $\mathbf{M}_1$, and for any homomorphism φ in that cluster set $\varphi(B) = 0$.

This means that the closure of U in $\mathbf{M}$ does not contain all of $\mathbf{M}_1$, since it omits such φ. $\square$

Remarks. The above proof suggests two additional facts: (1) The converse of Proposition 2 is false, even for open U. (2) If X is a nonempty proper subset of $(\mathbf{M} - \mathbf{D})$ which is a union of fibers $\mathbf{M}_\lambda$, then there is no subset of $\mathbf{D}$ with cluster set in $(\mathbf{M} - \mathbf{D})$ equal to X.

I shall leave the proof of these as an exercise for those interested.

References

[1] Hoffman K., *Banach Spaces of Analytic Functions*, Dover, New York, 2007.
[2] Garnett J. B., *Bounded Analytic Functions*, Graduate Texts in Mathematics, 236, Springer, New York, 2007.

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