Seeking a network characterization of Corson compacta

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Abstract. We say that a collection $\mathcal{A}$ of subsets of X has property (CC) if there is a set D and point-countable collections $\mathcal{C}$ of closed subsets of X such that for any $A \in \mathcal{A}$ there is a finite subcollection $\mathcal{F}$ of $\mathcal{C}$ such that $A = D \setminus \bigcup \mathcal{F}$. Then we prove that any compact space is Corson if and only if it has a point-σ-(CC) base. A characterization of Corson compacta in terms of (strong) point network is also given. This provides an answer to an open question in "A Biased View of Topology as a Tool in Functional Analysis" (2014) by B. Cascales and J. Orihuela and as in "Network characterization of Gul'ko compact spaces and their relatives" (2004) by F. Garcia, L. Oncina, J. Orihuela, which asked whether there is a network characterization of the class of Corson compacta.

Keywords: Corson compacta; point network; condition (F); almost subbase; additively $\aleph_0$-Noetherian

Classification: 54D30, 46B50

1. Introduction

A compact space K is *Corson* if K is homeomorphic to a subspace of a Σ-product of real lines. For any κ, the space $\Sigma(\mathbb{R}^\kappa)$ is called a Σ*-product* of real lines if $\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : \operatorname{supp}(x) \text{ is countable}\}$, where $\operatorname{supp}(x) = \{\gamma \in \kappa : x(\gamma) \neq 0\}$ for each $x \in \mathbb{R}^\kappa$. This class of compacta is an extension of Eberlein compacta, hence contains all metrizable compacta. A space is *Eberlein compact* if it is homeomorphic to a weakly compact subset (compact in the weak topology) of a Banach space.

The class of Corson compacta has an impressive list of natural topological properties, for example, Fréchet-Uryshon, monolithic, hereditarily meta-Lindelöf, hereditarily D, etc. A great number of topological characterizations of Corson compacta have been obtained by different mathematicians. The first one was given by E. Michael and M. E. Rudin in [15] followed by characterizations by G. Gruenhage in [11], A. P. Kombarov in [14], I. Bandlow in [1], S. Clontz and G. Gruenhage in [4], G. D. Dimov [8], and F. Casarrubias-Segura, S. García-Ferreira and R. Rojas-Hernández in [2]. The main purpose of this note is to give

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characterizations of Corson compacta using bases and (strong) point networks, see definition in Section 3. These provide an answer to an open question raised by F. García, L. Oncina, and J. Orihuela in [10], and P. J. Cascales and A. W. Orihula in [3], which asked whether there is a network characterization of Corson compacta. In what follows, all the spaces are assumed to be Tychonoff (completely regular and T_1).

Let X be a space. Let $\mathcal{A}$ be a collection of subsets of X and $\mathcal{A}_x = \{A\colon x \in A \text{ and } A \in \mathcal{A}\}$ for each $x \in X$. We say that $\mathcal{A}$ is *point-finite* (*point-countable,* respectively) if $\mathcal{A}_x$ is finite (countable) for each $x \in X$. A collection is *σ-point-finite* if it can be written as a countable union of point-finite subcollections.

We say that a collection $\mathcal{A}$ of subsets of X has *property* (CC) if there is a set $D \subseteq X$ and a point-countable collection $\mathcal{C}$ of closed subsets of X such that for any $A \in \mathcal{A}$ there is a finite subcollection $\mathcal{F}$ of $\mathcal{C}$ such that $A = D \setminus \bigcup \mathcal{F}$. We say that a collection $\mathcal{A}$ has *property* σ-(CC) if it can be written as a countable union of subcollections of $\mathcal{A}$ which have the property (CC). In this note, we say that a collection is (CC) (σ-(CC), respectively) if it has the property (CC) (property σ-(CC)). A collection $\mathcal{A}$ of subsets of X is called *point*-(CC) (*point*-σ-(CC), respectively) if $\mathcal{A}_x$ has property (CC) (property σ-(CC)) for each $x \in X$.

2. About the property (CC)

The following lemma directly follows from the definition above.

Lemma 1. *Any countable collection of open subsets of X has the property (CC).*

PROOF: Let $\mathcal{U}$ be a countable collection of open subsets of X. Let $D = X$ and $\mathcal{C} = \{X \setminus U\colon U \in \mathcal{U}\}$ which is clearly point-countable. It is straightforward to see that the property (CC) of $\mathcal{U}$ is guaranteed by D and $\mathcal{C}$. □

Hence, the result below follows naturally.

Corollary 2. *Any second countable space has a base with property (CC).*

Lemma 3. *Let $\mathcal{A}$ and $\mathcal{B}$, respectively, be (CC) collections of subsets of X and Y. Then the collection $\mathcal{A} \times \mathcal{B} = \{A \times B\colon A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ of subsets of $X \times Y$ has the property (CC).*

PROOF: Let D and $\mathcal{C}$, respectively, D' and $\mathcal{C}'$, be the set and collection which witness the (CC) properties of $\mathcal{A}$ and $\mathcal{B}$. Note that $\mathcal{C}$ and $\mathcal{C}'$ are point-countable.

We claim that the set $D \times D'$ and the collection $\mathcal{C} \times \{Y\} \cup \{X\} \times \mathcal{C}'$ guarantee the (CC) property of the collection $\mathcal{A} \times \mathcal{B}$. It is clear that $\mathcal{C} \times \{Y\} \cup \{X\} \times \mathcal{C}'$ is a point-countable collection of closed subsets of $X \times Y$. For each $A \times B \in \mathcal{A} \times \mathcal{B}$, choose finite subcollections $\mathcal{F} \subset \mathcal{C}$ and $\mathcal{F}' \subset \mathcal{C}'$ such that $A = D \setminus \bigcup \mathcal{F}$ and $B = D' \setminus \bigcup \mathcal{F}'$.

Then $(A\times B)=\big(D\setminus\bigcup\mathcal{F}\big)\times\big(D'\setminus\bigcup\mathcal{F}'\big)=(D\times D')\setminus\big(\big(X\times\bigcup\mathcal{F}'\big)\cup\big(\bigcup\mathcal{F}\times Y\big)\big)$. This finishes the proof of the collection $\mathcal{A}\times\mathcal{B}$ having the property (CC). $\square$

Let $\mathcal{C}$ be a collection of sets. Then $\mathcal{C}$ is said to be less than κ*-Noetherian* if every subcollection of $\mathcal{C}$ which is well-ordered by "$\subseteq$" has cardinality less than κ. The family is said to be *additively less than* κ*-Noetherian* if the collection of all unions of members of the family is less than κ-Noetherian. We say that $\mathcal{C}$ is additively κ-Noetherian (additively Noetherian) if $\mathcal{C}$ is additively less than κ^+-Noetherian (additively less than $\aleph_0$-Noetherian, respectively). We say $\mathcal{C}$ is σ-additively Noetherian if it is countable union of additively Noetherian collections. The results in the following lemma were proved in [13] and [16], respectively.

Lemma 4. *Let $\mathcal{A}$ be a family of sets. Then $\mathcal{A}$ is additively $\aleph_0$-Noetherian (additively Noetherian, respectively) if and only if any subcollection $\mathcal{A}'$ of $\mathcal{A}$ contains a countable (finite) subcollection $\mathcal{A}''$ such that $\bigcup\mathcal{A}'=\bigcup\mathcal{A}''$.*

Motivated by this result, we say a collection $\mathcal{A}$ is *weakly σ-additively Noetherian* if any subcollection $\mathcal{A}'$ of $\mathcal{A}$ can be written as $\bigcup\{\mathcal{A}'_n\colon n\in\omega\}$ such that for each $n\in\omega$, there exists a finite collection $\mathcal{A}''_n\subset\mathcal{A}'_n$ with $\bigcup\mathcal{A}''_n=\bigcup\mathcal{A}'_n$. It is clear that:

1. any countable collection is σ-additively Noetherian;
2. any σ-additively Noetherian collection is weakly σ-additively Noetherian;
3. any weakly σ-additively Noetherian collection is additively $\aleph_0$-Noetherian.

From the lemma above, it is straightforward to verify the corollary below.

Corollary 5. *Any countable union of additively $\aleph_0$-Noetherian families is additively $\aleph_0$-Noetherian.*

Lemma 6. *Any space with an additively $\aleph_0$-Noetherian base is Lindelöf.*

Proof: Let $\mathcal{B}$ be an additively $\aleph_0$-Noetherian base of X. Let $\mathcal{U}$ be an open cover of X. For each $x\in X$, pick $U_x\in\mathcal{U}$ and $B_x\in\mathcal{B}$ such that $x\in B_x\subset U_x$. Then $\{B_x\colon x\in X\}\subseteq\mathcal{B}$ is an open refinement of $\mathcal{U}$. Then there exists a countable subset C of X such that $\bigcup\{B_x\colon x\in C\}=X$. Then $\{U_x\colon x\in C\}$ is a countable subcollection of $\mathcal{U}$ which covers X. $\square$

Lemma 7. *Suppose that X is Lindelöf. Any (CC) collection of subsets of X is additively $\aleph_0$-Noetherian.*

Proof: Let $\mathcal{A}$ be a collection of subsets of X with property (CC). Let D and the point-countable collection $\mathcal{C}$ witness the (CC) property of $\mathcal{A}$. Take an arbitrary subcollection $\mathcal{A}'$ of $\mathcal{A}$. We show that there is a countable subcollection $\mathcal{A}''$ such that $\bigcup\mathcal{A}'=\bigcup\mathcal{A}''$. Then by Lemma 4, the collection $\mathcal{A}$ is additively $\aleph_0$-Noetherian.

For each $A \in \mathcal{A}'$, we fix $\mathcal{F}(A)$ to be a finite subcollection of $\mathcal{C}$ such that $A = D \setminus \bigcup \mathcal{F}(A)$. Let $\mathcal{S} = \{\mathcal{F}(A) \colon A \in \mathcal{A}'\}$. For each $S \in \mathcal{S}$, it is clear that there is a unique element, A_S, in $\mathcal{A}'$ such that $S = \mathcal{F}(A_S)$. Hence if $\mathcal{S}$ is countable, $\mathcal{A}'$ is countable too.

Now we assume that $\mathcal{S}$ is uncountable. We define $\mathcal{R}(\mathcal{S})$ to be the collection of all roots of uncountable Δ-system from $\mathcal{S}$, i.e. $R \in \mathcal{R}(\mathcal{S})$ if and only if there is an uncountable subcollection of $\mathcal{T}$ of $\mathcal{S}$ such that $S_0 \cap S_1 = R$ whenever S_0 and S_1 are distinct elements of $\mathcal{T}$. Then we define

$$\mathcal{M}_0(\mathcal{S}) = \{R \in \mathcal{R}(\mathcal{S}) \colon \nexists\, R' \in \mathcal{R}(\mathcal{S})(R' \subsetneq R)\}$$

and

$$\mathcal{M}_1(\mathcal{S}) = \{S \in \mathcal{S} \colon \nexists\, R \in \mathcal{R}(\mathcal{S})(R \subseteq S)\}.$$

For each $R \in \mathcal{M}_0(\mathcal{S})$, we denote $\mathcal{T}_R$ to be an uncountable Δ-system with root R. Then we define $\mathcal{A}'_{\mathcal{T}_R} = \{A_S \colon S \in \mathcal{T}_R\}$. Since $\mathcal{C}$ is point-countable and R is the root of $\mathcal{T}_R$, $\big\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\big\}$ is point-countable.

If $\mathcal{M}_0(\mathcal{S})$ is uncountable, then there is a root of a Δ-system which is contained in some element of $\mathcal{M}_0(\mathcal{S})$ which is impossible. For the similar reason, $\mathcal{M}_1(\mathcal{S})$ is also countable. For each $R \in \mathcal{M}_0(\mathcal{S})$, we define $\mathcal{S}_R = \{S \in \mathcal{S} \colon S \supset R\}$. It is straightforward to see that $\mathcal{S} = \bigcup\{\mathcal{S}_R \colon R \in \mathcal{M}_0(\mathcal{S})\} \cup \mathcal{M}_1(\mathcal{S})$. For each $R \in \mathcal{M}_0(\mathcal{S})$, we define $\mathcal{A}'_{\mathcal{S}_R} = \{A \in \mathcal{A}' \colon \mathcal{F}(A) \in \mathcal{S}_R\}$. Then,

$$\mathcal{A}' = \Big(\bigcup\{\mathcal{A}_{\mathcal{S}_R} \colon R \in \mathcal{M}_0(\mathcal{S})\}\Big) \cup \{A_S \colon S \in \mathcal{M}_1(\mathcal{S})\}.$$

Notice that $\mathcal{M}_1(\mathcal{S})$ is countable. It is sufficient to prove that there is a countable subcollection $\mathcal{A}''_{\mathcal{S}_R}$ of $\mathcal{A}'_{\mathcal{S}_R}$ with $\bigcup \mathcal{A}''_{\mathcal{S}_R} = \bigcup \mathcal{A}'_{\mathcal{S}_R}$ for each $R \in \mathcal{M}_0(\mathcal{S})$. Then we let $\mathcal{A}'' = \big(\bigcup\{\mathcal{A}''_{\mathcal{S}_R} \colon R \in \mathcal{M}_0(\mathcal{S})\}\big) \cup \{A_S \colon S \in \mathcal{M}_1(\mathcal{S})\}$ which is clearly countable and also $\bigcup \mathcal{A}'' = \bigcup \mathcal{A}'$.

Fix $R \in \mathcal{M}_0(\mathcal{S})$. First, we claim that $\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\}$ is point-countable. Suppose not. Then there exists $x \in X$ and $\{S_\alpha \colon \alpha < \omega_1 \text{ and } S_\alpha \in \mathcal{T}_R\}$ such that $x \in \bigcup(S_\alpha \setminus R)$ for each $\alpha < \omega_1$. Since $\mathcal{T}_R$ is a Δ-system with room R, $(S_\alpha \setminus R) \cap (S_{\alpha'} \setminus R) = \emptyset$ for any $\alpha \neq \alpha'$. Then for each $\alpha < \omega_1$, there exists $C_\alpha \in S_\alpha \subseteq \mathcal{C}$ such that $x \in C_\alpha$, and also, $C_\alpha \neq C_\beta$ if $\alpha \neq \beta$. This contradicts with the point-countable property of $\mathcal{C}$.

We claim that $D \setminus \bigcup R = \bigcup \mathcal{A}'_{\mathcal{S}_R} = \bigcup \mathcal{A}'_{\mathcal{T}_R}$. Since $R \subseteq S$ for any $S \in \mathcal{S}_R$ and $\mathcal{T}_R \subseteq \mathcal{S}_R$, it is clear that $D \setminus \bigcup R \supseteq \bigcup \mathcal{A}'_{\mathcal{S}_R} \supseteq \bigcup \mathcal{A}_{\mathcal{T}_R}$. Take any $x \in D \setminus \bigcup R$. Since $\big\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\big\}$ is point-countable, there is an $S_1 \in \mathcal{S}_R$ such that $x \notin \bigcup S_1 \setminus R$. Hence, $x \in D$ but $x \notin \bigcup S_1$, i.e. $x \in D \setminus \bigcup S_1$. Therefore, $D \setminus \bigcup R \subseteq \bigcup \mathcal{A}'_{\mathcal{T}_R}$.

Lastly, since the family $\big\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\big\}$ is point-countable and $\mathcal{T}_R$ is uncountable, it is straightforward to see that $\bigcap\big\{\bigcup(S \setminus R) \colon R \in \mathcal{T}_R\big\} = \emptyset$. Notice that $\bigcup(S \setminus R)$ is closed for each $S \in \mathcal{T}_R$. By the Lindelöf property of X, there exists a countable subset $\mathcal{L}_R$ of $\mathcal{T}_R$ such that $\bigcap\big\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\big\} = \emptyset$. We claim that $\bigcup \mathcal{A}'_{\mathcal{S}_R} = D \setminus \bigcup R \subseteq \bigcup\{A_S \colon S \in \mathcal{L}_R\}$. Take any $x \in D \setminus \bigcup R$. Since $\bigcap\big\{\bigcup(S \setminus R) \colon S \in \mathcal{L}_R\big\} = \emptyset$, there is an $S_2 \in \mathcal{L}_R$ such that $x \notin \bigcup(S_2 \setminus R)$. Therefore, $x \notin \bigcup S_2$. So $x \in D \setminus \bigcup S_2 = A_{S_2} \subseteq \bigcup\{A_S \colon S \in \mathcal{L}_R\}$. Hence, we have that $\bigcup \mathcal{A}'_R = D \setminus \bigcup R \subseteq \bigcup\{A_S \colon S \in \mathcal{L}_R\}$. Let $\mathcal{A}''_{\mathcal{S}_R} = \{A_S \colon S \in \mathcal{L}_R\}$ which clearly satisfies the requirement. This finishes the proof. □

By Corollary 5 and the lemma above, the following result holds.

Corollary 8. *Suppose that X is Lindelöf. Any σ-(CC) collection of subsets of X is additively $\aleph_0$-Noetherian.*

In next section, we will give an example (Example 16) of an additively $\aleph_0$-Noetherian collection of sets which is not σ-(CC).

Lemma 9. *Let X be a compact space. Any (CC) collection of subsets of X is weakly σ-additively Noetherian.*

Proof: Let $\mathcal{A}$ be a collection of subsets of X with property (CC). Let D and $\mathcal{C}$ be the set and the collection, respectively, witnessing the (CC) property of $\mathcal{A}$. Without loss of generality, we assume that $\mathcal{A}$ is uncountable. Let $\mathcal{A}'$ be any uncountable subcollection of $\mathcal{A}$. Next, we will show the following:

Property $(*)$. $\mathcal{A}'$ can be written as $\bigcup\{\mathcal{A}'_n \colon n \in \omega\}$ where for each $n \in \omega$ there exists a finite $\mathcal{A}''_n \subset \mathcal{A}'_n$ such that $\bigcup \mathcal{A}''_n = \bigcup \mathcal{A}'_n$.

We define $\mathcal{F}(A)$, $\mathcal{S}$, and A_S as in the proof of Lemma 7. Since $\mathcal{A}'$ is uncountable, $\mathcal{S}$ is uncountable too. Then, we define $\mathcal{R}(\mathcal{S})$, $\mathcal{T}_R$, $\mathcal{A}'_{\mathcal{T}_R}$, $\mathcal{M}_0(\mathcal{S})$, $\mathcal{A}'_{\mathcal{S}_R} = \{A_S \colon S \supset R\}$, and $\mathcal{M}_1(\mathcal{S})$ to be the same as in the proof of Lemma 7. Recall that:

1. $\mathcal{M}_0(\mathcal{S})$ and $\mathcal{M}_1(\mathcal{S})$ are countable;
2. $\mathcal{A}' = \big(\bigcup\{\mathcal{A}_{\mathcal{S}_R} \colon R \in \mathcal{M}_0(\mathcal{S})\}\big) \cup \{A_S \colon S \in \mathcal{M}_1(\mathcal{S})\}$.

Hence, to prove that $\mathcal{A}'$ has property $(*)$, it is sufficient to show that for each $R \in \mathcal{M}_0(\mathcal{S})$ there exists a finite subcollection $\mathcal{A}''_R \subset \mathcal{A}'_{\mathcal{S}_R}$ such that $\bigcup \mathcal{A}''_R = \bigcup \mathcal{A}'_{\mathcal{S}_R}$.

Fix $R \in \mathcal{M}_0(\mathcal{S})$. It is shown in the proof of Lemma 7 that $D \setminus \bigcup R = \bigcup \mathcal{A}'_{\mathcal{S}_R} = \bigcup \mathcal{A}'_{\mathcal{T}_R}$. Recall that the family $\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\}$ is point-countable and $\mathcal{T}_R$ is uncountable, so $\bigcap\{\bigcup(S \setminus R) \colon R \in \mathcal{T}_R\} = \emptyset$. Then, by the compact property of X, there exists a finite subset $\mathcal{L}_R$ of $\mathcal{T}_R$ such that $\bigcap\{\bigcup(S \setminus R) \colon S \in \mathcal{T}_R\} = \emptyset$. Let $\mathcal{A}''_R = \{A_S \colon S \in \mathcal{L}_R\}$ which is clearly finite. Using a similar approach as in the proof of Lemma 7, it is straightforward to verify that $\bigcup \mathcal{A}''_{\mathcal{S}_R} = \bigcup \mathcal{A}'_{\mathcal{S}_R}$. This finishes the proof. □

3. Characterizing Corson compacta

Almost subbases were introduced by G. Dimov in [7] where, among other things, he used them to characterize the subspaces of Eberlein compacta. We say that a family α of subsets of a space X is an *almost subbase* of X, with respect to a family $\{f_V \colon X \to [0,1]\}_{V\in\alpha}$ of continuous functions, if $V = f_V^{-1}(0,1]$ for $V \in \alpha$ and the family $\alpha \cup \{X \setminus f_V^{-1}[1/n, 1] \colon V \in \alpha,\ n \in \mathbb{N}\}$ is a subbase of X. An almost subbase of X is a family α for which such a family of functions exists.

Theorem 10. *If a space X has a point-countable almost subbase, then it has a point-σ-(CC) base.*

Proof: Let α be a point-countable almost subbase of X. Let $\mathcal{B}$ be the base generated by α. An element in $\mathcal{B}$ is of the form, $\bigcap\{U_i \colon i = 1,\dots,n\} \cap \left(\bigcap\{X \setminus f_{V_j}^{-1}[1/n_j, 1] \colon j = 1,\dots,m\}\right)$ where $U_i, V_j \in \mathcal{S}$ for $i = 1,\dots,n$ and $j = 1,\dots,m$. We show that $\mathcal{B}$ is point-σ-(CC).

Fix $x \in X$ and a finite subcollection $\{U_1,\dots,U_n\}$ of $\mathcal{S}$ with $x \in U_i$ for $i = 1,\dots,n$. We define $U = \bigcap\{U_i \colon i = 1,\dots,n\}$. Since $\mathcal{S}$ is point-countable, it is sufficient to show that the subcollection $\mathcal{B}_x^U$ of $\mathcal{B}_x$ is (CC), where $\mathcal{B}_x^U$'s elements are in the form $U \cap \left(\bigcap\{X \setminus f_{V_j}^{-1}[1/n_j, 1] \colon j = 1,\dots,m\}\right) = U \setminus \bigcup\{f_{V_j}^{-1}[1/n_j, 1] \colon j = 1,\dots,m\}$. Choose $D = U$ and $\mathcal{C} = \{f_V^{-1}[1/m, 1] \cap G_n \colon m \in \mathbb{N}$ and $V \in \mathcal{S}\}$. It is straightforward to verify that D and $\mathcal{C}$ satisfy the requirements for $\mathcal{B}_x^U$ being (CC). $\square$

A *point network* for X is a collection $\mathcal{W} = \{\mathcal{W}(x) \colon x \in X\}$ where each $\mathcal{W}(x)$ is a collection of subsets of X containing x such that whenever $x \in U$, U open, there is an open $V(x,U)$ with $x \in V(x,U) \subseteq U$ such that, whenever $y \in V(x,U)$ then $x \in W \subseteq U$ ($x \in W \subseteq V$, respectively) for some $W \in \mathcal{W}(y)$. Point networks are also known as "condition (F)", and as the "Collins–Roscoe structuring mechanism" after the authors who introduced them in [6]. The term "point network" was suggested by G. Gruenhage. In fact, in [12] G. Gruenhage pointed out that a collection $\mathcal{W} = \{\mathcal{W}(x) \colon x \in X\}$ is a point network if and only if for any subset A of X, $\bigcup\{\mathcal{W}(x) \colon x \in A\}$ contains a network at every point in $\bar{A}$. A space is said to satisfy condition (G) if it has a countable point network. It is a long-standing open problem raised in [5] whether a space satisfying open (G) (or, having a countable open point network) has a point-countable base. In [9], the author and P. Gartside gave uniform characterizations of Ebertein, Talagrand and Gul'ko compacta in terms of point networks, bases and almost bases, but their approach failed for the class of Corson compacta.

Next, we show that any compact space with σ-(CC) point network is Corson through several lemmas. Note that a σ-(CC) point network for a space X means that $\mathcal{W}(x)$ is σ-(CC) for each $x \in X$.

The following lemma is implied by the definitions directly.

Lemma 11. *Any space with a point-σ-(CC) base has a strong σ-(CC) point network.*

By Lemma 3, it is straightforward to show the following result.

Lemma 12. *If X has a σ-(CC) point network, so does X^2.*

Moreover, the property having a σ-(CC) point network is preserved by taking countable product.

Lemma 13. *If X_n has a σ-(CC) point network for each $n \in \omega$, then so does $\prod_{n\in\omega} X_n$.*

Proof: For each $n \in \omega$, let $\mathcal{W}_n = \{\mathcal{W}_n(x_n) \colon x_n \in X_n\}$ be the point network for X_n. Then for each $\mathbf{x} = (x_n) \in \prod_{n\in\omega} X_n$, define $\mathcal{W}(\mathbf{x}) = \big\{\prod_{i=0}^{n} W_i \times \prod_{m>n} X_m \colon n \in \omega$ and $W_i \in \mathcal{W}_i(x_i)$ for each $i = 0, 1, \ldots, n\big\}$. Since property (CC) is finitely productive, it is straightforward to verify that $\mathcal{W}(\mathbf{x})$ is σ-(CC) and $\{\mathcal{W}(\mathbf{x}) \colon \mathbf{x} \in \prod_{n\in\omega} X_n\}$ satisfies condition (F). □

The following game was introduced by G. Gruenhage in [11]. Consider the game $G(H, X)$ of length ω played in X, where H is a closed subset of X. There are two players, O and P. In nth round, O chooses an open set O_n containing H, and P chooses $p_n \in O_n$. We say that the player O wins the game if p_n converges to H, i.e. any open superset of H contains all but finite manly p_n's. In the same paper, G. Gruenhage proved that a compact space X is Corson if and only if O has a winning strategy in $G(\Delta, X^2)$ where Δ is the diagonal of X^2. In [12], G. Gruenhage proved that if X is countably compact and monotonically ω-monolithic, then O has a winning strategy in $G(H, X)$ for any closed subset H of X. Using a similar strategy, we show that the same result holds for countably compact spaces with an additively $\aleph_0$-Noetherian point network.

Lemma 14. *If X is a countably compact with an additively $\aleph_0$-Noetherian point network, then O has a winning strategy in space $G(H, X)$ for any closed subset H of X.*

Proof: Let $\mathcal{W} = \{\mathcal{W}(x) \colon x \in X\}$ be the point network of X such that $\mathcal{W}(x)$ is additively $\aleph_0$-Noetherian for each $x \in X$. Let H be a closed subset of X. We will show that O has a winning strategy in the game $G(H, X)$. The player O chooses $O_0 = X$, and P responses with p_0. Suppose $p_0, p_1, \ldots, p_{n-1}$ are P's choices so far. Fix $i < n$. Let $\mathcal{N}_i = \{W \colon W \in \mathcal{W}(p_i)$ and $\overline{W} \cap H = \emptyset\}$. Since $\mathcal{W}(p_i)$ is additively $\aleph_0$-Noetherian, we could fix a countable subcollection, listed as $\{N_{j,i} \colon j \in \omega\}$, of $\mathcal{N}_i$ such that $\bigcup\{N_{j,i} \colon j \in \omega\} = \bigcup \mathcal{N}_i$. Then we pick O_n such that $H \subset O_n \subset O_{n-1}$ and $N_{j,i} \cap \overline{O_n} = \emptyset$ for all $i, j < n$.

Next, we show that this is a winning strategy for the player O. Since X is countably compact, it is sufficient to show that any $A = \{p_n : n < \omega\}$ has no cluster point in $X \setminus H$. Suppose, for a contradiction, that p is a cluster point of A and not in H. Take an open neighborhood U of p such that $\overline{U} \cap H = \emptyset$. Then there exists an $m_0 \in \omega$ such that $p_{m_0} \in U$ and there is an $W^\star \in \mathcal{W}(p_{m_0})$ with $p \in W^\star \subset U$. Then there is an $m_1 \in \omega$ such that $W^\star \subseteq N_{m_1,m_0}$. Let $m_* = \max\{m_1, m_0\} + 1$. Then $p \notin \overline{O_{m_*}}$, i.e. $p \notin \overline{\{p_n : n > m_*\}}$ which contradicts with the fact that p is a cluster point of A. $\square$

Next, we will prove the main result.

Theorem 15. *Let X be compact. The following are equivalent:*

1) *X is Corson.*
2) *X has a point-σ-(CC) base.*
3) *X has a σ-(CC) strong point network.*
4) *X has a σ-(CC) point network.*

Proof: If X is Corson, then it has a point-countable almost subbase by [8]. Hence it has a point-σ-(CC) base by Lemma 10. This proves that 1) implies 2).

Suppose that X has a point-σ-(CC) base. By Lemma 11, it has a σ-(CC) point network. Hence 2) implies 3). It is clear that 3) implies 4).

Now we suppose that X has a σ-(CC) point network. By Lemma 12, X^2 has a σ-(CC) point network, hence an additively $\aleph_0$-Noetherian point network by Corollary 8. Then by Lemma 14, player O has a winning strategy in $G(\Delta, X^2)$. By [11], X is Corson. We finish the proof of 4) implies 1). $\square$

Example 16. There is a collection of subsets of a space which is additively $\aleph_0$-Noetherian but not σ-(CC).

Proof: Let X be the double arrow space. The authors in [9] pointed out that X is a non-Corson compact space with an additively $\aleph_0$-Noetherian base $\mathcal{B}$. Then the base $\mathcal{B}$ is not point-σ-(CC) by Theorem 15. Hence at some element x in X, $\mathcal{B}_x$ is additively $\aleph_0$-Noetherian, but not σ-(CC). $\square$

Question. Is there a non-Corson compact space which has a weakly σ-additively Noetherian point network?

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