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On non-normality points, Tychonoff products and Suslin number

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Abstract. Let a space X be Tychonoff product $\prod_{\alpha<\tau} X_\alpha$ of τ-many Tychonoff nonsingle point spaces X_α. Let Suslin number of X be strictly less than the cofinality of τ. Then we show that every point of remainder is a non-normality point of its Čech–Stone compactification βX. In particular, this is true if X is either R^τ or ω^τ and a cardinal τ is infinite and not countably cofinal.

Keywords: non-normality point; Čech–Stone compactification; Tychonoff product; Suslin number

Classification: 54D15, 54D35, 54D40, 54D80, 54E35, 54G20

1. Introduction

Let $X^* = \beta X \setminus X$ be a remainder of Čech–Stone compactification βX of Tychonoff space X. In 1960 L. Gillman [3] posed the following question for countable discrete space $\omega = \{0, 1, 2, \dots\}$, despite great efforts so far having only very particular or conditional solutions, see, for example, [1], [2] or [7]:

Is $\omega^* \setminus \{p\}$ not normal for any point p of ω^*?

But one turned to be more solvable for crowded spaces. Thus in 2007 the following result was obtained independently by J. Terasawa [6] and the author [4]:

Theorem A. *Let X be a non-compact metrizable crowded space. Then any point p of X^* is a butterfly-point in βX. Hence $\beta X \setminus \{p\}$ is not normal.*

In 2014 the following generalizations for Tychonoff products were obtained by the author [5]:

Theorem B. *Let τ be an arbitrary cardinal number and for every $k < \tau$ let $\mathcal{F}_k$ be a family of metrizable spaces with the following properties: $\mathcal{F}_k$ contains a crowded space and $\mathcal{F}_k$ contains at most one non-compact space. Let a space S be a free union $\bigcup_{k<\tau} S_k$ of Tychonoff products $S_k = \prod\{X : X \in \mathcal{F}_k\}$. Then every point p of S^* is a butterfly-point in βS. Hence $\beta S \setminus \{p\}$ is not normal.*

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Corollary A. *Let a space S be a free union of arbitrary powers of closed segment $\bigcup_{k<\tau} I^{\tau_k}$. Then every point p of S^* is a butterfly-point in βS. Hence $\beta S \setminus \{p\}$ is not normal.*

Corollary B. *Let $S = \omega \times I^C$. Then every point p of S^* is a butterfly-point in βS. Hence $\beta S \setminus \{p\}$ is not normal.*

Now we obtain the next facts on Tychonoff products. By $C(X)$ we denote Suslin number of a space X, i.e. the maximal size of cellular families of nonempty open sets. By $d(X)$ we denote density of X, i.e. the minimal size of everywhere dense subset of X and by $cf(\tau)$ – cofinality of a cardinal τ, i.e. the minimal size of unbounded subset of τ. By the Hewitt–Marczewski–Pondiczery theorem on the density of products and its corollary on the Suslin number we have $C(X) < cf(\tau)$ under the conditions of Corollaries 1–3.

Theorem 1. *Let a space $X = \prod_{\alpha<\tau} X_\alpha$ be Tychonoff product of τ-many nonsingle point Tychonoff spaces X_α. Let a point $p \in X^*$ be in the closure of some subset $G \subset X$ with $C(G) < cf(\tau)$. Then $\beta X \setminus \{p\}$ is not normal.*

Corollary 1. *The space $\beta(R^\tau) \setminus \{p\}$ is not normal if τ is not countably cofinal and $p \in (R^\tau)^*$.*

Corollary 2. *The space $\beta(\omega^\tau) \setminus \{p\}$ is not normal, if infinite τ is not countably cofinal and $p \in (\omega^\tau)^*$.*

Corollary 3. *The space $\beta(X^\tau) \setminus \{p\}$ is not normal if $d(X) < cf(\tau)$ and $p \in (X^\tau)^*$.*

2. Proofs

In our article all spaces are Tychonoff and R is a straight line. In what follows, we are in the conditions of Theorem 1. So by $X = \prod_{\alpha<\tau} X_\alpha$ we denote Tychonoff product of τ-many nonsingle point Tychonoff spaces X_α and by [ ] closure operator in its Čech–Stone compactification βX. We assume all the ordinals to be less than the number of factors τ. Our goal is to construct subsets F and G of $\beta X \setminus \{p\}$ so that $\{p\} = [F] \cap [G]$. This obviously implies the validity of Theorem 1.

Considering pairwise products, if necessary, we can assume that every factor X_α consists of at least three points. Therefore, there are points $a = (a_\alpha)_{\alpha<\tau}$, $b = (b_\alpha)_{\alpha<\tau}$ and $c = (c_\alpha)_{\alpha<\tau}$ in X, all coordinates of which a_α, b_α and c_α are pairwise different. For an arbitrary bases $\mathcal{B}_\alpha$ of X_α we define $\mathcal{B}$ to be a base of X, consisting of all products of the form $U = \prod_{\alpha<\tau} U_\alpha$, where $U_\alpha \in \mathcal{B}_\alpha$ for some finite $K \subset \tau$ and every $\alpha \in K$ and $U_\alpha = X_\alpha$ otherwise. For any $U \in \mathcal{B}$ we put $\lambda(U) = \max\{\alpha < \tau \colon U_\alpha \neq X_\alpha\}$ and

$$U(\alpha, a) = \prod_{\gamma \leq \alpha} U_\gamma \times \prod_{\gamma > \alpha} \{a_\gamma\}$$

for each $\alpha < \tau$. In other words, $(x_\gamma)_{\gamma<\tau} \in U(\alpha, a)$ if and only if $x_\gamma \in U_\gamma$ for every $\gamma \leq \alpha$ and $x_\gamma = a_\gamma$ otherwise. Let $\mathcal{O}$ be all open neighbourhoods of p in βX. For any $O \in \mathcal{O}$ define $\mathcal{F}(O)$ to be the family

$$\Big\{F \subset \mathcal{B} \colon \bigcup F \subset O,\ O \cap G \subset \Big[\bigcup F \cap G\Big] \text{ and } \mid F \mid \leq C(G)\Big\},$$

which is, obviously, nonempty. Let $\mathcal{F} = \bigcup_{O \in \mathcal{O}} \mathcal{F}(O)$. For every $F \in \mathcal{F}$ denote $\lambda(F) = \sup\{\lambda(U) \colon U \in F\}$ and

$$F(\alpha, a) = \{U(\alpha, a) \colon U \in F\}$$

for each $\alpha < \tau$. Then the condition $C(G) < cf(\tau)$ implies $\lambda(F) < \tau$.

Lemma 1. *If $U \in \mathcal{B}$ and $\alpha \geq \lambda(U)$, then $U(\alpha, a) \subset U$. If $F \in \mathcal{F}$ and $\alpha \geq \lambda(F)$, then $\bigcup F(\alpha, a) \subset \bigcup F$.*

PROOF: If $U(\alpha, a)_\gamma \neq U_\gamma$, then $\gamma > \alpha$. But then $U_\gamma = X_\gamma$ implies Lemma 1. $\square$

Lemma 2. *The family $\big\{\bigcup F(\alpha, a) \colon F \in \mathcal{F}\big\}$ is centred for every $\alpha < \tau$.*

PROOF: Let $F_0 \in \mathcal{F}(O_0)$, ..., $F_n \in \mathcal{F}(O_n)$ for some $n < \omega$ and $O_0, \dots, O_n \in \mathcal{O}$. Let $O = \bigcap_{i \leq n} O_i$. Then every $V_i = \bigcup F_i \cap G \cap O$ is open and everywhere dense subset of nonempty $O \cap G$. There is a point $x = (x_\gamma)_{\gamma<\tau}$ of X with $x \in \bigcap_{i \leq n} V_i$. Then $x \in U_i$ for some $U_i \in F_i$. Define $y = (y_\gamma)_{\gamma<\tau}$ as follows: $y_\gamma = x_\gamma$ if $\gamma \leq \alpha$ and $y_\gamma = a_\gamma$ otherwise. Then $y \in \bigcap_{i \leq n} U_i(\alpha, a)$. $\square$

For every $\alpha < \tau$ define $\xi_\alpha(a) \in \beta X$ to be an arbitrary point of $\bigcap_{F \in \mathcal{F}} \big[\bigcup F(\alpha, a)\big]$. Similarly, b and c generate the points $\xi_\alpha(b) \in \bigcap_{F \in \mathcal{F}} \big[\bigcup F(\alpha, b)\big]$ and $\xi_\alpha(c) \in \bigcap_{F \in \mathcal{F}} \big[\bigcup F(\alpha, c)\big]$, respectively. Denote $A = \{\xi_\alpha(a) \colon \alpha < \tau\}$, $B = \{\xi_\alpha(b) \colon \alpha < \tau\}$ and $C = \{\xi_\alpha(c) \colon \alpha < \tau\}$

Lemma 3. *The space $\beta X \setminus \{p\}$ is not normal.*

PROOF: Let $O \in \mathcal{O}$, $F \in \mathcal{F}(O)$ and $\alpha \geq \lambda(F)$. Then

$$\xi_\alpha(a) \in \Big[\bigcup F(\alpha, a)\Big] \subset \Big[\bigcup F\Big] \subset [O]$$

by Lemma 1. Therefore $\{\xi_\alpha(a) \colon \alpha \geq \lambda(F)\}$ and, quite similarly, $\{\xi_\alpha(b) \colon \alpha \geq \lambda(F)\}$ and $\{\xi_\alpha(c) \colon \alpha \geq \lambda(F)\}$ are subsets of $[O]$.

For any $\lambda < \tau$ let a continuous map $f_\lambda \colon X_\lambda \to [0, 2]$ satisfies $f_\lambda(a_\lambda) = 0$, $f_\lambda(b_\lambda) = 1$ and $f_\lambda(c_\lambda) = 2$. Denote its composition with the orthogonal projection

$\pi_\lambda\colon X \to X_\lambda$ by $f\colon X \to [0,2]$, i.e. put $f(x) = f_\lambda(x_\lambda)$ for each $x \in X$. There is a continuous extension $\tilde{f}\colon \beta X \to [0,2]$.

If $\alpha < \lambda$ and $F \in \mathcal{F}$ is arbitrary, then

$$f\Big(\bigcup F(\alpha,a)\Big) = \bigcup_{U\in F} f(U(\alpha,a)) = \bigcup_{U\in F} f_\lambda\{a_\lambda\} = \{0\}$$

implies

$$\xi_\alpha(a) \in \Big[\bigcup F(\alpha,a)\Big] \subset \tilde{f}^{-1}(0).$$

Therefore $\{\xi_\alpha(a)\colon \alpha < \lambda\} \subset \tilde{f}^{-1}(0)$ and, quite similarly, $\{\xi_\alpha(b)\colon \alpha < \lambda\} \subset \tilde{f}^{-1}(1)$ and $\{\xi_\alpha(c)\colon \alpha < \lambda\} \subset \tilde{f}^{-1}(2)$. Hence the closures of these sets are pairwise disjoint.

It implies that A, B and C are also pairwise disjoint and at most one of them contain p. For any $O \in \mathcal{O}$, $F \in \mathcal{F}(O)$ and $\lambda > \lambda(F)$ we can argue as above to show that A, B and C have pairwise disjoint closures outside $[O]$ and, therefore, outside $\{p\}$. Two of them not containing $\{p\}$ show that p is a non-normality point. $\square$

References

[1] Blaszczyk A., Szymański A., *Some non-normal subspaces of the Čech–Stone compactification of a discrete space*, Abstracta, Eighth Winter School on Abstract Analysis, Czechoslovak Academy of Sciences, Praha, 1980, pages 35–38.

[2] Bešlagić A., van Douwen E. K., *Spaces of nonuniform ultrafilters in space of uniform ultrafilters*, Topology Appl. **35** (1990), no. 2–3, 253–260.

[3] Fine N. J., Gillman L., *Extensions of continuous functions in βN*, Bull. Amer. Math. Soc. **66** (1960), 376–381.

[4] Logunov S., *On non-normality points and metrizable crowded spaces*, Comment. Math. Univ. Carolin. **48** (2007), no. 3, 523–527.

[5] Logunov S., *Non-normality points and big products of metrizable spaces*, Topology Proc. **46** (2015), 73–85.

[6] Terasawa J., *$\beta X - \{p\}$ are non-normal for non-discrete spaces X*, Topology Proc. **31** (2007), no. 1, 309–317.

[7] Warren N. M., *Properties of Stone–Čech compactifications of discrete spaces*, Proc. Amer. Math. Soc. **33** (1972), 599–606.

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