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Analysis of periodic solutions for nonlinear coupled integro-differential systems with variable delays

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Abstract. The objective of this work is the application of Krasnosel'skii's fixed point technique to prove the existence of periodic solutions of a system of coupled nonlinear integro-differential equations with variable delays. An example is given to illustrate this work.

Keywords: integro-differential equation; periodic solution; Krasnosel'skii's fixed point theorem

Classification: 34K20, 45J05, 45D05

1. Introduction

There are many papers written on the subject of existence of periodic solutions of nonlinear differential equations and nonlinear integro-differential equations, for such topics we refer the interested reader to [1]–[7], [10], [12] and the references therein. In 2007, in the paper [14] Y. Wang, H. Lian and W. Ge consider the second order nonlinear differential equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t)x'(t - \tau(t)) + f\bigl(t, x(t), x(t - \tau(t))\bigr),$$

and by using fixed point theorem technique, the authors obtained existence of periodic solutions. H. Deham in [8] considers the second order nonlinear integro-differential equation

$$x''(t) + p(t)x'(t) + q(t)h(x(t)) = \int_{-\infty}^{t} Q(t,s)f\bigl(s, x(s - g(s))\bigr)\,\mathrm{d}s,$$

and by Krasnosel'skii–Burton's fixed point theorem shows that the existence of periodic solutions is concluded. In the paper [11] Y. N. Raffoul studies the existence of periodic and asymptotically periodic solutions of the following system of

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coupled nonlinear Volterra integro-differential equations with infinite delay

$$\begin{cases} x'(t) = h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^{t} a(t,s)f(x(s),y(s))\,\mathrm{d}s, \\ y'(t) = p_1(t)y(t) + p_2(t)x(t) + \int_{-\infty}^{t} b(t,s)g(x(s),y(s))\,\mathrm{d}s, \end{cases}$$

the author uses Schauder's fixed point theorem to obtain his results.

Motivated by the papers [8], [11], [14] and the references therein and by using Krasnosel'skii's fixed point theorem, in this paper we study the existence of periodic solutions of the following system of coupled nonlinear integro-differential equations with variable delays

$$\begin{aligned} x_i''(t) + p_i(t)x_i'(t) + q_i(t)x_i(t) &\\ = g_i\big(t, x_1(t), x_2(t), x_1(t-\tau_1(t)), x_2(t-\tau_2(t))\big) &\\ + c_i(t)x_i'(t-\tau_i(t)) &\\ + \int_{-\infty}^{t} C_i(t,s) f_i(x_1(s), x_2(s))\,\mathrm{d}s, &\qquad i = 1,2, \end{aligned} \tag{1.1}$$

where p_i, q_i, $i = 1,2$, are positive continuous real-valued functions and the functions c_i, C_i, $i = 1,2$, are assumed to be continuous in their arguments throughout the paper. The functions $g_i(t,x,y,z,w)$, $i = 1,2$, are continuous, periodic in t and Lipschitz continuous in x, y, z and w, $f_i(x,y)$, $i = 1,2$, are continuous and Lipschitz continuous in x and y, and for some positive constants η_{ji}, $j = 1,\dots,4$, and $i = 1,2$, we have

$$|g_i(t,y_1,y_2,y_3,y_4) - g_i(t,x_1,x_2,x_3,x_4)| \le \sum_{j=1}^{4} \eta_{ji}|y_j - x_j|,$$

and for some positive constants ϱ_{ji}, $j = 1,2$, and $i = 1,2$, we have

$$|f_i(y_1,y_2) - f_i(x_1,x_2)| \le \sum_{j=1}^{2} \varrho_{ji}|y_j - x_j|,$$

we also assume that $g_i(t,0,0,0,0) = f_i(0,0) = 0$.

We assume that there exists a positive real number T, such that

$$\begin{cases} C_i(t+T, s+T) = C_i(t,s), \\ c_i(t+T) = c_i(t), \quad \tau_i(t+T) = \tau_i(t), \end{cases} \qquad i = 1,2, \tag{1.2}$$

for all $t \in \mathbb{R}$, with τ_i being scalar functions, continuous and $\tau_i(t) \ge \tau_i^* > 0$, $\tau_i'(t) \ne 1$.

To have a well behaved mapping we must assume that

$$\begin{cases} p_i(t+T) = p_i(t), \quad \int_0^T p_i(s)\,\mathrm{d}s > 0, \\ q_i(t+T) = q_i(t), \quad \int_0^T q_i(s)\,\mathrm{d}s > 0, \end{cases} \qquad i = 1, 2. \tag{1.3}$$

Define

$$P_T = \{x \in C(\mathbb{R}, \mathbb{R})\colon x(t+T) = x(t)\},$$
$$P_T^2 = P_T \times P_T = \{(x_1, x_2)\colon x_1 \in P_T,\ x_2 \in P_T\}.$$

Then P_T^2 is a Banach space when endowed with the maximum norm

$$\|(x_1, x_2)\| = \max\Big\{\max_{t\in[0,T]} |x_1(t)|, \max_{t\in[0,T]} |x_2(t)|\Big\}.$$

Lemma 1.1 ([9])**.** *Suppose that* (1.2) *and* (1.4) *hold and for* $i = 1, 2$,

$$\frac{R_i}{Q_i T}\big(\mathrm{e}^{\int_0^T p_i(u)\,\mathrm{d}u} - 1\big) \geq 1, \tag{1.4}$$

where

$$R_i = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\mathrm{e}^{\int_t^s p_i(u)\,\mathrm{d}u}}{\mathrm{e}^{\int_0^T p_i(u)\,\mathrm{d}u} - 1} q_i(s)\,\mathrm{d}s \right|, \qquad Q_i = (1 + \mathrm{e}^{\int_0^T p_i(u)\,\mathrm{d}u})^2 R_i^2.$$

Then there are continuous T-periodic functions a_i and b_i such that

$$b_i(t) > 0, \qquad \int_0^T a_i(u)\,\mathrm{d}u > 0,$$

and

$$a_i(t) + b_i(t) = p_i(t), \quad b_i'(t) + a_i(t)b_i(t) = q_i(t) \qquad \text{for all } t \in \mathbb{R}.$$

Lemma 1.2 ([14])**.** *Suppose the conditions of Lemma* 1.1 *hold and* $\varphi_i \in P_T$, $i = 1, 2$. *Then the equation*

$$x_i''(t) + p_i(t)x_i'(t) + q_i(t)x_i(t) = \varphi_i(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x_i(t) = \int_t^{t+T} G_i(t,s)\varphi_i(s)\,\mathrm{d}s,$$

where

$$G_i(t,s) = \frac{\int_t^s \mathrm{e}^{\int_t^u b_i(v)\,\mathrm{d}v + \int_u^s a_i(v)\,\mathrm{d}v}\,\mathrm{d}u + \int_s^{t+T} \mathrm{e}^{\int_t^u b_i(v)\,\mathrm{d}v + \int_u^{s+T} a_i(v)\,\mathrm{d}v}\,\mathrm{d}u}{\big(\mathrm{e}^{\int_0^T a_i(u)\,\mathrm{d}u} - 1\big)\big(\mathrm{e}^{\int_0^T b_i(u)\,\mathrm{d}u} - 1\big)}.$$

Corollary 1.3 ([14]). *Green's functions G_i, $i = 1, 2$, satisfies the following properties*

$$G_i(t, t+T) = G_i(t, t), \qquad G_i(t+T, s+T) = G_i(t, s),$$
$$\frac{\partial}{\partial s} G_i(t, s) = a_i(s) G_i(t, s) - H_i(t, s),$$
$$\frac{\partial}{\partial t} G_i(t, s) = -b_i(t) G_i(t, s) + H_i^*(t, s),$$

where

$$H_i(t, s) = \frac{\mathrm{e}^{\int_t^s b_i(v)\,\mathrm{d}v}}{\mathrm{e}^{\int_0^T b_i(v)\,\mathrm{d}v} - 1}, \qquad H_i^*(t, s) = \frac{\mathrm{e}^{\int_t^s a_i(v)\,\mathrm{d}v}}{\mathrm{e}^{\int_0^T a_i(v)\,\mathrm{d}v} - 1}.$$

Lemma 1.4. *Assume* (1.2)–(1.4). *If* $(x_1, x_2) \in P_T^2$, *then* x_i *is a solution of* (1.1) *if and only if*

$$\begin{aligned} x_i(t) = & \int_t^{t+T} G_i(t, u) g_i\big(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))\big)\,\mathrm{d}u \\ & + \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s))\,\mathrm{d}s\,\mathrm{d}u \\ & + \int_t^{t+T} [h_i(u) H_i(t, u) - r_i(u) G_i(t, u)] x_i(u - \tau_i(u))\,\mathrm{d}u, \quad i = 1, 2, \end{aligned} \tag{1.5}$$

where

$$h_i(u) = \frac{c_i(u)}{1 - \tau_i'(u)}, \qquad i = 1, 2, \tag{1.6}$$

$$r_i(u) = \frac{(a_i(u) c_i(u) + c_i'(u))(1 - \tau_i'(u)) + \tau_i''(u) c_i(u)}{(1 - \tau_i'(u))^2}, \qquad i = 1, 2. \tag{1.7}$$

Proof: Let $(x_1, x_2) \in P_T^2$ be a solution of (1.1). From Lemma 1.2 we have

$$\begin{aligned} x_i(t) = & \int_t^{t+T} G_i(t, u) g_i\big(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))\big)\,\mathrm{d}u \\ & + \int_t^{t+T} G_i(t, u) c_i(u) x_i'(u - \tau_i(u))\,\mathrm{d}u \\ & + \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s))\,\mathrm{d}s\,\mathrm{d}u, \qquad i = 1, 2. \end{aligned} \tag{1.8}$$

Letting

$$\int_t^{t+T} G_i(t, u) c_i(u) x_i'(u - \tau_i(u))\,\mathrm{d}u = \int_t^{t+T} \frac{G_i(t, u) c_i(u)}{1 - \tau_i'(u)} (1 - \tau_i'(u)) x_i'(u - \tau_i(u))\,\mathrm{d}u,$$

performing an integration by parts, we get

$$\int_t^{t+T} G_i(t,u)c_i(u)x_i'(u-\tau_i(u))\,\mathrm{d}u = \Big[\frac{G_i(t,u)c_i(u)}{1-\tau_i'(u)}\,x_i(u-\tau_i(u))\Big]_t^{t+T} - \int_t^{t+T}[r_i(u)G_i(t,u)-h_i(u)H_i(t,u)]x_i(u-\tau_i(u))\,\mathrm{d}u.$$

Since

$$\Big[\frac{G_i(t,u)c_i(u)}{1-\tau_i'(u)}\,x_i(u-\tau_i(u))\Big]_t^{t+T} = 0,$$

we obtain

$$\begin{aligned}\int_t^{t+T} & G_i(t,u)c_i(u)x_i'(u-\tau_i(u))\,\mathrm{d}u \\ &= \int_t^{t+T}[h_i(u)H_i(t,u)-r_i(u)G_i(t,u)]x_i(u-\tau_i(u))\,\mathrm{d}u,\end{aligned}\tag{1.9}$$

where h_i, r_i are given by (1.6) and (1.7). Substituting (1.9) into (1.8), we obtain

$$\begin{aligned}x_i(t) = & \int_t^{t+T} G_i(t,u)g_i\big(u,x_1(u),x_2(u),x_1(u-\tau_1(u)),x_2(u-\tau_2(u))\big)\,\mathrm{d}u \\ & + \int_t^{t+T} G_i(t,u)\int_{-\infty}^{u} C_i(u,s)f_i(x_1(s),x_2(s))\,\mathrm{d}s\,\mathrm{d}u \\ & + \int_t^{t+T}[h_i(u)H_i(t,u)-r_i(u)G_i(t,u)]x_i(u-\tau_i(u))\,\mathrm{d}u, \qquad i=1,2.\end{aligned}$$

□

Lemma 1.5 ([14]). *Let* $\Gamma_i = \int_0^T p_i(u)\,\mathrm{d}u$, $\Lambda_i = T^2\mathrm{e}^{(1/T)\int_0^T \ln(q_i(u))\,\mathrm{d}u}$, $i=1,2$. *If* $\Gamma_i^2 \geq 4\Lambda_i$, *then we have*

$$\min\Big\{\int_0^T a_i(u)\,\mathrm{d}u, \int_0^T b_i(u)\,\mathrm{d}u\Big\} \geq \frac{1}{2}\Big(\Gamma_i - \sqrt{\Gamma_i^2-4\Lambda_i}\Big) := l_i,$$

and

$$\max\Big\{\int_0^T a_i(u)\,\mathrm{d}u, \int_0^T b_i(u)\,\mathrm{d}u\Big\} \leq \frac{1}{2}\Big(\Gamma_i + \sqrt{\Gamma_i^2-4\Lambda_i}\Big) := m_i.$$

Corollary 1.6 ([14]). *Functions* G_i *and* H_i, $i=1,2$, *satisfy*

$$\frac{T}{(\mathrm{e}^{m_i}-1)^2} \leq G_i(t,s) \leq \frac{T\mathrm{e}^{\int_0^T p_i(v)\,\mathrm{d}v}}{(\mathrm{e}^{l_i}-1)^2}, \qquad |H_i(t,s)| \leq \frac{\mathrm{e}^{m_i}}{\mathrm{e}^{l_i}-1}.$$

To simplify notation, we introduce for $i = 1, 2$, the constants

$$\alpha_i = \frac{T \mathrm{e}^{\int_0^T p_i(v)\,\mathrm{d}v}}{(\mathrm{e}^{l_i} - 1)^2}, \qquad \gamma_i = \frac{\mathrm{e}^{m_i}}{\mathrm{e}^{l_i} - 1}, \qquad \theta_i = \max_{t\in[0,T]} |h_i(t)|,$$
$$\beta_i = \max_{t\in[0,T]} |r_i(t)|, \qquad \lambda_i = \max_{t\in[0,T]} |a_i(t)|, \qquad \delta_i = \max_{t\in[0,T]} |b_i(t)|.$$

2. Periodic solutions

Lemma 2.1 ([13]). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into S such that*

(i) *$x, y \in \mathbb{M}$ implies $Ax + By \in \mathbb{M}$;*
(ii) *A is compact and continuous;*
(iii) *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

We assume the existence of positive constants M_i, K_i and L_i, $i = 1, 2$, such that

$$|f_i(x, y)| \le M_i, \tag{2.1}$$

$$|g_i(t, x, y, z, w)| \le K_i, \tag{2.2}$$

and

$$\int_t^{t+T} \int_{-\infty}^{u} |C_i(u, s)|\,\mathrm{d}s\,\mathrm{d}u \le L_i. \tag{2.3}$$

Set

$$M = \max\Big\{ \frac{(TK_i + L_i M_i)\alpha_i}{1 - T(\theta_i\gamma_i + \beta_i\alpha_i)} \colon\ i = 1, 2 \Big\}, \tag{2.4}$$

with $0 < T(\theta_i\gamma_i + \beta_i\alpha_i) < 1$, $i = 1, 2$.

We define subset Ω_M of P_T^2 as follows

$$\Omega_M = \{(x_1, x_2) \in P_T^2 \colon \|(x_1, x_2)\| \le M\}.$$

Then Ω_M is a bounded, closed and convex subset of P_T^2.

Now for $(x_1, x_2) \in \Omega_M$ we can define an operator $E\colon \Omega_M \to P_T^2$ by

$$E(x_1, x_2)(t) = (E_1(x_1, x_2)(t), E_2(x_1, x_2)(t)),$$

where

$$
\begin{aligned}
E_i(x_1,x_2)(t) = {} & \int_t^{t+T} G_i(t,u) g_i\big(u, x_1(u), x_2(u), \\
& \qquad x_1(u-\tau_1(u)), x_2(u-\tau_2(u))\big)\,\mathrm{d}u \\
& + \int_t^{t+T} G_i(t,u) \int_{-\infty}^{u} C_i(u,s) f_i(x_1(s), x_2(s))\,\mathrm{d}s\,\mathrm{d}u \\
& + \int_t^{t+T} [h_i(u)H_i(t,u) - r_i(u)G_i(t,u)] x_i(u-\tau_i(u))\,\mathrm{d}u, \\
& \qquad i = 1,2.
\end{aligned}
\tag{2.5}
$$

To apply Lemma 2.1, we need to construct two mappings, one is a contraction and the other is compact. Therefore, we state (2.5) as

$$
E_i(x_1,x_2)(t) = B_i(x_1,x_2)(t) + A_i(x_1,x_2)(t), \qquad i = 1,2,
$$

where $B_i, A_i \colon \Omega_M \to P_T$ are given by

$$
B_i(x_1,x_2)(t) = \int_t^{t+T} G_i(t,u) g_i\big(u, x_1(u), x_2(u), x_1(u-\tau_1(u)), x_2(u-\tau_2(u))\big)\,\mathrm{d}u,
$$

and

$$
\begin{aligned}
A_i(x_1,x_2)(t) = {} & \int_t^{t+T} G_i(t,u) \int_{-\infty}^{u} C_i(u,s) f_i(x_1(s), x_2(s))\,\mathrm{d}s\,\mathrm{d}u \\
& + \int_t^{t+T} [h_i(u)H_i(t,u) - r_i(u)G_i(t,u)] x_i(u-\tau_i(u))\,\mathrm{d}u.
\end{aligned}
$$

Now for $(x_1,x_2) \in \Omega_M$ we can define the operators $B, A \colon \Omega_M \to P_T^2$ by

$$
\begin{aligned}
B(x_1,x_2)(t) &= (B_1(x_1,x_2)(t), B_2(x_1,x_2)(t)), \\
A(x_1,x_2)(t) &= (A_1(x_1,x_2)(t), A_2(x_1,x_2)(t)).
\end{aligned}
$$

Observe that, since the functions $g_i(t,x_1,x_2,x_3,x_4)$, $i=1,2$, is Lipschitz continuous in x_1,x_2,x_3,x_4 and $f_i(x_1,x_2)$, $i=1,2$, is Lipschitz continuous in x_1,x_2 we have

$$
\begin{aligned}
|g_i(t,x_1,x_2,x_3,x_4)| &= |g_i(t,x_1,x_2,x_3,x_4) - g_i(t,0,0,0,0) + g_i(t,0,0,0,0)| \\
&\le |g_i(t,x_1,x_2,x_3,x_4) - g_i(t,0,0,0,0)| + |g_i(t,0,0,0,0)| \\
&\le \sum_{j=1}^{4} \eta_{ji} |x_j|,
\end{aligned}
$$

and

$$|f_i(x_1,x_2)| = |f_i(x_1,x_2) - f_i(0,0) + f_i(0,0)|$$
$$\leq |f_i(x_1,x_2) - f_i(0,0)| + |f_i(0,0)| \leq \sum_{j=1}^{2} \varrho_{ji}|x_j|.$$

Theorem 2.2. *Suppose* (1.2)–(1.4) *and* (2.1)–(2.3) *hold. Suppose that*

$$L_i\alpha_i \sum_{j=1}^{2} \varrho_{ji} + T(\theta_i\gamma_i + \beta_i\alpha_i) \leq 1, \qquad i = 1,2,$$

and $2T\alpha_i V_i < 1$, $i = 1,2$, *where* $V_i = \max(\eta_{1i} + \eta_{3i}, \eta_{2i} + \eta_{4i})$. *Then* (1.1) *has a T-periodic solution.*

Proof: In order to prove that (1.1) has a T-periodic solution, we shall make sure that A and B satisfy the conditions of Lemma 2.1. For all $(x_1,x_2) \in \Omega_M$, we have $(x_1,x_2)(t+T) = (x_1,x_2)(t)$ and $\|(x_1,x_2)\| \leq M$. Now let us discuss $B(x_1,x_2) + A(x_1,x_2)$. We have

$$\begin{aligned}
B_i(x_1,&x_2)(t+T) \\
&= \int_{t+T}^{t+2T} G_i(t+T,u) g_i\Big(u, x_1(u), x_2(u), x_1(u-\tau_1(u)), x_2(u-\tau_2(u))\Big)\,\mathrm{d}u \\
&= \int_{t}^{t+T} G_i(t,u) g_i\big(u, x_1(u), x_2(u), x_1(u-\tau_1(u)), x_2(u-\tau_2(u))\big)\,\mathrm{d}u \\
&= B_i(x_1,x_2)(t), \qquad i = 1,2,
\end{aligned}$$

and

$$\begin{aligned}
A_i(x_1,&x_2)(t+T) \\
&= \int_{t+T}^{t+2T} G_i(t+T,u) \int_{-\infty}^{u} C_i(u,s) f_i(x_1(s),x_2(s))\,\mathrm{d}s\,\mathrm{d}u \\
&\quad + \int_{t+T}^{t+2T} [h_i(u)H_i(t+T,u) - r_i(u)G_i(t+T,u)] x_i(u-\tau_i(u))\,\mathrm{d}u \\
&= \int_{t}^{t+T} G_i(t,u) \int_{-\infty}^{u} C_i(u,s) f_i(x_1(s),x_2(s))\,\mathrm{d}s\,\mathrm{d}u \\
&\quad + \int_{t}^{t+T} [h_i(u)H_i(t,u) - r_i(u)G_i(t,u)] x_i(u-\tau_i(u))\,\mathrm{d}u \\
&= A_i(x_1,x_2)(t), \qquad i = 1,2.
\end{aligned}$$

Then $E_i(x_1,x_2)(t+T)=E_i(x_1,x_2)(t)$, $i=1,2$. Therefore, $E(x_1,x_2)(t+T)=E(x_1,x_2)(t)$.

For any $(x_1,x_2)\in\Omega_M$, we will show that $|E(x_1,x_2)(t)|\le M$. In view of the above estimates, we have for $i=1,2$,

$$\begin{aligned}
|B_i(x_1,&x_2)(t)|\\
&\le\int_t^{t+T}G_i(t,u)\big|g_i\big(u,x_1(u),x_2(u),x_1(u-\tau_1(u)),x_2(u-\tau_2(u))\big)\big|\,\mathrm{d}u\\
&\le TK_i\alpha_i,
\end{aligned}$$

and

$$\begin{aligned}
|A_i(x_1,x_2)(t)|\le&\int_t^{t+T}G_i(t,u)\int_{-\infty}^{u}|C_i(u,s)||f_i(x_1(s),x_2(s))|\,\mathrm{d}s\,\mathrm{d}u\\
&+\int_t^{t+T}[\,|h_i(u)||H_i(t,u)|+|r_i(u)|G_i(t,u)]|x_i(u-\tau_i(u))|\,\mathrm{d}u\\
\le&\ L_iM_i\alpha_i+T(\theta_i\gamma_i+\beta_i\alpha_i)M.
\end{aligned}$$

As a consequence of (2.4), we have

$$\frac{(TK_i+L_iM_i)\alpha_i}{1-T(\theta_i\gamma_i+\beta_i\alpha_i)}\le M,$$

so,

$$(TK_i+L_iM_i)\alpha_i\le(1-T(\theta_i\gamma_i+\beta_i\alpha_i))M.$$

This implies that

$$\begin{aligned}
|E_i(x_1,x_2)(t)|&\le TK_i\alpha_i+L_iM_i\alpha_i+T(\theta_i\gamma_i+\beta_i\alpha_i)M\\
&\le(1-T(\theta_i\gamma_i+\beta_i\alpha_i))M+T(\theta_i\gamma_i+\beta_i\alpha_i)M=M.
\end{aligned}$$

Thus, E maps Ω_M into itself, i.e. $E(\Omega_M)\subseteq\Omega_M$.

We will now show that A is continuous. For $n\in\mathbb{N}$, let $\{(x_{1n},x_{2n})\}$ be a sequence in Ω_M such that

$$\lim_{n\to\infty}\|(x_{1n},x_{2n})-(x_1,x_2)\|=0.$$

Since Ω_M is closed, we have $(x_1,x_2)\in\Omega_M$. Then by the definition of A we have

$$\begin{aligned}
\|A(x_{1n},x_{2n})-A(x_1,x_2)\|=\max\Big\{&\max_{t\in[0,T]}|A_1(x_{1n},x_{2n})(t)-A_1(x_1,x_2)(t)|,\\
&\max_{t\in[0,T]}|A_2(x_{1n},x_{2n})(t)-A_2(x_1,x_2)(t)|\Big\},
\end{aligned}$$

in which for $i = 1, 2$,

$$
\begin{aligned}
&|A_i(x_{1n}, x_{2n})(t) - A_i(x_1, x_2)(t)| \\
&\le \int_t^{t+T} G_i(t,u) \int_{-\infty}^{u} |C_i(u,s)||f_i(x_{1n}(s), x_{2n}(s)) - f_i(x_1(s), x_2(s))| \,\mathrm{d}s\,\mathrm{d}u \\
&\quad + \int_t^{t+T} [\,|h_i(u)||H_i(t,u)| + |r_i(u)|G_i(t,u)]|x_{in}(u-\tau_i(u)) - x_i(u-\tau_i(u))| \,\mathrm{d}u.
\end{aligned}
$$

The continuity of f_i along with the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \max_{t\in[0,T]} |A_i(x_{1n}, x_{2n})(t) - A_i(x_1, x_2)(t)| = 0.$$

Thus

$$\lim_{n\to\infty} \|A(x_{1n}, x_{2n}) - A(x_1, x_2)\| = 0.$$

This result proves that A is continuous.

We now have to show that A is compact. For $n \in \mathbb{N}$, let $\{(x_{1n}, x_{2n})\}$ be a sequence in Ω_M, then we have for $i = 1, 2$,

$$
\begin{aligned}
|A_i(x_{1n}, x_{2n})(t)| &\le \int_t^{t+T} G_i(t,u) \int_{-\infty}^{u} |C_i(u,s)||f_i(x_{1n}(s), x_{2n}(s))| \,\mathrm{d}s\,\mathrm{d}u \\
&\quad + \int_t^{t+T} [\,|h_i(u)||H_i(t,u)| + |r_i(u)|G_i(t,u)]|x_{in}(u-\tau_i(u))| \,\mathrm{d}u \\
&\le \left(L_i\alpha_i \sum_{j=1}^{2} \varrho_{ji} + T(\theta_i\gamma_i + \beta_i\alpha_i) \right) M \le M.
\end{aligned}
$$

Thus

$$\|A(x_{1n}, x_{2n})\| \le M.$$

If we calculate $(A(x_{1n}, x_{2n}))'(t)$, then for $i = 1, 2$,

$$
\begin{aligned}
&(A_i(x_{1n}, x_{2n}))'(t) \\
&\quad = \int_t^{t+T} [-b_i(t)G_i(t,u) + H_i^*(t,u)] \int_{-\infty}^{u} C_i(u,s) f_i(x_{1n}(s), x_{2n}(s)) \,\mathrm{d}s\,\mathrm{d}u \\
&\qquad + h_i(t)x_{in}(t-\tau_i(t)) - \int_t^{t+T} [b_i(t)(h_i(u)H_i(t,u) - r_i(u)G_i(t,u)) \\
&\qquad + r_i(u)H_i^*(t,u)]x_{in}(u-\tau_i(u)) \,\mathrm{d}u.
\end{aligned}
$$

Hence, for some positive constant D_i, we obtain

$$
\begin{aligned}
&|(A_i(x_{1n}, x_{2n}))'(t)| \\
&\quad \le \int_t^{t+T} (|b_i(t)|G_i(t,u) + |H_i^*(t,u)|) \int_{-\infty}^{u} |C_i(u,s)||f_i(x_{1n}(s), x_{2n}(s))| \,\mathrm{d}s\,\mathrm{d}u
\end{aligned}
$$

$$+ |h_i(t)||x_{in}(t-\tau_i(t))| + \int_t^{t+T} [\,|b_i(t)|(|h_i(u)||H_i(t,u)| + |r_i(u)|G_i(t,u))$$
$$+ |r_i(u)||H_i^*(t,u)|\,]\,|x_{in}(u-\tau_i(u))|\,\mathrm{d}u$$
$$\le (\delta_i\alpha_i + \gamma_i)L_i M \sum_{j=1}^{2} \varrho_{ji} + \theta_i M + T[\delta_i(\theta_i\gamma_i + \beta_i\alpha_i) + \beta_i\gamma_i]M \le D_i.$$

Thus

$$\|(A(x_{1n}, x_{2n}))'\| \le D,$$

where $D = \max(D_1, D_2)$. Thus, the sequence $(A(x_{1n}, x_{2n}))$ is uniformly bounded and equi-continuous. The Arzelà–Ascoli theorem implies that there exists a subsequence $(A(x_{1n_k}, x_{2n_k}))$ of $(A(x_{1n}, x_{2n}))$ converging uniformly to a continuous T-periodic function. Thus, A is compact.

For all $(x_{11}, x_{21}), (x_{12}, x_{22}) \in \Omega_M$, and for $i = 1, 2$,

$$|B_i(x_{11}, x_{21})(t) - B_i(x_{12}, x_{22})(t)|$$
$$\le \int_t^{t+T} G_i(t,u)\big|g_i\big(u, x_{11}(u), x_{21}(u), x_{11}(u-\tau_1(u)), x_{21}(u-\tau_2(u))\big)$$
$$- g_i\big(u, x_{21}(u), x_{22}(u), x_{21}(u-\tau_1(u)), x_{22}(u-\tau_2(u))\big)\big|\,\mathrm{d}u$$
$$\le T\alpha_i(\eta_{1i}|x_{11}(t) - x_{21}(t)| + \eta_{3i}|x_{11}(t-\tau_1(t)) - x_{21}(t-\tau_1(t))|$$
$$+ \eta_{2i}|x_{21}(t) - x_{22}(t)| + \eta_{4i}|x_{21}(t-\tau_2(t)) - x_{22}(t-\tau_2(t))|)$$
$$\le T\alpha_i\Big(\eta_{1i}\max_{t\in[0,T]}|x_{11}(t) - x_{21}(t)| + \eta_{3i}\max_{t\in[0,T]}|x_{11}(t-\tau_1(t)) - x_{21}(t-\tau_1(t))|$$
$$+ \eta_{2i}\max_{t\in[0,T]}|x_{21}(t) - x_{22}(t)| + \eta_{4i}\max_{t\in[0,T]}|x_{21}(t-\tau_2(t)) - x_{22}(t-\tau_2(t))|\Big)$$
$$\le T\alpha_i\Big((\eta_{1i} + \eta_{3i})\max_{t\in[0,T]}|x_{11}(t) - x_{21}(t)| + (\eta_{2i} + \eta_{4i})\max_{t\in[0,T]}|x_{21}(t) - x_{22}(t)|\Big)$$
$$\le 2T\alpha_i V_i \max\Big(\max_{t\in[0,T]}|x_{11}(t) - x_{21}(t)|, \max_{t\in[0,T]}|x_{21}(t) - x_{22}(t)|\Big),$$

hence B_i is a contraction because $2T\alpha_i V_i < 1$. Then

$$|B(x_{11}, x_{21})(t) - B(x_{12}, x_{22})(t)|$$
$$= \max\{|B_1(x_{11}, x_{21})(t) - B_1(x_{12}, x_{22})(t)|,$$
$$|B_2(x_{11}, x_{21})(t) - B_2(x_{12}, x_{22})(t)|\},$$

this implies that

$$\|B(x_{11}, x_{21}) - B(x_{12}, x_{22})\|$$
$$\le 2T\alpha V \max\Big(\max_{t\in[0,T]}|x_{11}(t) - x_{21}(t)|, \max_{t\in[0,T]}|x_{21}(t) - x_{22}(t)|\Big),$$

where $\alpha V = \max(\alpha_1 V_1, \alpha_2 V_2)$. Hence B is a contraction.

Thus, the conditions of Lemma 2.1 are satisfied and there is a $(x_1, x_2) \in \Omega_M$, such that $(x_1, x_2) = A(x_1, x_2) + B(x_1, x_2)$. □

In the next theorem for $i = 2$ we relax condition (2.1).

Theorem 2.3. *Suppose* (1.2)–(1.4), (2.1) *for* $i = 1$, (2.2) *and* (2.3) *hold. Suppose that*

$$L_i \alpha_i \sum_{j=1}^{2} \varrho_{ji} + T(\theta_i \gamma_i + \beta_i \alpha_i) \leq 1, \qquad i = 1, 2,$$

and $2T\alpha_i V_i < 1$, $i = 1, 2$, *where* $V_i = \max(\eta_{1i} + \eta_{3i}, \eta_{2i} + \eta_{4i})$. *In addition, we assume the existence of continuous nondecreasing function* W_2 *such that*

$$|f_2(x_1, x_2)| \leq f_2(|x_1|, x_2) \leq N_2 W_2(|x_1|)$$

for some positive constant N_2, *and for* $u > 0$ *we ask that*

$$\frac{W_2(u)}{u} + \frac{TK_2}{L_2 N_2 u} \leq \frac{1 - T(\theta_2 \gamma_2 + \beta_2 \alpha_2)}{L_2 N_2 \alpha_2}. \tag{2.6}$$

Then (1.1) *has a* T*-periodic solution.*

Proof: Set

$$\sigma = \frac{(TK_1 + L_1 M_1)\alpha_1}{1 - T(\theta_1 \gamma_1 + \beta_1 \alpha_1)}. \tag{2.7}$$

For any $(x_1, x_2) \in \Omega_\sigma$, we have by the proof of Theorem 2.2 that

$$|E_1(x_1, x_2)(t)| \leq \sigma.$$

Thus

$$\begin{aligned}
&|B_2(x_1,\, x_2)(t)| \\
&\qquad \leq \int_t^{t+T} G_2(t,u) \big| g_2\big(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))\big) \big| \,\mathrm{d}u \\
&\qquad \leq TK_2 \alpha_2,
\end{aligned}$$

and

$$\begin{aligned}
|A_2(x_1, x_2)(t)| \leq &\int_t^{t+T} G_2(t,u) \int_{-\infty}^{u} |C_2(u,s)| f_2(|x_1(s)|, x_2(s)) \,\mathrm{d}s \,\mathrm{d}u \\
&+ \int_t^{t+T} \big[\, |h_2(u)| |H_2(t,u)| + |r_2(u)| G_2(t,u) \big] |x_2(u - \tau_2(u))| \,\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
&\le N_2 W_2(\sigma) \int_t^{t+T} G_2(t,u) \int_{-\infty}^{u} |C_2(u,s)| \,\mathrm{d}s \,\mathrm{d}u \\
&\quad + \sigma \int_t^{t+T} \big[\, |h_2(u)||H_2(t,u)| + |r_2(u)| G_2(t,u)\big] \,\mathrm{d}u \\
&\le L_2 N_2 \alpha_2 W_2(\sigma) + T(\theta_2\gamma_2 + \beta_2\alpha_2)\sigma.
\end{aligned}$$

As a consequence of (2.6), we get

$$\frac{(TK_2 + L_2 N_2 W_2(\sigma))\alpha_2}{1 - T(\theta_2\gamma_2 + \beta_2\alpha_2)} \le \sigma,$$

so, we have

$$(TK_2 + L_2 N_2 W_2(\sigma))\alpha_2 \le (1 - T(\theta_2\gamma_2 + \beta_2\alpha_2))\sigma.$$

This implies that

$$\begin{aligned}
|E_2(x_1,x_2)(t)| &\le TK_2\alpha_2 + L_2\alpha_2 N_2 W_2(\sigma) + T(\theta_2\gamma_2 + \beta_2\alpha_2)\sigma \\
&\le (1 - T(\theta_2\gamma_2 + \beta_2\alpha_2))\sigma + T(\theta_2\gamma_2 + \beta_2\alpha_2)\sigma = \sigma.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

In the next theorem for $i = 1$ we relax condition (2.1).

Theorem 2.4. *Suppose* (1.2)–(1.4), (2.1) *for* $i = 2$, (2.2) *and* (2.3) *hold. Suppose that*

$$L_i\alpha_i \sum_{j=1}^{2} \varrho_{ji} + T(\theta_i\gamma_i + \beta_i\alpha_i) \le 1, \qquad i = 1,2,$$

and $2T\alpha_i V_i < 1$, $i = 1,2$, *where* $V_i = \max(\eta_{1i} + \eta_{3i}, \eta_{2i} + \eta_{4i})$. *In addition, we assume the existence of continuous nondecreasing function* W_1 *such that*

$$|f_1(x_1,x_2)| \le f_1(x_1,|x_2|) \le N_1 W_1(|x_2|)$$

for some positive constant N_1, *and for* $u > 0$ *we ask that*

$$\frac{W_1(u)}{u} + \frac{TK_1}{L_1 N_1 u} \le \frac{1 - T(\theta_1\gamma_1 + \beta_1\alpha_1)}{L_1 N_1 \alpha_1}. \tag{2.8}$$

Then (1.1) *has a* T*-periodic solution.*

The proof follows along the lines of the proof of Theorem 2.3, and hence we omit it here.

3. An example

Example 3.1. Consider the following coupled integro-differential system

$$
\begin{aligned}
x_1''(t) + \frac{1}{\pi}x_1'(t) + \frac{1}{10^3}x_1(t) &= \frac{1}{10^9}\sin(x_1(t)) + \frac{2}{10^9}\sin(x_2(t)) \\
&+ \frac{1}{10^8}\sin(x_1(t-2\pi)) + \frac{3}{10^8}\sin(x_2(t-4\pi)) \\
&+ \frac{2}{10^8}\sin(t)x_1'(t-2\pi) \\
&+ \int_{-\infty}^{t} \frac{1-\mathrm{e}^{-2\pi}}{\pi\, 10^8}\mathrm{e}^{-2t+2s}\Big(\frac{1}{10}\sin(x_1(s)) + \frac{1}{10^3}\sin(x_2(s))\Big)\,\mathrm{d}s, \\
x_2''(t) + \frac{1}{\pi}x_2'(t) + \frac{1}{10^3}x_2(t) &= \frac{2}{10^8}\sin(x_1(t)) + \frac{1}{10^8}\sin(x_2(t)) \\
&+ \frac{3}{10^9}\sin(x_1(t-2\pi)) + \frac{1}{10^9}\sin(x_2(t-4\pi)) \\
&+ \frac{3}{10^8}\sin(t)x_2'(t-4\pi) \\
&+ \int_{-\infty}^{t} \frac{1}{10^9}\mathrm{e}^{-t+s}\Big(\frac{1}{10^2}\sin(x_1(s)) + \frac{1}{10^4}\sin(x_2(s))\Big)\,\mathrm{d}s.
\end{aligned} \tag{3.1}
$$

Then

$$
p_1(t) = p_2(t) = \frac{1}{\pi}, \qquad q_1(t) = q_2(t) = \frac{1}{10^3}, \qquad T = 2\pi,
$$

$$
\tau_1(t) = 2\pi, \qquad \tau_2(t) = 4\pi, \qquad c_1(t) = \frac{2}{10^8}\sin(t),
$$

$$
c_2(t) = \frac{3}{10^8}\sin(t), \qquad C_1(t,s) = \frac{1-\mathrm{e}^{-2\pi}}{\pi\, 10^8}\mathrm{e}^{-2t+2s}, \qquad C_2(t,s) = \frac{1}{10^9}\mathrm{e}^{-t+s},
$$

$$
\begin{aligned}
&g_1(t, x_1(t), x_2(t), x_1(t-2\pi), x_2(t-4\pi)) \\
&\quad = \frac{1}{10^9}\sin(x_1(t)) + \frac{2}{10^9}\sin(x_2(t)) + \frac{1}{10^8}\sin(x_1(t-2\pi)) + \frac{3}{10^8}\sin(x_2(t-4\pi)), \\
&g_2(t, x_1(t), x_2(t), x_1(t-2\pi), x_2(t-4\pi)) \\
&\quad = \frac{2}{10^8}\sin(x_1(t)) + \frac{1}{10^8}\sin(x_2(t)) + \frac{3}{10^9}\sin(x_1(t-2\pi)) + \frac{1}{10^9}\sin(x_2(t-4\pi)),
\end{aligned}
$$

and

$$
\begin{aligned}
f_1(x_1(t), x_2(t)) &= \frac{1}{10}\sin(x_1(t)) + \frac{1}{10^3}\sin(x_2(t)), \\
f_2(x_1(t), x_2(t)) &= \frac{1}{10^2}\sin(x_1(t)) + \frac{1}{10^4}\sin(x_2(t)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|g_1(t,x_1(t),\, x_2(t),x_1(t-2\pi),x_2(t-4\pi))| \\
&\qquad\qquad \le \frac{1}{10^9}|x_1(t)|+\frac{2}{10^9}|x_2(t)|+\frac{1}{10^8}|x_1(t-2\pi)|+\frac{3}{10^8}|x_2(t-4\pi)|,\\
&|g_2(t,x_1(t),x_2(t),x_1(t-2\pi),x_2(t-4\pi))| \\
&\qquad\qquad \le \frac{2}{10^8}|x_1(t)|+\frac{1}{10^8}|x_2(t)|+\frac{3}{10^9}|x_1(t-2\pi)|+\frac{1}{10^9}|x_2(t-4\pi)|,
\end{aligned}
$$

$$
|f_1(x_1,x_2)| \le \frac{1}{10}|x_1(t)|+\frac{1}{10^3}|x_2(t)|,
$$

and

$$
|f_1(x_1,x_2)| \le \frac{1}{10^2}|x_1(t)|+\frac{1}{10^4}|x_2(t)|.
$$

So,

$$
\begin{array}{llll}
\eta_{11}=\dfrac{1}{10^9}, & \eta_{21}=\dfrac{2}{10^9}, & \eta_{31}=\dfrac{1}{10^8}, & \eta_{41}=\dfrac{3}{10^8},\\
\eta_{12}=\dfrac{2}{10^8}, & \eta_{22}=\dfrac{1}{10^8}, & \eta_{32}=\dfrac{3}{10^9}, & \eta_{42}=\dfrac{1}{10^9},\\
\varrho_{11}=\dfrac{1}{10}, & \varrho_{21}=\dfrac{1}{10^3}, & \varrho_{12}=\dfrac{1}{10^2}, & \varrho_{22}=\dfrac{1}{10^4}.
\end{array}
$$

We check the conditions of Lemma 1.1 for $i=1,2$,

$$
\begin{aligned}
R_i &= \max_{t\in[0,2/\pi]}\left|\int_t^{t+2\pi}\frac{e^{\int_t^s(1/\pi)\,du}}{e^{\int_0^{2\pi}(1/\pi)\,du}-1}\frac{1}{10^3}\,ds\right| \simeq 0.003,\\
Q_i &= \left(1+e^{\int_0^{2\pi}(1/\pi)\,du}\right)^2 R_i^2 \simeq 0.0006,
\end{aligned}
$$

and

$$
\frac{R_i}{2\pi Q_i}\left(e^{\int_0^{2\pi}(1/\pi)\,du}-1\right)\simeq 5.0842 \ge 1,
$$

this implies

$$
a_i(t)=0.0032,\quad b_i(t)=0.3152,\qquad i=1,2.
$$

We check the conditions of Lemma 1.5

$$
\Gamma_i=\int_0^{2\pi}\frac{1}{\pi}\,du=2,\qquad \Lambda_i=(2\pi)^2 e^{(1/2\pi)\int_0^{2\pi}\ln(1/10^3)\,du}\simeq 0.0395,
$$

and

$$
2^2 \ge 4\cdot 0.0395 \Rightarrow \Gamma_i^2 \ge 4\Lambda_i,\qquad i=1,2,
$$

then we have for $i=1,2$,

$$\begin{aligned}\min\left\{\int_0^{2\pi}0.0032\,\mathrm{d}u,\int_0^{2\pi}0.3152\,\mathrm{d}u\right\} &\geq \frac{1}{2}\Big(\Gamma_i-\sqrt{\Gamma_i^2-4\Lambda_i}\Big)\\ &\simeq \frac{1}{2}\Big(2-\sqrt{2^2-4\cdot 0.0395}\Big)=l_i\simeq 0.0199,\\ \max\left\{\int_0^{2\pi}0.0032\,\mathrm{d}u,\int_0^{2\pi}0.3152\,\mathrm{d}u\right\} &\leq \frac{1}{2}\Big(\Gamma_i+\sqrt{\Gamma_i^2-4\Lambda_i}\Big)\\ &\simeq \frac{1}{2}\Big(2+\sqrt{2^2-4\times 0.0395}\Big)=m_i\simeq 1.9801.\end{aligned}$$

By Corollary 1.6, we get

$$\begin{aligned}\frac{2\pi}{(\mathrm{e}^{1.9801}-1)^2}\simeq 0.1611\leq G_i(t,s)\leq \frac{2\pi\,\mathrm{e}^{\int_0^{2\pi}(1/\pi)\,\mathrm{d}v}}{(\mathrm{e}^{0.0199}-1)^2}\simeq 1.15\cdot 10^5, \qquad i=1,2,\\ |H_i(t,s)|\leq \frac{\mathrm{e}^{1.9801}}{\mathrm{e}^{0.0199}-1}\simeq 3.61\cdot 10^2, \qquad i=1,2.\end{aligned}$$

We obtain

$$\begin{aligned}&\alpha_i=1.15\cdot 10^5,\quad \gamma_i=3.61\cdot 10^2,\quad \lambda_i=0.0032,\quad \delta_i=0.3152, \qquad i=1,2,\\ &\theta_1=\max_{t\in[0,2\pi]}|h_1(t)|=\max_{t\in[0,2\pi]}|c_1(t)|=\max_{t\in[0,2\pi]}\left|\frac{2}{10^8}\sin(t)\right|=\frac{2}{10^8},\\ &\theta_2=\max_{t\in[0,2\pi]}|h_2(t)|=\max_{t\in[0,2\pi]}|c_2(t)|=\max_{t\in[0,2\pi]}\left|\frac{3}{10^8}\sin(t)\right|=\frac{3}{10^8},\\ &\beta_1=\max_{t\in[0,2\pi]}|r_1(t)|=\max_{t\in[0,2\pi]}|a_1(t)c_1(t)+c_1'(t)|\\ &\quad=\max_{t\in[0,2\pi]}\left|0.0032\cdot\frac{2}{10^8}\sin(t)+\frac{2}{10^8}\cos(t)\right|=\frac{2.0064}{10^8},\\ &\beta_2=\max_{t\in[0,2\pi]}|r_2(t)|=\max_{t\in[0,2\pi]}|a_2(t)c_2(t)+c_2'(t)|\\ &\quad=\max_{t\in[0,2\pi]}\left|0.0032\cdot\frac{3}{10^8}\sin(t)+\frac{3}{10^8}\cos(t)\right|=\frac{3.0096}{10^8},\end{aligned}$$

$$|f_1(x_1(t),x_2(t))|\leq M_1=\frac{101}{10^3},\qquad |f_2(x_1(t),x_2(t))|\leq M_2=\frac{101}{10^4},$$

$$\begin{aligned}|g_1(t,x_1(t),x_2(t),x_1(t-2\pi),x_2(t-4\pi))|\leq K_1=\frac{43}{10^9},\\ |g_2(t,x_1(t),x_2(t),x_1(t-2\pi),x_2(t-4\pi))|\leq K_2=\frac{34}{10^9},\end{aligned}$$

$$\int_t^{t+2\pi} \int_{-\infty}^{u} \left| \frac{1-\mathrm{e}^{-2\pi}}{\pi\, 10^8} \mathrm{e}^{-2u+2s} \right| \mathrm{d}s\, \mathrm{d}u \leq L_1 = \frac{0.9982}{10^8},$$
$$\int_t^{t+2\pi} \int_{-\infty}^{u} \left| \frac{1}{10^9} \mathrm{e}^{-u+s} \right| \mathrm{d}s\, \mathrm{d}u \leq L_2 = \frac{2\pi}{10^9}.$$

Therefore,

$$L_1\alpha_1 \sum_{j=1}^{2} \varrho_{j1} + T(\theta_1\gamma_1 + \beta_1\alpha_1) = 0.0145 \leq 1,$$
$$L_2\alpha_2 \sum_{j=1}^{2} \varrho_{j2} + T(\theta_2\gamma_2 + \beta_2\alpha_2) = 0.0218 \leq 1,$$

and

$$V_1 = \max(\eta_{11} + \eta_{31}, \eta_{21} + \eta_{41}) = \max\left(\frac{1}{10^9} + \frac{1}{10^8}, \frac{2}{10^9} + \frac{3}{10^8}\right) = \frac{32}{10^9},$$
$$V_2 = \max(\eta_{12} + \eta_{32}, \eta_{22} + \eta_{42}) = \max\left(\frac{2}{10^8} + \frac{3}{10^9}, \frac{1}{10^8} + \frac{1}{10^9}\right) = \frac{23}{10^9},$$

so,

$$2T\alpha_1 V_1 = 4\pi \cdot 1.15 \cdot 10^5 \cdot \frac{32}{10^9} = 0.0462 < 1,$$
$$2T\alpha_2 V_2 = 4\pi \cdot 1.15 \cdot 10^5 \cdot \frac{23}{10^9} = 0.0332 < 1.$$

The conditions of Theorem 2.2 are satisfied, then (3.1) has a 2π-periodic solution.

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