Comment.Math.Univ.Carolin. 63,1 (2022) 21–31 

# On the probability that two elements of a finite semigroup have the same right matrix

Attila Nagy, Csaba Tóth

*Abstract.* We study the probability that two elements which are selected at random with replacement from a finite semigroup $S$ have the same right matrix.

*Keywords:* congruence; equivalence relation; probability; semigroup

*Classification:* 20M10, 60B99

## 1. Introduction and motivation

There are many papers in the mathematical literature which use probabilistic methods to study special algebraic structures. Only the papers [1], [3]–[6] and [8] are cited here, because we refer only to them. All of these papers deal with special cases of the following problem. For a given finite algebraic structure $A$ and a given binary relation $\sigma$ on $A$, find the probability $P_\sigma(A)$ that $(a,b)\in\sigma$ is satisfied for two elements $a$ and $b$ of $A$ which are selected at random with replacement. We note that random elements are chosen independently with the uniform distribution on $A$. Thus every couple $(a,b)\in A\times A$ has the same probability $1/|A|^2$ of being chosen and so

$$P_\sigma(A)=\frac{|\{(a,b)\in A\times A\colon (a,b)\in\sigma\}|}{|A|^2}.$$

In [1], [6] and [8] the probability $P_\sigma(A)$ is examined in the cases, when $A$ is a finite noncommutative semigroup, a noncommutative group and a noncommutative ring, respectively. In all three cases $\sigma$ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if $xy=yx$. In [4], the probability $P_\sigma(A)$ is investigated in that case when $A$ is a finite simple group and $\sigma$ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if $w(x,y)=e$, where $w$ is a given nontrivial element of the free group $F_2$ and $e$ is the identity element of the group $A$. In both [3] and [5], the probability $P_\sigma(A)$ is examined in that case when $A$ is the symmetric group $S_n$ of degree $n$. In [3], $\sigma$ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if $x$ and $y$ generate the group $S_n$.

DOI 10.14712/1213-7243.2022.008

This work was supported by the National Research, Development and Innovation Office – NKFIH, 115288.

In [5], $\sigma$ is defined by $(x, y) \in \sigma$ for $x, y \in A$ if and only if $x$ and $y$ generate the alternating group $A_n$ or the group $S_n$.

In the theory of semigroups the right regular matrix representation plays a very important role. The above investigations motivate us to examine the probability $P_{\theta_S}(S)$, where $S$ is a finite semigroup and $\theta_S$ is the kernel of the right regular matrix representation of $S$. In other words, we examine the following problem. Two elements $a$ and $b$ are selected at random with replacement from a finite semigroup $S$. What is the probability that $a$ and $b$ have the same right matrix? We show that $P_{\theta_S}(S) \geq 1/|S/\theta_S|$ for every finite semigroup $S$, where $S/\theta_S$ denotes the factor semigroup of $S$ modulo $\theta_S$. In the paper we also deal with the following question. How does the structure of a finite semigroup $S$ depend on the probability $P_{\theta_S}(S)$. This question is very general to answer. For example, if $G$ is a group of order 3 and $S$ is a semigroup defined by the Cayley table then $P_{\theta_G}(G) =$

| | $e$ | $a$ | $b$ | $x$ | $x^2$ | $x^3$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $e$ | $e$ | $x$ | $x^2$ | $x^3$ |
| $a$ | $e$ | $e$ | $e$ | $x$ | $x^2$ | $x^3$ |
| $b$ | $e$ | $e$ | $e$ | $x$ | $x^2$ | $x^3$ |
| $x$ | $x$ | $x$ | $x$ | $x^2$ | $x^3$ | $e$ |
| $x^2$ | $x^2$ | $x^2$ | $x^2$ | $x^3$ | $e$ | $x$ |
| $x^3$ | $x^3$ | $x^3$ | $x^3$ | $e$ | $x$ | $x^2$ |

TABLE 1.

$P_{\theta_S}(S) = 1/3$, but the structures of $G$ and $S$ are very different. In this paper we deal with a special case of the above question. Our main goal is to describe the structure of a finite semigroup $S$ if $P_{\theta_S}(S) = 1/|S/\theta_S|$. In this paper we give solutions in two cases. In the first case $S$ is an arbitrary finite semigroup with $|S/\theta_S| = 1$; in the second case $S$ is a finite commutative semigroup with $|S/\theta_S| = 2$.

## 2. Preliminaries

By a semigroup we mean a nonempty set together with an associative multiplication. Let $S$ be a semigroup and $G^0$ be a semigroup arising from a one-element group $G = \{1\}$ by the adjunction of a zero element 0. By an *$S \times S$ matrix over $G^0$*, we mean a mapping of $S \times S$ into $G^0$. Let $A$ be an $S \times S$ matrix over $G^0$. For an element $s \in S$, the set $\{A(s, x) \colon x \in S\}$ is called a *row* (the *s-row*) of $A$. An

$S\times S$ matrix $A$ over $G^0$ is called *strictly row-monomial* if each row of $A$ contains exactly one nonzero element of $G^0$.

For an element $a$ of a semigroup $S$, let $R^{(a)}$ denote the strictly row-monomial $S$-matrix over $G^0$ defined by

$$R^{(a)}((x,y)) = \begin{cases} 1, & \text{if } \ xa = y, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix is called the *right matrix over* $G^0$ *defined by* $a$. (The right matrices were investigated, for example, in [11] and [14], in the case when 1 is the identity element and 0 is the zero element of a field.) It is known that

$$a \mapsto R^{(a)}$$

is a homomorphism of a semigroup $S$ into the multiplicative semigroup of all strictly row-monomial $S$-matrices over $G^0$, see [2, Exercise 4 (b) for Section 3.5]. This homomorphism is called the *right regular matrix representation* of the semigroup $S$.

A semigroup $S$ is said to be *left reductive*, see [10], if the following condition is satisfied for arbitrary elements $a$ and $b$ of $S$: $sa=sb$ for all $s\in S$ implies $a=b$. The right regular matrix representation of a semigroup $S$ is *faithful* (*injective*) if and only if $S$ is left reductive.

Let $\theta_S$ denote the kernel of the right regular matrix representation of a semigroup $S$. It is obvious that

$$\theta_S = \{(a,b)\in S\times S\colon\ \forall x\in S \ \ xa=xb\}.$$

In this paper, we investigate the probability

$$P_{\theta_S}(S) = \frac{|\{(a,b)\in S\times S\colon (a,b)\in\theta_S\}|}{|S|^2}.$$

In our investigation, the following construction (which is a special case of the construction defined in part (a) of [12, Theorem 2]) plays an important role.

**Construction 2.1** ([13, Construction 1])**.** Let $T$ be a left cancellative semigroup (that is, a semigroup with the property that $xa=xb$ implies $a=b$ for every $x,a,b\in T$). For each $t\in T$, associate a nonempty set $S_t$ such that $S_x\cap S_y=\emptyset$ for all $x,y\in T$ with $x\neq y$.

For an arbitrary couple $(t,x)\in T\times T$, let $(\cdot)\varphi_{t,tx}$ be a mapping of $S_t$ into $S_{tx}$ acting on the right. For all $t,x,y\in T$, and $a\in S_t$, assume

$$(a)(\varphi_{t,tx}\circ\varphi_{tx,txy}) = (a)\varphi_{t,txy}.$$

On the set $S=\bigcup_{t\in T} S_t$ define an operation "$\star$" as follows: for arbitrary $a\in S_t$ and $b\in S_x$ let

$$a\star b=(a)\varphi_{t,tx}.$$

As every left cancellative semigroup is left reductive, [12, Theorem 2] implies that $(S;\star)$ is a semigroup such that the sets $S_t$, $t\in T$, are the $\theta_S$-classes of $S$.

## 3. Results

Let $A$ be a nonempty set and $\sigma$ be a binary relation on $A$. Let $P_\sigma(A)$ denote the probability that $(a,b)\in\sigma$, where $a$ and $b$ are selected at random with replacement from the set $A$.

**Theorem 3.1.** *Let $p$ be an arbitrary rational number with $0\leq p\leq 1$. Then the following assertions are equivalent:*

(i) *There is a finite semigroup $S$ such that $P_{\theta_S}(S)=p$.*

(ii) *There is a nonempty finite set $A$ and an equivalence relation $\sigma$ on $A$ such that $P_\sigma(A)=p$.*

Proof: It is sufficient to show that (ii) implies (i). Assume (ii). We use Construction 2.1. Let $T$ be a commutative group of order $|A/\sigma|$. Such group always exists. Let $S_t$, $t\in T$, denote the $\sigma$-classes of $A$. For every $t\in T$ fix an element $s_t$ in $S_t$. For every $t,x\in T$ let $(\cdot)\varphi_{t,tx}$ be the mapping of $S_t$ into $S_{tx}$ which maps the elements of $S_t$ to $s_{tx}$. It is easy to see that the family $\{\varphi_{t,tx}\colon t,x\in T\}$ of mappings satisfies the following condition: for every $t,x,y\in T$ and every $a\in S_t$

$$(a)(\varphi_{t,tx}\circ\varphi_{tx,txy})=(a)\varphi_{t,txy}.$$

Thus $S=\bigcup_{t\in T} S_t$ forms a semigroup under the operation "$\star$" defined by the following way: for every $t,x\in T$ and $a\in S_t$, $b\in S_x$

$$a\star b=(a)\varphi_{t,tx}=s_{tx}.$$

As $T$ is left reductive, [12, Theorem 2] implies that the $\theta_S$-classes of the semigroup $S$ are the sets $S_t$ $(t\in T)$. Consequently $P_{\theta_S}(S)=P_\sigma(A)=p$. $\square$

We note that the semigroup $S$ defined in the proof of Theorem 3.1 is commutative, because

$$a\star b=(a)\varphi_{t,tx}=s_{tx}=s_{xt}=(b)\varphi_{x,xt}=b\star a$$

is satisfied for every $t,x\in T$ and $a\in S_t$, $b\in S_x$.

Let $A$ be a nonempty finite set and $\sigma$ be an equivalence relation on $A$. If $m$ denotes the cardinality of the factor set $A/\sigma$ (which is called the index of $\sigma$) and

$t_1, \ldots, t_m$ are the cardinalities of the $\sigma$-classes of $A$, then

$$P_\sigma(A) = \frac{t_1^2 + \cdots + t_m^2}{(t_1 + \cdots + t_m)^2}.$$

By the well known connection between the root mean square and the arithmetic mean, we have

$$\sqrt{\frac{t_1^2 + \cdots + t_m^2}{m}} \geq \frac{t_1 + \cdots + t_m}{m},$$

that is,

$$\frac{t_1^2 + \cdots + t_m^2}{m} \geq \frac{(t_1 + \cdots + t_m)^2}{m^2}$$

from which we get

$$P_\sigma(A) = \frac{t_1^2 + \cdots + t_m^2}{(t_1 + \cdots + t_m)^2} \geq \frac{1}{m} = \frac{1}{|A/\sigma|}.$$

The equation $P_\sigma(A) = 1/|A/\sigma|$ holds if and only if

$$t_1 = \cdots = t_m.$$

In particular, $P_{\theta_S}(S) \geq 1/|S/\theta_S|$ for every finite semigroup $S$. In addition, $P_{\theta_S}(S) = 1/|S/\theta_S|$ if and only if each $\theta_S$-class contains the same number of elements.

In the next part of the paper, we deal with a special case of the following problem: How does the structure of a finite semigroup $S$ depend on the probability $P_{\theta_S}(S)$? Our question is the following: What can we say about the structure of a finite semigroup $S$, if the index of $\theta_S$ is $m$ and $P_{\theta_S}(S) = 1/m$?

We deal with this question for $m = 1, 2$. In the case of $m = 1$, we give a solution to the question in the class of all semigroups. For $m = 2$, we answer the question in the class of all commutative semigroups.

**3.1 Case** $m = 1$**.** For a finite semigroup $S$, the assumption $P_{\theta_S}(S) = 1$ is satisfied if and only if the index of $\theta_S$ is 1, that is, $\theta_S$ is the universal relation on $S$. The next theorem characterizes not necessarily finite semigroups $S$ in which $\theta_S$ is the universal relation on $S$. We shall use the following notions.

A homomorphism $\varphi$ of a semigroup $S$ onto an ideal $I \subseteq S$ is called a *retract homomorphism* [9, Definition 1.44] if $\varphi$ leaves the elements of $I$ fixed. An ideal $I$ of a semigroup $S$ is called a *retract ideal* if there is a retract homomorphism of $S$ onto $I$. In this case, we say that $S$ is a *retract* (*ideal*) *extension of* $I$ *by the Rees factor semigroup* $S/I$.

A semigroup satisfying the identity $ab = a$ is called a *left zero semigroup*, see [7]. A semigroup with a zero element 0 is called a *zero semigroup* if it satisfies the identity $ab = 0$.

**Theorem 3.2.** *For a semigroup $S$, $\theta_S$ is the universal relation on $S$ if and only if $S$ is a retract extension of a left zero semigroup by a zero semigroup.*

Proof: Let $S$ be a semigroup in which $\theta_S$ is the universal relation on $S$. Then $xa = xb$ holds for every $x, a, b \in S$. Let $a \in S$ be an arbitrary element. Then

$$a^2 = aa = aa^2 = a^3,$$

and so

$$(a^2)^2 = aa^3 = aa^2 = a^3 = a^2,$$

that is, $a^2$ is an idempotent element. Let $E(S)$ denote the set of all idempotent elements of $S$. As $ab = a^2 \in E(S)$ for every $a, b \in S$, the set $E(S)$ is an ideal of $S$, and the Rees factor semigroup $Q = S/E(S)$ is a zero semigroup. For arbitrary $e, f \in E(S)$,

$$ef = ee = e.$$

Hence $E(S)$ is a left zero semigroup. For every $a \in S$, we have $aS \subseteq E(S)$ and $|aS| = 1$. For every $a \in S$, let $(a)\varphi$ denote the element of $aS$. By the above,

$$(a)\varphi \in E(S)$$

for every $a \in S$. Moreover, $(e)\varphi = e$ for every idempotent element $e$ of $S$. Let $x^* \in S$ be an arbitrary fixed element. Then, for every $a, b \in S$ we have

$$(ab)\varphi = abx^* = ax^*bx^* = (a)\varphi(b)\varphi.$$

Hence $\varphi$ is a homomorphism of $S$ onto $E(S)$. As $\varphi$ leaves the elements of $E(S)$ fixed, it is a retract homomorphism of $S$ onto $E(S)$. Thus $S$ is a retract extension of the left zero semigroup $E(S)$ by the zero semigroup $Q = S/E(S)$.

Conversely, let $S$ be a semigroup and $I$ be an ideal of $S$ such that $I$ is a left zero semigroup, the Rees factor semigroup $S/I$ is a zero semigroup, and there is a retract homomorphism $\varphi$ of $S$ onto $I$. Then, for arbitrary $x, a, b \in S$, we have $xa, xb \in I$ and so

$$xa = (xa)\varphi = (x)\varphi(a)\varphi = (x)\varphi = (x)\varphi(b)\varphi = (xb)\varphi = xb.$$

Hence $\theta_S$ is the universal relation on $S$. Thus the theorem is proved. $\square$

In the next part of this subsection, we show how to construct semigroups $S$ in which $\theta_S$ is the universal relation.

**Construction 3.3.** Let $S$ be a nonempty set and $L$ be a nonempty subset of $S$. Let $(\cdot)\varphi$ be an arbitrary mapping of $S$ onto $L$ which leaves the elements of $L$ fixed. Define an operation "$\star$" on $S$ as follows: for arbitrary $a, b \in S$, let

$a \star b = (a)\varphi$. For every $a, b, c \in S$,

$$a \star (b \star c) = a \star (b)\varphi = (a)\varphi = ((a)\varphi)\varphi = (a \star b)\varphi = (a \star b) \star c,$$

that is, $S$ is a semigroup with the operation "$\star$". This semigroup is denoted by $(S, L, \varphi, \star)$.

**Theorem 3.4.** *In the semigroup $S = (S, L, \varphi, \star)$, the equation $\theta_S = \omega_S$ is satisfied. Conversely, every semigroup $S$ in which $\theta_S = \omega_S$ is satisfied is isomorphic to a semigroup defined in Construction* 3.3.

Proof: For every $a, b \in (S, L, \varphi, \star)$, we have $a \star b \in L$. Thus $L$ is an ideal of $S$ and the Rees factor semigroup $S/L$ is a zero semigroup. For every $a, b \in L$, we have

$$a \star b = (a)\varphi = a.$$

Thus $L$ is a left zero semigroup. As

$$(a \star b)\varphi = ((a)\varphi)\varphi = (a)\varphi = ((a)\varphi)((b)\varphi),$$

$\varphi$ is a retract homomorphism of $S$ onto $L$. Thus $S = (S, L, \varphi, \star)$ is a retract extension of the left zero semigroup $L$ by the zero semigroup $S/L$. Consequently $\theta_S = \omega_S$ by Theorem 3.2.

Conversely, assume that $S$ is a semigroup in which $\theta_S = \omega_S$. By Theorem 3.2, there is an ideal $L$ of $S$ such that $L$ is a left zero semigroup, the Rees factor semigroup $S/L$ is a zero semigroup, and there is a retract homomorphism $\varphi$ of $S$ onto $L$. Consider the semigroup $(S, L, \varphi, \star)$ defined as in Construction 3.3. As

$$ab = (ab)\varphi = (a)\varphi(b)\varphi = (a)\varphi = a \star b$$

for every $a, b \in S$, the semigroups $S$ and $(S, L, \varphi, \star)$ are isomorphic. □

**3.2 Case $m = 2$.** In our investigation, the following three examples play an important role.

**Example 3.5.** Let $A$ and $B$ be zero semigroups such that $A \cap B = \emptyset$. Let $e$ and $f$ denote the zero elements of $A$ and $B$, respectively. Let $S = A \cup B$. We define an operation on $S$ as follows:

$$xy = \begin{cases} e, & \text{if } \ x, y \in A; \\ f, & \text{otherwise.} \end{cases}$$

It is easy to see that $S$ is a commutative semigroup whose $\theta_S$-classes are $A$ and $B$. The factor semigroup $S/\theta_S$ is a two-element semilattice, see [15], (that is, a two-element commutative semigroup in which every element is an idempotent element).

**Example 3.6.** Let $A$ be a zero semigroup with a zero element $e$. Let $B$ be a nonempty set with $A \cap B = \emptyset$. Let $S = A \cup B$. We define an operation on $S$ as follows: fix an element $b^*$ in $B$, and let

$$xy = \begin{cases} e, & \text{if } \ x, y \in A \ \text{ or } \ x, y \in B; \\ b^*, & \text{otherwise.} \end{cases}$$

It is a matter of checking to see that $S$ is a commutative semigroup whose $\theta_S$-classes are $A$ and $B$. The factor semigroup $S/\theta_S$ is a two-element group.

**Example 3.7.** Let $A$ be a zero semigroup with a zero element $e$. Let $B$ be a nonempty set with $A \cap B = \emptyset$. Let $S = A \cup B$. We define an operation on $S$ as follows: fix an element $a^* \in A$ with $e \neq a^*$, and let

$$xy = \begin{cases} a^*, & \text{if } \ x, y \in B; \\ e, & \text{otherwise.} \end{cases}$$

It is a matter of checking to see that $S$ is a commutative semigroup whose $\theta_S$-classes are $A$ and $B$. The factor semigroup $S/\theta_S$ is a two-element zero semigroup.

**Theorem 3.8.** *On an arbitrary semigroup $S$, the following conditions are equivalent:*

(1) *$S$ is a commutative semigroup such that the index of $\theta_S$ is* 2.
(2) *$S$ is isomorphic to one of the semigroups defined in Example* 3.5, *Example* 3.6, *and Example* 3.7.

Proof: As the semigroups $S$ defined in Example 3.5, Example 3.6 and Example 3.7 are commutative such that the index of $\theta_S$ is 2, it is sufficient to show that (1) implies (2). Let $S$ be a commutative semigroup such that the index of $\theta_S$ is 2. Then the factor semigroup $S/\theta_S$ is a two-element commutative semigroup. Thus $S/\theta_S$ is either a two-element semilattice or a two-element group or a two-element zero semigroup. Let $A$ and $B$ denote the $\theta_S$-classes of $S$.

First we consider the case when $S/\theta_S$ is a two-element semilattice. Then $A$ and $B$ are subsemigroups of $S$ such that one of $A$ and $B$, say $B$, is an ideal of $S$. It is easy to see that $\theta_A = \omega_A$ and $\theta_B = \omega_B$. Then, by Theorem 3.2, $A$ and $B$ are zero semigroups. Let $e$ and $f$ denote the zero element of $A$ and $B$, respectively. For every $x, y \in A$, we have $xy = e$. For every $x, y \in B$, we have $xy = f$. For every $x \in A$ and $y \in B$, we have

$$yx = xy = xf = xff = f,$$

because $(y, f) \in \theta_S$, $xf \in B$ and $f$ is the zero element of $B$.

Summarizing our results, we have

$$xy = \begin{cases} e, & \text{if } x, y \in A; \\ f, & \text{otherwise.} \end{cases}$$

Thus $S$ is isomorphic to a semigroup defined in Example 3.5.

In the next part of the proof, we consider the case when $S/\theta_S$ is a two-element group. Let $A$ be the $\theta_S$-class corresponding to the identity element of $S/\theta_S$. Then $A$ is a subsemigroup of $S$. Since $\theta_A = \omega_A$, Theorem 3.2 implies that $A$ is a zero semigroup. Let $e$ denote the zero element of $A$. As $B$ is a $\theta_S$-class, we have that $eB \subseteq B$ is a singleton. Let $b^*$ denote the element of $eB$. Then, for every $x \in A$ and $y \in B$, we have

$$xy = yx = ye = ey = b^*,$$

because $(x, e) \in \theta_S$. For every $x, y \in B$, we get $ex \in B$ and $x^2 \in A$, hence

$$xy = xex = x^2 e = e.$$

Summarizing our results, we have

$$xy = \begin{cases} e, & \text{if } x, y \in A \text{ or } x, y \in B; \\ b^*, & \text{otherwise.} \end{cases}$$

Thus $S$ is isomorphic to a semigroup defined in Example 3.6.

Consider the case when $S/\theta_S$ is a two-element zero semigroup. Let $A$ be the $\theta_S$-class corresponding to the zero element of $S/\theta_S$. Then, by Theorem 3.2, $A$ is a zero semigroup. Let $e$ denote the zero element of $A$. Then $xy = e$ for every $x, y \in A$. For an arbitrary $y \in B$,

$$ey = eey = e,$$

because $ey \in A$ and $e$ is the zero element of $A$. Thus, for every $x \in A$ and $y \in B$, we have

$$xy = yx = ye = ey = e,$$

because $(x, e) \in \theta_S$. Let $x_0, y_0 \in B$ be fixed arbitrary elements, and let

$$a^* = x_0 y_0 = y_0 x_0.$$

Then, for every $x, y \in B$, we have

$$xy = xy_0 = y_0 x = y_0 x_0 = a^*,$$

because $B$ is a $\theta_S$-class.

Summarizing our results, we have

$$xy = \begin{cases} a^*, & \text{if } x, y \in B; \\ e, & \text{otherwise.} \end{cases}$$

If $a^*$ was the zero element of $A$, then we would have

$$xa = xb = e$$

for every $a \in A$, $b \in B$ and $x \in S$, which would imply that $\theta_S = \omega_S$. It would be a contradiction. Hence $a^* \neq e$. Thus $S$ is isomorphic to a semigroup defined in Example 3.7.

In a finite semigroup $S$, both of the conditions that the index of $\theta_S$ is $m$ and $P_{\theta_S}(S) = 1/m$ are satisfied if and only if each $\theta_S$-class contains the same number of elements. Thus the following result is a consequence of Theorem 3.8. □

**Theorem 3.9.** *On an arbitrary semigroup $S$, the following conditions are equivalent:*

(1) *$S$ is a finite commutative semigroup such that the index of $\theta_S$ is* 2 *and* $P_{\theta_S}(S) = 1/2$.

(2) *$S$ is isomorphic to one of the semigroups defined in Example* 3.5, *Example* 3.6 *and Example* 3.7 *in which* $|A| = |B| < \infty$.

**Remark 3.10.** Commutative semigroups defined in Example 3.5, Example 3.6, and Example 3.7 can also be obtained by using the construction defined in part (a) of [12, Theorem 2]. It seems to be that this construction would also be a useful tool to investigate our problem in other cases.

**Acknowledgement.** The authors are thankful to the referee's suggestions on the first version of this paper.

## References

[1] Ahmedidelir K., Campbell C. M., Doostie H., *Almost commutative semigroups*, Algebra Colloq. **18** (2011), Special Issue no. 1, 881–888.

[2] Clifford A. H., Preston G. B., *The Algebraic Theory of Semigroups, Vol. I.*, Mathematical Surveys, 7, American Mathematical Society, Providence, 1961.

[3] Dixon J. D., *The probability of generating the symmetric group*, Math. Z. **110** (1969), 199–205.

[4] Dixon J. D., Pyber L., Seress Á., Shalev A., *Residual properties of free groups and probabilistic methods*, J. Reine Angew. Math. **556** (2003), 159–172.

[5] Eberhard S., Virchow S.-C., *The probability of generating the symmetric group*, Combinatorica **39** (2019), no. 2, 273–288.

[6] Gustafson W. H., *What is the probability that two group elements commute?*, Amer. Math. Monthly **80** (1973), 1031–1034.
[7] Howie J. M., *An Introduction to Semigroup Theory*, L. M. S. Monographs, 7, Academic Press, London, 1976.
[8] MacHale D., *Commutativity in finite rings*, Amer. Math. Monthly **83** (1975), no. 1, 30–32.
[9] Nagy A., *Special Classes of Semigroups*, Advances in Mathematics (Dordrecht), 1, Kluwer Academic Publishers, Dordrecht, 2001.
[10] Nagy A., *Left reductive congruences on semigroups*, Semigroup Forum **87** (2013), no. 1, 129–148.
[11] Nagy A., *On faithful representations of finite semigroups $S$ of degree $|S|$ over the fields*, Int. J. of Algebra **7** (2013), no. 1–4, 115–129.
[12] Nagy A., *Remarks on the paper "M. Kolibiar, On a construction of semigroups"*, Period. Math. Hungar. **71** (2015), no. 2, 261–264.
[13] Nagy A., *Left equalizer simple semigroups*, Acta Math. Hungar. **148** (2016), no. 2, 300–311.
[14] Nagy A., Rónyai L., *Finite semigroups whose semigroup algebra over a field has a trivial right annihilator*, Int. J. of Contemp. Math. Sci. **9** (2014), no. 1–4, 25–36.
[15] Petrich M., *Lectures in Semigroups*, John Wiley & Sons, London, 1977.

A. Nagy, C. Tóth:
Department of Algebra, Budapest University of Technology and Economics,
Egry József u. 3, 1111 Budapest, Hungary

*E-mail:* nagyat@math.bme.hu

*E-mail:* tcsaba94@gmail.com

(*Received* February 17, 2020, *revised* December 5, 2021)