Mersenne numbers as a difference of two Lucas numbers

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Abstract. Let $(L_n)_{n\geq 0}$ be the Lucas sequence. We show that the Diophantine equation $L_n - L_m = M_k$ has only the nonnegative integer solutions $(n,m,k) = (2,0,1)$, $(3,1,2)$, $(3,2,1)$, $(4,3,2)$, $(5,3,3)$, $(6,2,4)$, $(6,5,3)$ where $M_k = 2^k - 1$ is the kth Mersenne number and $n > m$.

Keywords: Lucas number; Mersenne number; Diophantine equation; linear forms in logarithm

Classification: 11B39, 11J86, 11D61

1. Introduction

Let $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ be the Fibonacci and Lucas sequences given by $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$, respectively. Because of their interesting properties, these two sequences have been investigated from various point of view in many papers. Especially over the last decade products, sums and differences of these two sequences have been investigated by a number of authors, see for example [4], [5], [7], [9], [12], [13], [14]. We refer [10] for a detailed information on these sequences.

Recall that Mersenne numbers are the numbers of the form $2^k - 1$, where k is a nonnegative integer. In [2], Mersenne numbers in a generalized Fibonacci numbers are found. In this study, we search the differences of two Lucas numbers which are Mersenne numbers. More precisely the aim of this paper is to prove the following theorem.

Theorem 1. *The equation*

$$L_n - L_m = 2^k - 1, \qquad n > m, \tag{1}$$

has only the nonnegative integer solutions $(n,m,k) \in \{(2,0,1), (3,1,2), (3,2,1), (4,3,2), (5,3,3), (6,2,4), (6,5,3)\}$.

The proof of the above theorem mainly depends on two results one of them gives us a general lower bound for linear forms in logarithms and the other one

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is a version of reduction method given in Baker–Davenport lemma, see [1]. We summarize these two results in the next chapter. It is worth to note that when it is needed we used the software Maple for all calculations and computations.

2. Preliminaries

It is well-known that all Lucas numbers L_n can be written as

$$L_n = \alpha^n + \beta^n, \qquad n \geq 0,$$

where

$$\alpha := \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta := \frac{1-\sqrt{5}}{2}$$

are the roots of the equation $x^2 - x - 1 = 0$. This is the Binet formula of Lucas numbers. By using the Binet formula, one can see that,

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \tag{2}$$

holds for all $n > 0$.

Lemma 2. If $n \equiv m \pmod 3$ then $L_n \equiv L_m \pmod 2$

Proof: Assume that $n \equiv m \pmod 3$. Assume also without loss of generality that $n \geq m$ and let $n = m + 3k$ for some nonnegative integer k. Then the proof follows by induction on k, since $L_{m+3} = L_{m+2} + L_{m+1} = 2L_{m+1} + L_m \equiv L_m \pmod 2$. □

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}),$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. The logarithmic height of η is defined by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\}) \right).$$

In particular, for a rational number p/q with $\gcd(p, q) = 1$ and $q > 0$, $h(p/q) = \log \max\{|p|, q\}$. The following properties are very useful for calculation of $h(\eta)$:

- $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$.
- $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$.
- $h(\eta^s) = |s| h(\eta)$.

Theorem 3 (Matveev's theorem, [11]). *Assume that* $\gamma_1, \ldots, \gamma_t$ *are positive real algebraic numbers in a real algebraic number field* $\mathbb{K}$ *of degree* $d_{\mathbb{K}}$, $b_1, \ldots, b_t$ *are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\big(- 1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \cdots A_t \big),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{d_{\mathbb{K}} h(\gamma_i), |\log \gamma_i|, 0.16\} \qquad \text{for all } \; i = 1, \ldots, t.$$

In [6, Lemma 5 (a)], A. Dujella and A. Pethő give a version of the reduction method based on the Baker–Davenport lemma, see [1]. The following lemma is from [3, Lemma 4] which is a variation of [6, Lemma 5 (a)] and it is useful for reducing some upper bounds on the variables. For a real number θ, we put $\|\theta\| = \min\{|\theta - n| \colon n \in \mathbb{Z}\}$, the distance from θ to the nearest integer.

Lemma 4. Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. If $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$, then there is no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \qquad \text{and} \qquad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Proof of Theorem 1

Proof of Theorem 1: Assume that the equation (1) holds. A quick computation with Maple shows that the equation (1) has the solutions $(n, m, k) \in \{(2, 0, 1), (3, 1, 2), (3, 2, 1), (4, 3, 2), (5, 3, 3), (6, 2, 4), (6, 5, 3)\}$ in nonnegative integers in the range $0 \leq m < n \leq 200$. If $n - m = 1$ and $m \neq 0$ or $n - m = 2$, then the equation (1) turns into the equations $L_{m-1} = 2^k - 1$ and $L_{m+1} = 2^k - 1$, respectively. All solutions of these two equations only come from the equalities $L_1 + 1 = 2$, $L_2 + 1 = 2^2$ and $L_4 + 1 = 2^8$, see [4], which corresponds already known solutions $(n, m, k) \in \{(3, 2, 1), (4, 3, 2), (6, 5, 3)\}$ and $(n, m, k) \in \{(2, 0, 1), (3, 1, 2),$

$(5,3,3)\}$ of (1). Also, by Lemma 2, the case $n-m=3$ is clearly false. So we will take $n-m \geq 4$ and $n > 200$.

From (1), we may get a relation between variables n and k for future reference. Indeed from (1) and (2) we have that

$$2^{k-1} \leq 2^k - 1 = L_n - L_m < L_n \leq 2\alpha^n < 2^{n+1},$$

and hence we observe that

$$k < n+2. \tag{3}$$

Using Binet formula for Lucas sequences we rewrite equation (1) as

$$\alpha^n + \beta^n - L_m = 2^k - 1,$$
$$\alpha^n - 2^k = L_m - \beta^n - 1.$$

So

$$\left|1 - \frac{2^k}{\alpha^n}\right| < \frac{L_m}{\alpha^n} + \frac{|\beta|^n}{\alpha^n} + \frac{1}{\alpha^n} < \frac{2\alpha^m}{\alpha^n} + \frac{1}{\alpha^{2n}} + \frac{1}{\alpha^n} < \frac{4}{\alpha^{n-m}}.$$

Hence we have that

$$|\Lambda_1| := \left|1 - \frac{2^k}{\alpha^n}\right| < \frac{4}{\alpha^{n-m}}. \tag{4}$$

Now we may apply Theorem 3 to the left side of the inequality (4) with $\eta_1 := 2$, $\eta_2 := \alpha$ and $b_1 := k$, $b_2 := -n$. Clearly $\eta_1, \eta_2 \in \mathbb{K} = \mathbb{Q}(\sqrt{5})$, and hence we take $d_{\mathbb{K}} = 2$, the degree of the number field $\mathbb{K}$. Since $\alpha^n \neq 2^k \in \mathbb{Z}$ for $n > 0$, we see that Λ_1 is nonzero. Further since $h(\eta_1) = \log 2$ and $h(\eta_2) = (1/2)\log\alpha$, we take $A_1 = 2\log 2$, $A_2 = \log\alpha$. By (3) we know that $k < n+2$, so we take

$$B := \max\{|b_i|\} = \max\{k, n\} < n+2.$$

Now we are ready to apply Theorem 3 to the inequality (4) and we get that

$$\log|\Lambda_1| > -1.4 \times 30^5 \times 2^{4.5} \times 2^2(1+\log 2)(1+\log(n+2))2\log 2 \cdot \log\alpha,$$

which implies that

$$\log|\Lambda_1| > 3.48 \times 10^9(1+\log(n+2)). \tag{5}$$

On the other hand, from (4), we know that

$$\log|\Lambda_1| < \log 4 - (n-m)\log\alpha. \tag{6}$$

From (5) and (6) we find that

$$n - m < 7.24 \times 10^9 \times (1+\log(n+2)). \tag{7}$$

Now we rewrite equation (1) in the Binet formula as

$$\begin{aligned} \alpha^n + \beta^n - \alpha^m - \beta^m &= 2^k - 1, \\ \alpha^n - \alpha^m - 2^k &= -\beta^n + \beta^m - 1, \\ \alpha^n(1 - \alpha^{m-n}) - 2^k &= -\beta^n + \beta^m - 1. \end{aligned}$$

So from the last equation above we write

$$\left|1 - \frac{2^k}{\alpha^n(1 - \alpha^{m-n})}\right| < \frac{1}{\alpha^n(1 - \alpha^{m-n})}(|\beta|^n + |\beta|^m + 1) < \frac{3}{\alpha^n},$$

where we used the fact that $|\beta|^n + |\beta|^m + 1 < 2.01$ and $(1 - \alpha^{m-n})^{-1} < 1.4$ for $m \geq 0$, $n \geq 200$ and $n - m \geq 3$. So

$$|\Lambda_2| := \left|1 - \frac{2^k}{\alpha^n(1 - \alpha^{m-n})}\right| < \frac{3}{\alpha^n}. \tag{8}$$

Let $\eta_1 := 2$, $\eta_2 := \alpha$, $\eta_3 := (1 - \alpha^{m-n})^{-1}$ and $b_1 := k, b_2 := -n, b_3 := 1$. Since η_1, η_2 and η_3 are elements of the quadratic number field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, we take $d_{\mathbb{K}} = 2$, the degree of the number field $\mathbb{K}$. Since $h(\eta_1) = \log 2$, $h(\eta_2) = (1/2)\log\alpha$ and

$$\begin{aligned} h(\eta_3) &= h((1 - \alpha^{m-n})^{-1}) = h(1 - \alpha^{m-n}) \\ &\leq |m - n|h(\alpha) + \log 2 = \frac{n-m}{2}\log\alpha + \log 2 \end{aligned}$$

we take

$$A_1 = 2\log 2, \qquad A_2 = \log\alpha \qquad \text{and} \qquad A_3 = (n - m)\log\alpha + 2\log 2.$$

By (3), $k < n + 2$, so we take

$$B := \max\{|b_i|\} = \max\{n, k, 1\} < n + 2.$$

Note that Λ_2 is nonzero. Indeed if it were zero then we would have that $\alpha^n - \alpha^m = 2^k$. Conjugating in $\mathbb{K}$, we get that $\beta^n - \beta^m = -2^k$ and adding these two equation we get that $L_n - L_m = 0$, which is false since $n > m$. Thus we apply Theorem 3 to Λ_2 given in (8) and we get that

$$\begin{aligned} \log(|\Lambda_2|) > \; & -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(n + 2)) \\ & \times 2\log 2 \cdot \log\alpha \cdot ((n - m)\log\alpha + 2\log 2). \end{aligned} \tag{9}$$

On the other hand, from (8), we know that

$$\log|\Lambda_2| < \log 3 - n\log\alpha. \tag{10}$$

Combining (9) and (10) we get that

$$n < 1.35 \cdot 10^{12} \times (1 + \log(n+2)) \times ((n-m)\log\alpha + 2\log 2). \tag{11}$$

From the inequality (11) together with (7), it follows that

$$n < 1.8 \cdot 10^{25}. \tag{12}$$

Now we use the theory of continued fractions to reduce the upper bound on $n-m$. More precisely we show that $n-m < 137$. To this end, assume that $n-m \geq 137$. Let

$$\Gamma_1 := k\log 2 - n\log\alpha \tag{13}$$

so that

$$|\Lambda_1| := |\exp(\Gamma_1) - 1| < \frac{4}{\alpha^{n-m}} < \frac{1}{2},$$

and hence we get that

$$|\Gamma_1| < \frac{8}{\alpha^{n-m}}.$$

So from (13), we write

$$\left|\frac{\log\alpha}{\log 2} - \frac{k}{n}\right| < \frac{8}{n\alpha^{n-m}\log 2}.$$

Since $2.8 \cdot 10^{28} < \alpha^{137}\log 2$, we have the inequality

$$16n < 16 \cdot 1.8 \cdot 10^{25} < 2.9 \cdot 10^{26} < \alpha^{137}\log 2 \leq \alpha^{n-m}\log 2.$$

Thus we write

$$\left|\frac{\log\alpha}{\log 2} - \frac{k}{n}\right| < \frac{8}{n\alpha^{n-m}\log 2} < \frac{1}{2n^2},$$

which means that k/n is a convergent of continued fractions of irrational $\tau := \log\alpha/\log 2$, say p_k/q_k. Since p_k and q_k are relatively prime integers we get that $q_k \leq n < 1.8 \times 10^{25}$. Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [0, 1, 2, 3, 1, \ldots]$ be the continued fraction expansion of τ. By Maple, we see that $k \leq 56$ and $\max\{a_i\} = 134$ for $i = 1, 2, \ldots, 56$. Thus, from the well known property of continued fractions [8, Theorem 163], we write

$$\frac{1}{136n^2} < \frac{1}{(a_k+2)n^2} < \left|\frac{\log\alpha}{\log 2} - \frac{k}{n}\right| < \frac{8}{n\alpha^{n-m}\log 2},$$

which means

$$2.7 \cdot 10^{25} < \frac{\alpha^{137}\log 2}{136 \cdot 8} \leq \frac{\alpha^{n-m}\log 2}{136 \cdot 8} < n,$$

a contradiction since $n < 1.8 \times 10^{25}$. So we conclude that $n - m < 137$. By substituting this upper bound for $n - m$ into (11) we get that

$$n < 1.35 \cdot 10^{12} \times (1 + \log(n+2)) \times (137 \log \alpha + 2 \log 2), \tag{14}$$

which implies that

$$n < 4 \cdot 10^{15}. \tag{15}$$

Let

$$\Gamma_2 := k \log 2 - n \log \alpha - \log\left(1 - \alpha^{m-n}\right) \tag{16}$$

so that

$$|\Lambda_2| := |\exp \Gamma_2 - 1| < \frac{3}{\alpha^n} < \frac{1}{2}.$$

Then

$$|\Gamma_2| < \frac{6}{\alpha^n}.$$

So, from (16),

$$0 < \left| k \frac{\log 2}{\log \alpha} - n - \frac{\log\left(1 - \alpha^{m-n}\right)}{\log \alpha} \right| < \frac{6}{\alpha^n \log \alpha}. \tag{17}$$

Now we take

$$M := 4 \cdot 10^{15} + 2 > n + 2 > k \qquad \text{and} \qquad \tau := \frac{\log 2}{\log \alpha}.$$

Then, in the continued fraction expansion of τ, we take q_{50}, the denominator of the 50th convergent of τ, which exceeds $6M$. Now with the help of Maple we calculate

$$\varepsilon_{n-m} := \|\mu_{n-m} q_{50}\| - M \|\tau q_{50}\|$$

for each $n - m \in \{4, \ldots, 137\}$ where

$$\mu_{n-m} := -\frac{\log\left(1 - \alpha^{m-n}\right)}{\log \alpha}, \qquad q_{50} = 90411515862404307875 39$$

and we get that

$$0.000012 < \varepsilon_{n-m} \qquad \text{for all} \;\; n - m \in \{4, \ldots, 137\}$$

except for $n - m = 6$, since for $n - m = 6$, $\varepsilon_6 < 0$. So we need to handle this case separately. But, from Lemma 2, this is not the case. So $n - m \neq 6$.

Let $A := 6/\log\alpha$, $B := \alpha$ and $\omega := n$. Thus from Lemma (4) we get that the inequality (17) has no solution for

$$n > \frac{\log\left(Aq_{50}/0.000012\right)}{\log B} \geq 133.8.$$

So we get that $n < 134$ which is a contradiction since $n > 200$. This completes the proof. □

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