The common division topology on $\mathbb{N}$

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Abstract. A topological space X is totally Brown if for each $n \in \mathbb{N}\setminus\{1\}$ and every nonempty open subsets $U_1, U_2, \dots, U_n$ of X we have $\mathrm{cl}_X(U_1) \cap \mathrm{cl}_X(U_2) \cap \cdots \cap \mathrm{cl}_X(U_n) \neq \emptyset$. Totally Brown spaces are connected. In this paper we consider a topology τ_S on the set $\mathbb{N}$ of natural numbers. We then present properties of the topological space $(\mathbb{N}, \tau_S)$, some of them involve the closure of a set with respect to this topology, while others describe subsets which are either totally Brown or totally separated. Our theorems generalize results proved by P. Szczuka in 2013, 2014, 2016 and by P. Szyszkowska and M. Szyszkowski in 2018.

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1. Introduction

We denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and of natural numbers, respectively. We define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_b = \{n \in \mathbb{N} \colon n \geq b\}$ for each $b \in \mathbb{N}$. The symbol $\mathbb{P}$ denotes the set of prime numbers. We assume that $\mathbb{P} \subset \mathbb{N}$. Given nonzero integers a and b, the symbol $\langle a, b\rangle$ denotes the greatest common divisor of a and b. Note that $\langle a, b\rangle \in \mathbb{N}$.

In this paper we consider arithmetic progressions in both $\mathbb{N}$ and $\mathbb{Z}$. Namely, for each $a, b \in \mathbb{N}$ we define

$$P(a,b) = \{b + an \colon n \in \mathbb{N}_0\} = b + a\mathbb{N}_0 \qquad \text{and} \qquad M(a) = \{an \colon n \in \mathbb{N}\}.$$

If $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ we also define

$$P_F(a,b) = \{b + az \colon z \in \mathbb{Z}\} = b + a\mathbb{Z}.$$

In both [8, page 663] and [9, page 179], S. W. Golomb showed that the family

$$\mathcal{B}_G = \{P(a,b) \colon (a,b) \in \mathbb{N} \times \mathbb{N} \text{ and } \langle a, b\rangle = 1\}$$

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is a base for a topology τ_G on $\mathbb{N}$. In [11] A. M. Kirch considered the family

$$\mathcal{B}_K = \{P(a,b) \in \mathcal{B}_G \colon a \text{ is square-free}\},$$

which is a base for a topology τ_K on $\mathbb{N}$ so that $\tau_K \subset \tau_G$. In [1] the first two authors study properties of the spaces $(\mathbb{N}, \tau_G)$ and $(\mathbb{N}, \tau_K)$. In this paper we consider a topology τ_S on $\mathbb{N}$ and present new properties of the space $(\mathbb{N}, \tau_S)$.

We divide the paper in four sections. After this Introduction in Section 2 we present notions and results on topological spaces. In Subsection 2.1 we consider the notion of a Brown space and of a totally Brown space, as well as properties different from the ones shown in [1, Section 3]. In Section 3 we present results on arithmetic progressions, different from the ones that appear in [1, Section 4].

In Section 4 we consider properties of the topological space $(\mathbb{N}, \tau_S)$. For example this space is totally Brown, not homogeneous and concerning its separation axioms it is T_D but not $T_{\frac{1}{2}}$. In Subsection 4.1 we show that $P(a,b)$ is totally separated when $a \in \mathbb{N}_2$. In Subsection 4.2 we study the points at which $(\mathbb{N}, \tau_S)$ is either locally connected or connected im kleinen or almost connected im kleinen. In Subsection 4.3 we present results that involve the closure of an arithmetic progression with respect to $(\mathbb{N}, \tau_S)$. The important results of this subsection are Theorems 4.15, 4.16 and 4.17 since with them we can calculate in a simple way the closure in $(\mathbb{N}, \tau_S)$ of an arithmetic progression. In Subsection 4.4 we characterize in Theorem 4.24 the arithmetic progressions that are totally Brown in $(\mathbb{N}, \tau_S)$. It turns out that each arithmetic progression is either totally separated or totally Brown in $(\mathbb{N}, \tau_S)$. The same two possibilities occur in $(\mathbb{N}, \tau_G)$, see [1, Corollary 5.15], but not in $(\mathbb{N}, \tau_K)$, since all arithmetic progressions are totally Brown in $(\mathbb{N}, \tau_K)$, see [1, Theorem 6.9].

2. Topological spaces

In this section we collect several notions, results and terminology from general topology that we use in the paper. The symbol $|Z|$ denotes the cardinality of the set Z. If (X, τ) is a topological space and $A \subset X$, then the symbols $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ denote the closure and the interior of A in (X, τ), respectively. If $A \subset Y \subset X$, then $\mathrm{cl}_Y(A) = Y \cap \mathrm{cl}_X(A)$. If we need to specify the topology τ on X we write $\mathrm{cl}_{(X,\tau)}(A)$ and $\mathrm{int}_{(X,\tau)}(A)$, respectively.

We say that $x \in X$ is an *indiscrete point* of X if $\{U \in \tau \colon x \in U\} = \{X\}$. The topological space (X, τ) is said to be

1) $T_{2\frac{1}{2}}$ or *Urysohn* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau$ so that $x \in U$, $y \in V$ and $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$;

2) *hereditarily disconnected* if X does not contain any connected subset of cardinality larger than one;
3) *totally separated* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau$ so that $x \in U$, $y \in V$, $X = U \cup V$ and $U \cap V = \emptyset$;
4) *connected im kleinen at* $x \in X$ if for each $U \in \tau$ with $x \in U$, there is a connected subset V of X such that $x \in \operatorname{int}_X(V) \subset V \subset U$;
5) *almost connected im kleinen at* $x \in X$ if for each $U \in \tau$ with $x \in U$, there is a closed and connected subset V of X such that $\operatorname{int}_X(V) \neq \emptyset$ and $V \subset U$;
6) *locally connected at* $x \in X$ if for each $U \in \tau$ with $x \in U$, there is $V \in \tau$ connected so that $x \in V \subset U$; and *locally connected* if X is locally connected at each of its points;
7) *homogeneous* if for each $x, y \in X$, there exists a homeomorphism $f\colon X \to X$ so that $f(x) = y$.

Urysohn spaces are also called completely Hausdorff spaces. By [6, Theorem 6.1.22, page 356] totally separated spaces are hereditarily disconnected. Being totally separated is hereditary. Though the notions are not equivalent, in the literature both totally separated spaces as well as hereditarily disconnected spaces have been called totally disconnected. If X is locally connected at $x \in X$, then X is connected im kleinen at x. A space which is connected im kleinen at some point y but not locally connected at y is shown in [14, Examples 119 and 120, page 139].

Let (X, τ) be a topological space and $A \subset X$. The derived set of A is

$$A' = \{y \in X\colon \text{ for each } U \in \tau \text{ with } y \in U \text{ we have } A \cap (U \setminus \{y\}) \neq \emptyset\}.$$

Note that $\operatorname{cl}_X(A) = A \cup A'$. In [3, page 29] it is said that X is T_D if for each $x \in X$ the derived set $\{x\}'$ is closed in X. In [12, Definition 2.1, page 90] the set A is said to be *g-closed* if for each open subset U of X with $A \subset U$ it follows that $\operatorname{cl}_X(A) \subset U$. In [12, Definition 5.1, page 93] it is said that X is $T_{\frac{1}{2}}$ if every g-closed set is closed in X. In both [5] and [10] it is proved that X is $T_{\frac{1}{2}}$ if and only if for every $x \in X$ the one-point-set $\{x\}$ is either open or closed in X. It is known that

$$T_1 \implies T_{\frac{1}{2}} \implies T_D \implies T_0$$

and that the reverse inclusions do not hold. For notions and results related with general topology and not given here, we refer the reader to [6].

2.1 Totally Brown spaces. Let X be a topological space. In [1, Definition 3.1] it is said that X is a *Brown space* if for every nonempty open subsets U and V of X, we have $\operatorname{cl}_X(U) \cap \operatorname{cl}_X(V) \neq \emptyset$. The space X is *totally Brown* if for every

$n \in \mathbb{N}_2$ and each nonempty open subsets $U_1, U_2, \ldots, U_n$ of X we have

$$\mathrm{cl}_X(U_1) \cap \mathrm{cl}_X(U_2) \cap \cdots \cap \mathrm{cl}_X(U_n) \neq \emptyset.$$

Clearly

$$\text{totally Brown} \implies \text{Brown} \implies \text{connected.}$$

In [1, Section 3] examples are given to show that the above implications are not reversible. The following properties are easy to show:

1) nondegenerate Brown spaces are not Urysohn;
2) nondegenerate Brown T_1 spaces are infinite.

In this subsection we present more properties of both Brown and totally Brown spaces. Let X be a topological space and $Y \subset X$. Then Y is totally Brown in X if and only if for every $n \in \mathbb{N}_2$ and each nonempty open subsets $O_1, O_2, \ldots, O_n$ of Y we have

$$Y \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2) \cap \cdots \cap \mathrm{cl}_X(O_n) \neq \emptyset.$$

Similarly Y is Brown in X if and only if for every nonempty open subsets U and V of Y we have $Y \cap \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \emptyset$.

Theorem 2.1. *If X contains an indiscrete point, then each nonempty closed subset of X is totally Brown in X. In particular, X is totally Brown.*

Proof: Let $x \in X$ be an indiscrete point of X. Then $x \in \mathrm{cl}_X(U)$ for every nonempty subset U of X. Let C be a nonempty closed subset of X. Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of C. Then $x \in C \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2) \cap \cdots \cap \mathrm{cl}_X(O_n)$, so C is totally Brown in X. □

Theorem 2.2. *Let X be a topological space whose nonempty open sets are dense in X. Then each nonempty open set is totally Brown in X. In particular, X is totally Brown.*

Proof: Let U be a nonempty open subset of X. Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of U. For every $i \in \{1, 2, \ldots, n\}$ the set O_i is open in X so it is dense in X. Hence $U \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2) \cap \cdots \cap \mathrm{cl}_X(O_n) = U \neq \emptyset$. This shows that U is totally Brown in X. □

If X is infinite and we consider the cofinite topology

$$\tau_C = \{\emptyset\} \cup \{U \subset X : |X \setminus U| < \aleph_0\},$$

then any nonempty open subset of X is dense in (X, τ_C), so by Theorem 2.2 each nonempty open subset of X is totally Brown in (X, τ_C).

The following result presents a condition under which a union of Brown spaces is a Brown space.

Theorem 2.3. *Let X be a topological space and $\{B_i\colon i \in I\}$ be a family of Brown spaces in X. Assume that*

($\star$) *for each $i, j \in I$ with $i \neq j$ if U and V are nonempty open subsets of B_i and B_j, respectively, then either $\mathrm{cl}_X(V) \cap \mathrm{cl}_{B_i}(U) \neq \emptyset$ or $\mathrm{cl}_X(U) \cap \mathrm{cl}_{B_j}(V) \neq \emptyset$.*

Then $B = \bigcup_{i\in I} B_i$ is Brown in X.

Proof: Let O_1 and O_2 be two nonempty open subsets of B. Take open subsets U_1 and U_2 of X so that

$$O_1 = B \cap U_1 = \bigcup_{i\in I}(B_i \cap U_1) \qquad \text{and} \qquad O_2 = B \cap U_2 = \bigcup_{i\in I}(B_i \cap U_2).$$

Since both O_1 and O_2 are nonempty, there exist $i, j \in I$ such that $U = B_i \cap U_1 \neq \emptyset$ and $V = B_j \cap U_2 \neq \emptyset$. Clearly $B_i, B_j \subset B$, $U \subset O_1$ and $V \subset O_2$. If $i = j$ then, using that U and V are nonempty open subsets of the Brown space B_i, we have $B_i \cap \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \emptyset$. Then

$$\emptyset \neq B_i \cap \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \subset B \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2).$$

If $i \neq j$ then by ($\star$) we can assume without loss of generality that $\mathrm{cl}_X(V) \cap \mathrm{cl}_{B_i}(U) \neq \emptyset$. Hence

$$\emptyset \neq \mathrm{cl}_X(V) \cap \mathrm{cl}_{B_i}(U) = B_i \cap \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \subset B \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2).$$

We deduce in both cases that $B \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2) \neq \emptyset$, so B is Brown in X. $\square$

3. Arithmetic progressions

In this section we present results on arithmetic progression that we use in the rest of the paper. For $a, b \in \mathbb{Z}$, the symbol $a|b$ means that $b = ac$ for some $c \in \mathbb{Z}$. If $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$, the symbol $a \equiv b \pmod{m}$ means that $m|(a-b)$. Note that $x \in P_F(a,b)$ if and only if $a|(x-b)$, i.e., $x \equiv b \pmod{a}$. Similarly $x \in P(a,b)$ if and only if $a|(x-b)$ and $x \geq b$, i.e., $x \equiv b \pmod{a}$ and $x \in \mathbb{N}_b$. We also have $P(a,b) \subset \mathbb{N}_b$, $M(a) = P(a,a)$, $M(1) = \mathbb{N}$ and

$$P(a,b) = P_F(a,b) \cap \mathbb{N}_b.$$

Hence $P(a,b) \subset P_F(a,b)$. We say that $a \in \mathbb{N}_2$ is *square-free* if its standard prime decomposition is of the form $\prod_{i=1}^k p_i$.

Given two arithmetic progressions in $\mathbb{N}$ the following result characterizes when one of these is contained in the other one.

Theorem 3.1. *Let* $a, b, c, d \in \mathbb{N}$. *Then*

$$P(c,d) \subset P(a,b) \quad \text{if and only if } a|c \text{ and } d \in P(a,b). \tag{1}$$

In particular,

1) *for each* $c \in P(a,b)$, *we have* $P(a,c) \subset P(a,b)$;
2) $P(ac,b) \subset P(a,b) \cap P(c,b)$;
3) $P(a^n,b) \subset P(a,b)$ *for every* $n \in \mathbb{N}$;
4) *if* $b, c \in M(a)$, *then* $P(c,b) \subset M(a)$;
5) $P(a,b) = P(c,d)$ *if and only if* $a = c$ *and* $b = d$.

PROOF: Assume first that $P(c,d) \subset P(a,b)$. Then $d, d+c \in P(a,b)$, so $a|(d-b)$ and $a|[(d+c)-b]$. Hence $a|[(d+c-b)-(d-b)]$, i.e., $a|c$. Now consider that $a|c$ and $d \in P(a,b)$. Then $a|(d-b)$ and $d \geq b$. If $z \in P(c,d)$, then $c|(z-d)$ and $z \geq d$. Thus $a|[(z-d)+(d-b)]$, i.e., $a|(z-b)$ and $z \geq b$. Then $z \in P(a,b)$. This completes the proof of (1) and from this it is straightforward to show 1)–5). □

Theorem 3.2. *Let* $a, b, c, d \in \mathbb{N}$ *be so that* $a|c$ *and* $d \equiv b \pmod a$. *Then* $P(c,d) \subset \mathbb{N} \cap P_F(a,b)$ *and if* $a \geq b$, *then* $\mathbb{N} \cap P_F(c,d) \subset P(a,b)$. *In particular*

$$P(a,b) = \mathbb{N} \cap P_F(a,b) \qquad \text{if } a \geq b.$$

PROOF: Let $z \in P(c,d)$. Then $c|(z-d)$ and $z \geq d$. Therefore $a|(z-d)$ and $a|(d-b)$, so $a|[(z-d)+(d-b)]$, i.e., $a|(z-b)$. Hence $z \in \mathbb{N} \cap P_F(a,b)$. This shows the first part. Now assume that $a \geq b$ and take $x \in \mathbb{N} \cap P_F(c,d)$. Then $c|(x-d)$ and $a|(d-b)$, so $a|[(x-d)+(d-b)]$, i.e., $a|(x-b)$. Hence $x \in P_F(a,b)$. If $x < b$, then $a, b-x \in \mathbb{N}$ and since $a|(b-x)$ we infer that $a \leq b-x < b$, a contradiction to the fact that $a \geq b$. Thus $x \geq b$ so $x \in P_F(a,b) \cap \mathbb{N}_b = P(a,b)$. This completes the second part. Since $a|a$ and $b \equiv b \pmod a$, the third part follows from the first two. □

Corollary 3.3. *Let* $a \in \mathbb{N}_2$ *and* $b \in \mathbb{N}$ *be so that* $\langle a,b \rangle = 1$. *Then*

$$\mathbb{N} \cap P_F(c,b) \subset \mathbb{N} \setminus M(a) \qquad \text{for each } c \in M(a). \tag{2}$$

In particular, for every $d \in \mathbb{N}$ *with* $d \equiv b \pmod a$ *and each* $n \in \mathbb{N}$ *we have*

$$P(a^n,d) \subset \mathbb{N} \setminus M(a). \tag{3}$$

PROOF: Fix $c \in M(a)$ and take $z \in \mathbb{N} \cap P_F(c,b)$. If $z \in M(a)$, then $a|z$, $a|c$ and $c|(z-b)$, so $a|(z-b)$ which implies that $a|[z-(z-b)]$, i.e., $a|b$ contradicting the fact that $\langle a,b \rangle = 1$. Then $z \in \mathbb{N} \setminus M(a)$. This shows (2). Now let $n \in \mathbb{N}$ and $d \in \mathbb{N}$ be so that $d \equiv b \pmod a$. Since $a|a^n$ and $d \equiv b \pmod a$ by the first part of Theorem 3.2 and (2) we have $P(a^n,d) \subset \mathbb{N} \cap P_F(a,b) \subset \mathbb{N} \setminus M(a)$. □

The next result appears in [1, Theorem 4.7].

Theorem 3.4. *Let $k \in \mathbb{N}_2$, $a_1, b_1, a_2, b_2, \dots, a_k, b_k \in \mathbb{N}$. Then the following conditions are equivalent.*

1) $\bigcap_{i=1}^{k} P(a_i, b_i) \neq \emptyset$;
2) $\langle a_i, a_j\rangle | (b_i - b_j)$ *for each* $i, j \in \{1, 2, \dots, k\}$ *with* $i \neq j$;
3) $P(a_i, b_i) \cap P(a_j, b_j) \neq \emptyset$ *for each* $i, j \in \{1, 2, \dots, k\}$ *with* $i \neq j$.

Theorem 3.4 remains true if we replace P by P_F in 1) and 3), i.e., if we consider arithmetic progressions in $\mathbb{Z}$. Hence if $a, c \in \mathbb{N}$ and $b, d \in \mathbb{Z}$ then

$$P_F(a, b) \cap P_F(c, d) \neq \emptyset \qquad \text{if and only if } \langle a, c\rangle | (b - d), \tag{4}$$

and if $b, d \in \mathbb{N}$ then

$$P(a, b) \cap P(c, d) \neq \emptyset \qquad \text{if and only if } \langle a, c\rangle | (b - d). \tag{5}$$

As an application of Theorem 3.4 we obtain a simple proof of the following result, which is [17, Lemma 3.2, page 777].

Theorem 3.5. *For each $a, b \in \mathbb{N}$ and $c \in P(a, b)$ we have $P(a, b) \cap M(b) \cap M(c) \neq \emptyset$.*

PROOF: Since $\langle a, c\rangle = \langle a, b\rangle$ we have $\langle a, b\rangle | (b - b), \langle a, c\rangle | (b - c)$ and $\langle b, c\rangle | (b - c)$ so the result follows from Theorem 3.4. $\square$

The next theorem is proved in [1, Theorem 4.14].

Theorem 3.6. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. If $b \in \mathbb{N}$, then*

$$P(a, b) = \bigcap_{i=1}^{k} P(p_i^{\alpha_i}, b) \qquad \textit{and} \qquad M(a) = \bigcap_{i=1}^{k} M(p_i^{\alpha_i}). \tag{6}$$

The following result is proved in [1, Theorem 4.20].

Theorem 3.7. *Let $a \in \mathbb{N}_2$, $b \in \mathbb{N}$ and $x, y \in P(a, b)$ with $x < y$. Write $x = am + b$, $y = an + b$ with $0 \le m < n$. Then $P(a, b) = U \cup V$, where*

$$U = \bigcup_{k=0}^{m} P(a^{n+1}, ak + b) \qquad \textit{and} \qquad V = \bigcup_{k=m+1}^{a^n - 1} P(a^{n+1}, ak + b). \tag{7}$$

Moreover, $x \in U$, $y \in V$ and the members of the family

$$\mathcal{F} = \{P(a^{n+1}, ak + b) \colon k \in \{0, 1, \dots, a^n - 1\}\}$$

are pairwise disjoint. In particular, $U \cap V = \emptyset$.

The next result is the Dirichlet theorem. A proof of it appears in [2, Chapter 7].

Theorem 3.8. *Let $a, b \in \mathbb{N}$ be so that $\langle a, b\rangle = 1$. Then the set $P(a, b) \cap \mathbb{P}$ is infinite.*

For each $a \in \mathbb{N}$ we define
$$\Theta(a) = \{p \in \mathbb{P}\colon\ p|a\}.$$
Note that $\Theta(a)$ is finite and $\Theta(a) = \emptyset$ if and only if $a = 1$. The proof of the following result is straightforward.

Proposition 3.9. *For each $a, b, c \in \mathbb{N}$ we have:*

1) $\Theta(ab) = \Theta(a) \cup \Theta(b)$. *In particular, for each $n \in \mathbb{N}$, $\Theta(a^n) = \Theta(a)$ and $\Theta(a^n) = \{a\}$ if and only if $a \in \mathbb{P}$;*
2) $\Theta(\langle a, b\rangle) = \Theta(a) \cap \Theta(b)$. *In particular, $\Theta(a) \cap \Theta(b) = \emptyset$ if and only if $\langle a, b\rangle = 1$;*
3) *if $d \in P(a, b)$ and $\Theta(a) \subset \Theta(b)$, then $\Theta(a) \subset \Theta(d)$;*
4) *if $\Theta(a) \subset \Theta(c)$ and $\Theta(b) \subset \Theta(c)$, then $\Theta(ab) \subset \Theta(c)$.*

For notions and results related with number theory that are not defined here, we refer the reader to [7].

4. The Szczuka space

In [15, Section 3, page 877] P. Szczuka (also known as P. Szyszkowska) consider the family
$$\mathcal{B}_S = \{P(a, b)\colon\ a, b \in \mathbb{N} \text{ and } \Theta(a) \subset \Theta(b)\}$$
and show that it is a base for a topology τ_S in $\mathbb{N}$. In [16, page 1009] P. Szczuka named τ_S the *common division topology* on $\mathbb{N}$. We name the topological space $(\mathbb{N}, \tau_S)$ the *Szczuka space*. Clearly
$$\begin{aligned}\tau_S = \{\emptyset\} \cup \{U \subset \mathbb{N}\colon\ &\text{for each } b \in U \text{ there is } P(a, b) \in \mathcal{B}_S\\ &\text{so that } P(a, b) \subset U\}.\end{aligned}$$
Note that nonempty open subsets of $(\mathbb{N}, \tau_S)$ are infinite. Note also that, for each $b \in \mathbb{N}$, the sets $P(1, b) = \mathbb{N}_b$ and $M(b) = P(b, b)$ are open in $(\mathbb{N}, \tau_S)$. In [15, Propositions 3.1–3.2, pages 877–878] it is shown that $(\mathbb{N}, \tau_S)$ is a connected compact space so that every nonempty closed subset of it contains 1. The last assertion implies the following result (compare with [18, Lemma 3.1, page 93]).

Theorem 4.1. *1 is the only indiscrete point of* $(\mathbb{N}, \tau_S)$.

PROOF: Let U be an open subset of $(\mathbb{N}, \tau_S)$ so that $1 \in U$. If $U \neq \mathbb{N}$, then $\mathbb{N} \setminus U$ is a nonempty closed subset of $(\mathbb{N}, \tau_S)$ that does not contains 1, contradicting [15, Proposition 3.1, page 877]. Hence 1 is an indiscrete point of $(\mathbb{N}, \tau_S)$. If $a \in \mathbb{N}_2$, then $M(a)$ is an open subset of $(\mathbb{N}, \tau_S)$ such that $a \in M(a)$ and $M(a) \neq \mathbb{N}$, so a is not an indiscrete point of $(\mathbb{N}, \tau_S)$. □

Corollary 4.2. $(\mathbb{N}, \tau_S)$ *is totally Brown. In particular, it is connected.*

PROOF: The result follows from Theorems 2.1 and 4.1. □

Corollary 4.3. $(\mathbb{N}, \tau_S)$ *is not homogeneous.*

PROOF: The image under a homeomorphism of an indiscrete point is an indiscrete point too, so no homeomorphism from $(\mathbb{N}, \tau_S)$ onto itself can map 1 onto 2. □

Concerning separation axioms, in [15, Proposition 3.3, page 878] it is shown that $(\mathbb{N}, \tau_S)$ is a T_0 space which is not T_1.

Theorem 4.4. $(\mathbb{N}, \tau_S)$ *is* T_D *but not* $T_{\frac{1}{2}}$.

PROOF: Since nonempty open sets are infinite and nonempty closed sets contain 1, for any $b \in \mathbb{N}_2$ the one-point-set $\{b\}$ is neither open nor closed, so $(\mathbb{N}, \tau_S)$ is not $T_{\frac{1}{2}}$. To show that $(\mathbb{N}, \tau_S)$ is T_D let $a \in \mathbb{N}$. Let us assume that $a \in \mathbb{N}_2$. We will prove that $\{a\}' = \{1\}$. Assume, by the way of contradiction, that $c \in \{a\}'$ and $c \neq 1$. Then $c \neq a$. If $1 < c < a$ take $n \in \mathbb{N}$ so that $a < c^n$. Then $P(c^n, c)$ is an open subset of $(\mathbb{N}, \tau_S)$ that contains c and $a \notin P(c^n, c) \setminus \{c\}$. If $a < c$ then $M(c)$ is an open subset of $(\mathbb{N}, \tau_S)$ that contains c and $a \notin M(c) \setminus \{c\}$. In any case we deduce that $c \notin \{a\}'$. This and Theorem 4.1 imply that $\{a\}' = \{1\}$. Since $P(1,2) \in \mathcal{B}_S$ and $\mathbb{N} \setminus P(1,2) = \{1\}$ the set $\{1\}$ is closed in $(\mathbb{N}, \tau_S)$. Hence $\{a\}' = \{1\}$ is closed in $(\mathbb{N}, \tau_S)$. Now assume that $a = 1$. Then $\{a\}' = \emptyset$ is closed in $(\mathbb{N}, \tau_S)$. This shows that $(\mathbb{N}, \tau_S)$ is T_D. □

By the proof of Theorem 4.4 if $a \in \mathbb{N}_2$, then

$$\mathrm{cl}_{(\mathbb{N}, \tau_S)}(\{a\}) = \{1, a\} \qquad \text{and} \qquad \mathrm{cl}_{(\mathbb{N}, \tau_S)}(\{1\}) = \{1\},$$

so $\{1\}$ is the only one-point-set which is closed in $(\mathbb{N}, \tau_S)$. Moreover, 1 is in the closure in $(\mathbb{N}, \tau_S)$ of every one-point-set, and then in every nonempty closed set.

In [13] a topological space X is said to be superconnected if it contains no disjoint nonempty open sets. Since $P(9,3), P(27,6) \in \mathcal{B}_S$ and $\langle 9, 27 \rangle = 9$ do not divide $6 - 3 = 3$, by (5) we have $P(9,3) \cap P(27,6) = \emptyset$. Hence $(\mathbb{N}, \tau_S)$ is not superconnected in the sense of [13]. Indeed for any $p_1, p_2 \in \mathbb{P} \setminus \{2\}$ we have $P(p_1^2, p_1), P(p_1^3, p_1 p_2) \in \mathcal{B}_S$ and, by (5),

$$P(p_1^2, p_1) \cap P(p_1^3, p_1 p_2) = \emptyset.$$

Combining 1) of Theorem 3.1 and 3) of Proposition 3.9 we obtain the following result.

Theorem 4.5. *If $P(a,b) \in \mathcal{B}_S$ and $c \in P(a,b)$, then $P(a,c) \subset P(a,b)$ and $\Theta(a) \subset \Theta(c)$. Hence $P(a,c) \in \mathcal{B}_S$ and*

$$\tau_S = \{\emptyset\} \cup \{U \subset \mathbb{N}\colon \ \textit{for each } b \in U \ \textit{there is } P(a,c) \in \mathcal{B}_S \\ \textit{so that } b \in P(a,c) \subset U\}.$$

Now we present two closed subsets of $(\mathbb{N}, \tau_S)$ that are important in order to determine properties related with the closure in $(\mathbb{N}, \tau_S)$ of an arithmetic progression.

Theorem 4.6. *If $a \in \mathbb{N}_2$ and $b \in M(a)$, then for each $n \in \mathbb{N}$ the sets*

$$(\mathbb{N} \cap P_F(a^n, b)) \cup (\mathbb{N} \setminus M(a)) \qquad \textit{and} \qquad P(a^n, b) \cup (\mathbb{N} \setminus M(a))$$

are closed in $(\mathbb{N}, \tau_S)$.

Proof: Fix $n \in \mathbb{N}$ and note that

$$\mathbb{N} \setminus \big(M(a) \setminus (\mathbb{N} \cap P_F(a^n, b))\big) = (\mathbb{N} \setminus M(a)) \cup (\mathbb{N} \cap P_F(a^n, b)),$$

so we will show that $U = M(a) \setminus (\mathbb{N} \cap P_F(a^n, b))$ is open in $(\mathbb{N}, \tau_S)$. Let $z \in U$. Then $a|z$ and $\Theta(a^n) = \Theta(a) \subset \Theta(z)$, so $P(a^n, z)$ is an open subset of $(\mathbb{N}, \tau_S)$ such that $z \in P(a^n, z) \subset M(a)$. Since $a^n \nmid (z-b)$, by (4), $P(a^n, z) \cap P_F(a^n, b) = \emptyset$ so $P(a^n, z) \subset U$ and then U is open in $(\mathbb{N}, \tau_S)$. Since

$$\mathbb{N} \setminus (M(a) \setminus P(a^n, b)) = (\mathbb{N} \setminus M(a)) \cup P(a^n, b)$$

proceeding as before we show that $M(a) \setminus P(a^n, b)$ is open in $(\mathbb{N}, \tau_S)$. □

The following result was observed in the proof of [18, Theorem 4.3, page 96].

Theorem 4.7. *The family*

$$\overline{\mathcal{B}}_S = \{P(a,b) \in \mathcal{B}_S \colon b \le a\}$$

is a base for τ_S.

Proof: Let $P(a,b) \in \mathcal{B}_S$ with $a < b$ and $c \in P(a,b)$. By Theorem 4.5 $P(a,c) \subset P(a,b)$ and $P(a,c) \in \mathcal{B}_S$. Moreover $1 \le a < b \le c$. Let $p_c, k_c \in \mathbb{P}$ be so that $p_c|c$ and $p_c^{k_c} \ge c$. Since $\Theta(a) \subset \Theta(c)$, applying 1) of Proposition 3.9,

$$\Theta(p_c^{k_c} a) = \{p_c\} \cup \Theta(a) \subset \{p_c\} \cup \Theta(c) = \Theta(c).$$

This shows that $P(p_c^{k_c} a, c) \in \overline{\mathcal{B}}_S$ and since

$$c \in P(p_c^{k_c} a, c) \subset P(a,c) \subset P(a,b),$$

we have

$$P(a,b) = \bigcup_{c\in P(a,b)} P(p_c^{k_c}a, c).$$

We have seen that each element of $\mathcal{B}_S$ is a union of members of $\overline{\mathcal{B}}_S$, so $\overline{\mathcal{B}}_S$ is a base for τ_S. □

4.1 Totally separated subsets of the Szczuka space. By [1, Theorem 5.12] the members of the base $\mathcal{B}_S$ of τ_S are totally Brown in $(\mathbb{N}, \tau_G)$. In particular such members are connected in $(\mathbb{N}, \tau_G)$. Now we show that every $P(a,b) \in \mathcal{B}_S$ with $a \in \mathbb{N}_2$ is totally separated in $(\mathbb{N}, \tau_S)$.

Theorem 4.8. *Let $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ be so that $P(a,b) \in \mathcal{B}_S$. Then $P(a,b)$ is totally separated in $(\mathbb{N}, \tau_S)$. In particular, $P(a,b)$ is hereditarily disconnected.*

Proof: Let $x, y \in P(a,b)$ with $x \neq y$. Assume, without loss of generality, that $x < y$. Write $x = am + b$, $y = an + b$ with $0 \leq m < n$ and consider the sets U and V defined in (7). By Theorem 3.7 we have $P(a,b) = U \cup V$, $U \cap V = \emptyset$, $x \in U$ and $y \in V$. Fix $k \in \mathbb{N}_0$. Since $\Theta(a) \subset \Theta(b)$ and $ak + b \in P(a,b)$, by 1) and 3) of Proposition 3.9 we have $\Theta(a^{n+1}) = \Theta(a) \subset \Theta(ak+b)$. Then both U and V are open in $(\mathbb{N}, \tau_S)$. □

Corollary 4.9. *Let $a, b \in \mathbb{N}$ be such that $\langle a, b\rangle \neq 1$. Then $P(a,b)$ is totally separated in $(\mathbb{N}, \tau_S)$.*

Proof: Since $\langle a, b\rangle \neq 1$, by 2) of Proposition 3.9 there exists $p \in \Theta(a) \cap \Theta(b)$. Then $P(a,b) \subset M(p)$. Now, since $M(p) \in \mathcal{B}_S$ by Theorem 4.8 the set $M(p)$ is totally separated in $(\mathbb{N}, \tau_S)$. Hence $P(a,b)$ is totally separated in $(\mathbb{N}, \tau_S)$. □

By Theorem 4.8 it follows that $M(a) = P(a,a) \in \mathcal{B}_S$ is totally separated in $(\mathbb{N}, \tau_S)$ for each $a \in \mathbb{N}_2$. The next result is [18, Theorem 3.2, page 93]. Using Theorem 4.8 we present a simple proof.

Theorem 4.10. *If $f\colon (\mathbb{N}, \tau_S) \to (\mathbb{N}, \tau_S)$ is a continuous and nonconstant function, then $f(1) = 1$.*

Proof: Let $a = f(1)$ and assume, by the way of contradiction, that $a \in \mathbb{N}_2$. We claim that

$$f(\mathbb{N}) \subset M(a). \tag{8}$$

Since $M(a) = P(a,a)$ is an open subset of $(\mathbb{N}, \tau_S)$ that contains $a = f(1)$, by the continuity of f, there is an open subset U of $(\mathbb{N}, \tau_S)$ so that $1 \in U$ and $f(U) \subset M(a)$. Since by Theorem 4.1 the point 1 is indiscrete in $(\mathbb{N}, \tau_S)$, we have $U = \mathbb{N}$. Hence $f(\mathbb{N}) \subset M(a)$ and (8) holds.

Since $(\mathbb{N}, \tau_S)$ is connected and f is continuous, by (8), $f(\mathbb{N})$ is a connected subset of $M(a)$, which by Theorem 4.8 is hereditarily disconnected. This implies that f is constant, a contradiction. Hence $a = 1$ and then $f(1) = 1$. □

From Theorem 4.10 it follows that $(\mathbb{N}, \tau_S)$ has the fixed point property, i.e., for each continuous function $f\colon (\mathbb{N}, \tau_S) \to (\mathbb{N}, \tau_S)$ there is $b \in \mathbb{N}$ so that $f(b) = b$.

4.2 Local connectedness. Now we study the points at which the space $(\mathbb{N}, \tau_S)$ is either locally connected or connected im kleinen or almost connected im kleinen. If $A \subset Y \subset \mathbb{N}$ we denote by $\operatorname{int}_{\mathbb{N}}(A)$ the interior of A in $(\mathbb{N}, \tau_S)$ and by $\operatorname{int}_Y(A)$ the interior of A in the subspace Y of $(\mathbb{N}, \tau_S)$.

Theorem 4.11. *Let $a, b \in \mathbb{N}$ be such that $P(a, b) \in \mathcal{B}_S$. Hence*

1) *if $a \in \mathbb{N}_2$, then $P(a, b)$ is neither connected im kleinen nor almost connected im kleinen at each of its points;*
2) *if $a = 1$, then $P(a, b)$ is neither connected im kleinen nor almost connected im kleinen at each point $c \in P(a, b) \setminus \{1\}$.*

Proof: To show 1) fix $a \in \mathbb{N}_2$ and assume that $P(a, b)$ is either connected im kleinen or almost connected im kleinen at $c \in P(a, b)$. By Theorem 4.5 we have $P(a, c) \in \mathcal{B}_S$ and $P(a, c) \subset P(a, b)$. Since $P(a, c)$ is an open subset of $P(a, b)$ that contains c, there is a connected subset C of $P(a, c)$ so that $\operatorname{int}_{P(a,b)}(C) \neq \emptyset$. Hence $\operatorname{int}_{\mathbb{N}}(C) \neq \emptyset$ and since nonempty open subsets of $(\mathbb{N}, \tau_S)$ are infinite, the set C is infinite. This contradicts the fact that, by Theorem 4.8, $P(a, c)$ is hereditarily disconnected. Therefore 1) holds.

To show 2) assume that $a = 1$ and that $P(a, b) = \mathbb{N}_b$ is either connected im kleinen or almost connected im kleinen at $c \in P(a, b) \setminus \{1\}$. Then $M(c)$ is an open subset of $P(a, b)$ that contains c, so there is a connected subset D of $M(c)$ so that $\operatorname{int}_{P(a,b)}(C) \neq \emptyset$. Hence $\operatorname{int}_{\mathbb{N}}(D) \neq \emptyset$ and since nonempty open subsets of $(\mathbb{N}, \tau_S)$ are infinite, the set D is infinite. This contradicts the fact that, by Theorem 4.8, $M(c)$ is hereditarily disconnected. □

Corollary 4.12. *The space $(\mathbb{N}, \tau_S)$ is locally connected at 1 and neither connected im kleinen nor almost connected im kleinen at each point $c \in \mathbb{N}_2$. In particular, $(\mathbb{N}, \tau_S)$ is not locally connected.*

Proof: By Theorem 4.1, 1 is an indiscrete point of the connected space $(\mathbb{N}, \tau_S)$, so $(\mathbb{N}, \tau_S)$ is locally connected at 1. Since $\mathbb{N} = P(1, 1)$ the rest of the proof follows from 2) of Theorem 4.11. □

4.3 The closure in the Szczuka space. We present in this subsection several results that involve the closure of an arithmetic progression with respect to the

Szczuka space. If $A \subset \mathbb{N}$ we denote by $\mathrm{cl}_{\mathbb{N}}(A)$ the closure of A in $(\mathbb{N}, \tau_S)$. In [16, Remark 3.3, page 1010] it is mentioned that

$$\mathrm{cl}_{\mathbb{N}}(P(1,b)) = \mathbb{N} \qquad \text{for every } b \in \mathbb{N}.$$

Now we show that the closure in $(\mathbb{N}, \tau_S)$ of each arithmetic progression contains infinitely many prime numbers.

Theorem 4.13. *If $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$, then*

$$\bigcap_{p \in \Theta(a)} (\mathbb{N} \setminus M(p)) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b)). \tag{9}$$

In particular,

$$\mathbb{P} \setminus \Theta(a) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b)). \tag{10}$$

Proof: Let c be in the left side of (9) and U be a nonempty open subset of $(\mathbb{N}, \tau_S)$ so that $c \in U$. Then there is $d \in \mathbb{N}$ with $\Theta(d) \subset \Theta(c)$ so that $P(d,c) \subset U$. If $\langle d, a \rangle \neq 1$, then for some $p \in \mathbb{P}$ we have $p|d$ and $p|a$. Note that $p \in \Theta(a) \cap \Theta(c)$ so $c \in M(p)$ contradicting the choice of c. Then $\langle d, a \rangle = 1$, so $\langle d, a \rangle | (c-b)$. This implies, by (5), that

$$\emptyset \neq P(d,c) \cap P(a,b) \subset U \cap P(a,b).$$

Hence $c \in \mathrm{cl}_{\mathbb{N}}(P(a,b))$ and (9) holds. The inclusion (10) follows from (9) and the fact that

$$\mathbb{P} \setminus \Theta(a) \subset \{z \in \mathbb{N} \colon \langle z, p \rangle = 1 \text{ for each } p \in \Theta(a)\} = \bigcap_{p \in \Theta(a)} (\mathbb{N} \setminus M(p)).$$

$\square$

If $a = 1$, then $\Theta(a) = \emptyset$ and $\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathbb{N}$, so the inclusion (10) is valid for each $a \in \mathbb{N}$.

Theorem 4.14. *Let $a, b, c, d \in \mathbb{N}$ be so that $a|c$ and $d \equiv b \pmod{a}$. Then*

$$\mathbb{N} \cap P_F(c,d) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b)). \tag{11}$$

In particular

$$\mathbb{N} \cap P_F(a^n, b) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b)) \qquad \textit{for each } n \in \mathbb{N}. \tag{12}$$

Proof: Note that if $a \geq b$ then, by the second part of Theorem 3.2,

$$\mathbb{N} \cap P_F(c,d) \subset \mathrm{cl}_{\mathbb{N}}(\mathbb{N} \cap P_F(c,d)) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b)).$$

We will show that inclusion (11) holds independently of the relation between a and b. Let $z \in \mathbb{N} \cap P_F(c,d)$ and U be a nonempty open subset of $(\mathbb{N}, \tau_S)$ so that $z \in U$. Then there is $q \in \mathbb{N}$ with $\Theta(q) \subset \Theta(z)$ so that $P(q,z) \subset U$. Note that $c|(z-d)$, $\langle q,a\rangle|a$ and $a|c$. Then $\langle q,a\rangle|(z-d)$. From $a|(d-b)$ and $\langle q,a\rangle|a$ we infer that $\langle q,a\rangle|(d-b)$. Hence $\langle q,a\rangle|[(z-d)+(d-b)]$, i.e., $\langle q,a\rangle|(z-b)$. This implies, by (5), that

$$\emptyset \neq P(q,z) \cap P(a,b) \subset U \cap P(a,b),$$

so $z \in \mathrm{cl}_{\mathbb{N}}(P(a,b))$ and (11) is satisfied. The inclusion (12) follows from (11) and the facts that $b \equiv b \pmod a$ and $a|a^n$ for every $n \in \mathbb{N}$. $\square$

The following result generalizes [16, Lemma 3.2, page 1009].

Theorem 4.15. *Let $a,b,c \in \mathbb{N}$ be so that $b \equiv c \pmod a$. Then*

$$\mathrm{cl}_{\mathbb{N}}(\mathbb{N} \cap P_F(a,c)) = \mathrm{cl}_{\mathbb{N}}(P(a,b)). \tag{13}$$

In particular, $\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathrm{cl}_{\mathbb{N}}(P(a,c))$ and if $b \in M(a)$, then $\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathrm{cl}_{\mathbb{N}}(M(a))$.

Proof: Since $a|a$ and $b \equiv c \pmod a$ by Theorem 3.2 we have $P(a,b) \subset \mathbb{N} \cap P_F(a,c)$. Taking closures in $(\mathbb{N}, \tau_S)$, the right side of (13) is a subset of its left side. Since $a|a$ and $c \equiv b \pmod a$ by (11) we have $\mathbb{N} \cap P_F(a,c) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b))$. Hence the left side of (13) is a subset of its right side. This shows (13). Now, since $b \equiv b \pmod a$ and $b \equiv c \pmod a$ applying (13) two times we have

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathrm{cl}_{\mathbb{N}}(\mathbb{N} \cap P_F(a,b)) = \mathrm{cl}_{\mathbb{N}}(P(a,c)).$$

Now assume that $b \in M(a)$. Then $b \equiv a \pmod a$ and by (13)

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathrm{cl}_{\mathbb{N}}(\mathbb{N} \cap P_F(a,a)) = \mathrm{cl}_{\mathbb{N}}(M(a)).$$

$\square$

Let $a,b,c \in \mathbb{N}$ be so that $c \in P(a,b)$. Then $b \equiv c \pmod a$ and $P(a,c) \subset P(a,b)$. The inclusion might be proper but, by Theorem 4.15, $\mathrm{cl}_{\mathbb{N}}(P(a,c)) = \mathrm{cl}_{\mathbb{N}}(P(a,c))$.

The following result generalizes [16, Theorem 3.4, page 1010] since we do not use the condition $c \leq p^n$ as claimed in [16].

Theorem 4.16. *Let $p \in \mathbb{P}$ and $b,c \in \mathbb{N}$. For each $n \in \mathbb{N}$ so that $b \equiv c \pmod{p^n}$ we have*

$$\mathrm{cl}_{\mathbb{N}}(P(p^n,b)) = (\mathbb{N} \cap P_F(p^n,c)) \cup (\mathbb{N} \setminus M(p)). \tag{14}$$

In particular,

1) *if* $\langle p,b\rangle = 1$, *then* $\mathrm{cl}_{\mathbb{N}}(P(p^n,b)) = \mathbb{N}\setminus M(p)$*;*
2) *if* $p|b$, *then* $\mathrm{cl}_{\mathbb{N}}(P(p,b)) = \mathbb{N}$*;*
3) $P(2,1)$ *is closed in* $(\mathbb{N},\tau_S)$*;*
4) *for each* $q\in\mathbb{P}$ *we have* $\mathrm{cl}_{\mathbb{N}}(M(q)) = \mathbb{N}$ *and if* $P(q,b)\in\mathcal{B}_S$, *then* $\mathrm{cl}_{\mathbb{N}}(P(q,b)) = \mathbb{N}$.

Proof: Fix $n\in\mathbb{N}$ and let

$$C = (\mathbb{N}\cap P_F(p^n,c))\cup(\mathbb{N}\setminus M(p)).$$

Clearly $\Theta(p^n)=\{p\}$ and $p^n|(c-b)$. Then, by (9) and (11) we have

$$\mathbb{N}\setminus M(p)\subset \mathrm{cl}_{\mathbb{N}}(P(p^n,b)) \qquad\text{and}\qquad \mathbb{N}\cap P_F(p^n,c)\subset \mathrm{cl}_{\mathbb{N}}(P(p^n,b)).$$

Hence the right side of (14) is contained in its left side. To show the reverse inclusion we divide the proof in two cases. Assume first that $p|c$. Then by Theorem 4.6, C is closed in $(\mathbb{N},\tau_S)$ and, by Theorem 3.2,

$$P(p^n,b)\subset\mathbb{N}\cap P_F(p^n,c)\subset C.$$

Then the left side of (14) is contained in its right side. Now assume that $p\nmid c$. Then $\langle p,c\rangle=1$ and since $b\equiv c \pmod p$, by (3), we have $P(p^n,b)\subset\mathbb{N}\setminus M(p)$. Since $\mathbb{N}\setminus M(p)$ is closed in $(\mathbb{N},\tau_S)$ we get

$$\mathrm{cl}_{\mathbb{N}}(P(p^n,b))\subset\mathbb{N}\setminus M(p)\subset C.$$

Hence (14) holds.

To show 1) assume that $\langle p,b\rangle=1$. Then $\langle p^n,b\rangle=1$ and, by (2),

$$\mathbb{N}\cap P_F(p^n,b)\subset\mathbb{N}\setminus M(p)$$

so, by (14), we have

$$\mathrm{cl}_{\mathbb{N}}(P(p^n,b)) = (\mathbb{N}\cap P_F(p^n,b))\cup(\mathbb{N}\setminus M(p)) = \mathbb{N}\setminus M(p).$$

This shows 1). To show 2) assume that $p|b$. Then $b\equiv p \pmod p$ and by (14)

$$\begin{aligned}\mathrm{cl}_{\mathbb{N}}(P(p,b)) &= (\mathbb{N}\cap P_F(p,p))\cup(\mathbb{N}\setminus M(p)) = P(p,p)\cup(\mathbb{N}\setminus M(p))\\ &= M(p)\cup(\mathbb{N}\setminus M(p)) = \mathbb{N}.\end{aligned}$$

This shows 2). Since $\langle 2,1\rangle = 1$ by 1) we have

$$\mathrm{cl}_{\mathbb{N}}(P(2,1)) = \mathbb{N}\setminus M(2) = P(2,1),$$

so 3) holds. To show 4) let $q\in\mathbb{P}$. Since $q|q$, by 2),

$$\mathrm{cl}_{\mathbb{N}}(M(q)) = \mathrm{cl}_{\mathbb{N}}(P(q,q)) = \mathbb{N}.$$

Now assume that $P(q,b)$ is open in $(\mathbb{N},\tau_S)$. Then $\{q\}=\Theta(q)\subset\Theta(b)$ so $q|b$ and by 2) $\mathrm{cl}_{\mathbb{N}}(P(q,b))=\mathbb{N}$. This shows 4). □

In [4] a topological space X is said to be superconnected if it is connected and every subset which contains a nonempty open subset is open. Note that $P(4,2)\in\mathcal{B}_S$ and, by (14)

$$\mathrm{cl}_{\mathbb{N}}(P(4,2))=(\mathbb{N}\cap P_F(2^2,2))\cup(\mathbb{N}\setminus M(2))=P(4,2)\cup P(2,1).$$

Hence $\mathrm{cl}_{\mathbb{N}}(P(4,2))$ is a nonempty proper closed subset of $(\mathbb{N},\tau_S)$ that contains a nonempty open set. Since $(\mathbb{N},\tau_S)$ is connected, $\mathrm{cl}_{\mathbb{N}}(P(4,2))$ is not open. Hence $(\mathbb{N},\tau_S)$ is not superconnected in the sense of [4].

Now we show that when the intersection of finitely many arithmetic progressions in $\mathbb{N}$ is nonempty, the closure in $(\mathbb{N},\tau_S)$ of such intersection is the intersection of the closures of the arithmetic progressions. By [1, Theorems 5.9 and 6.3] the equality (15) holds in both $(\mathbb{N},\tau_G)$ and $(\mathbb{N},\tau_K)$.

Theorem 4.17. *Let* $a_1,b_1,a_2,b_2,\dots,a_k,b_k\in\mathbb{N}$ *be so that* $\bigcap_{i=1}^k P(a_i,b_i)\neq\emptyset$. *Then*

$$\mathrm{cl}_{\mathbb{N}}\left(\bigcap_{i=1}^k P(a_i,b_i)\right)=\bigcap_{i=1}^k \mathrm{cl}_{\mathbb{N}}(P(a_i,b_i)). \tag{15}$$

Proof: Clearly the left side of (15) is contained in its right side, so to show the reverse inclusion let b be a member in the right side of (15) and U be an open subset of $(\mathbb{N},\tau_S)$ so that $b\in U$. Take $a\in\mathbb{N}$ with $\Theta(a)\subset\Theta(b)$ and $P(a,b)\subset U$. Then

$$P(a,b)\cap P(a_i,b_i)\neq\emptyset \qquad \text{for each } i\in\{1,2,\dots,k\}. \tag{16}$$

Since $\bigcap_{i=1}^k P(a_i,b_i)\neq\emptyset$, by Theorem 3.4, we have

$$P(a_i,b_i)\cap P(a_j,b_j)\neq\emptyset \qquad \text{for each } i,j\in\{1,2,\dots,k\} \text{ with } i\neq j. \tag{17}$$

Combining (16) and (17) we infer, applying again Theorem 3.4, that

$$P(a,b)\cap\left(\bigcap_{i=1}^k P(a_i,b_i)\right)\neq\emptyset,$$

so $U\cap\left(\bigcap_{i=1}^k P(a_i,b_i)\right)\neq\emptyset$ and then b is in the left side of (15). □

Now we write some consequences of Theorem 4.17. The following result appears in [16, Theorem 3.5, page 1011] with a very different proof.

Theorem 4.18. *If $a \in \mathbb{N}_2$ and $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a, then*

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(p_i^{\alpha_i}, b)). \tag{18}$$

Proof: The result follows from the first part of (6) and (15). □

In the proof of [1, Theorems 5.10 and 6.4] it is shown that equality (18) is valid in both $(\mathbb{N}, \tau_G)$ and $(\mathbb{N}, \tau_K)$. The following result was proved differently in the second part of [16, Theorem 3.6, page 1012].

Theorem 4.19. *Let $a, b \in \mathbb{N}_2$ be such that a is square-free and $a|b$. Then $P(a,b)$ and $M(a)$ are open subsets of $(\mathbb{N}, \tau_S)$ so that $\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathbb{N} = \mathrm{cl}_{\mathbb{N}}(M(a))$.*

Proof: Clearly $P(a,b)$ and $M(a)$ are open subsets of $(\mathbb{N}, \tau_S)$. By the last part of Theorem 4.15, $\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathrm{cl}_{\mathbb{N}}(M(a))$. Let $a = \prod_{i=1}^{k} p_i$ be the standard prime decomposition of a. For each $i \in \{1, 2, \ldots, k\}$ we have $p_i|b$ so by 2) of Theorem 4.16, $\mathrm{cl}_{\mathbb{N}}(P(p_i, b)) = \mathbb{N}$. Then by the first part of (6) and (15),

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathrm{cl}_{\mathbb{N}}\left(\bigcap_{i=1}^{k} P(p_i, b)\right) = \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(p_i, b)) = \mathbb{N}.$$

Alternatively we can use the second part of (6), (15) and 4) of Theorem 4.16, to deduce that $\mathrm{cl}_{\mathbb{N}}(M(a)) = \mathbb{N}$. □

In the following result we calculate the closure in $(\mathbb{N}, \tau_S)$ of an arithmetic progression $P(a,b)$ with $a \in \mathbb{N}_2$.

Theorem 4.20. *Let $a \in \mathbb{N}_2$ and $b, c \in \mathbb{N}$ so that $b \equiv c \pmod{a}$. If $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a, then*

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcap_{i=1}^{k} [(\mathbb{N} \cap P_F(p_i^{\alpha_i}, c)) \cup (\mathbb{N} \setminus M(p_i))]. \tag{19}$$

Proof: For each $i \in \{1, 2, \ldots, k\}$ we have $p_i^{\alpha_i}|a$ so $b \equiv c \pmod{p_i^{\alpha_i}}$. Hence, by (14) and (18)

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(p_i^{\alpha_i}, b)) = \bigcap_{i=1}^{k} [(\mathbb{N} \cap P_F(p_i^{\alpha_i}, c)) \cup (\mathbb{N} \setminus M(p_i))].$$

□

Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. Given $b \in \mathbb{N}$ since $b \equiv b \pmod{a}$ by (19) we have

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcap_{i=1}^{k} [(\mathbb{N} \cap P_F(p_i^{\alpha_i}, b)) \cup (\mathbb{N} \setminus M(p_i))]. \tag{20}$$

Let A_b be the set of $c \in \mathbb{N}$ so that $c \le a$ and for each $i \in \{1, 2, \ldots, k\}$ either $\langle p_i, c\rangle = 1$ or $c \equiv b \pmod{p_i^{\alpha_i}}$. Take $x \in \mathrm{cl}_{\mathbb{N}}(P(a,b))$ and choose $n \in \mathbb{N}_0$ and $c \in \mathbb{N}$ such that $x = an + c$ and $c \le a$. Given $i \in \{1, 2, \ldots, k\}$ by (20) either $x \in P_F(p_i^{\alpha_i}, b)$ or $x \in \mathbb{N} \setminus M(p_i)$. Since $an \equiv 0 \pmod{p_i^{\alpha_i}}$ in the first case we infer that $c \equiv b \pmod{p_i^{\alpha_i}}$ and, in the second case, we get $\langle p_i, c\rangle = 1$. This shows that $c \in A_b$ and $x \in P(a,c)$, so

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) \subset \bigcup_{c \in A_b} P(a,c).$$

Now take $z \in \bigcup_{c \in A_b} P(a,c)$. Then $z \in \mathbb{N}$ and there exist $c \in A_b$ and $m \in \mathbb{N}_0$ such that $z = am + c$. Given $i \in \{1, 2, \ldots, k\}$ since $c \in A_b$ either $\langle p_i, c\rangle = 1$ or $c \equiv b \pmod{p_i^{\alpha_i}}$. In the first case $z \in \mathbb{N} \setminus M(p_i)$ and, in the second case, using that $am \equiv 0 \pmod{p_i^{\alpha_i}}$ we get $z \in P_F(p_i^{\alpha_i}, b)$. Then, by (20), $z \in \mathrm{cl}_{\mathbb{N}}(P(a,b))$ and then

$$\bigcup_{c \in A_b} P(a,c) \subset \mathrm{cl}_{\mathbb{N}}(P(a,b)).$$

In this way we obtain the following result that was proved differently in the first part of [16, Theorem 3.6, page 1012].

Theorem 4.21. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. Then, for each $b \in \mathbb{N}$,*

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcup_{c \in A_b} P(a,c). \tag{21}$$

By (21) the right side of (20) is the union of finitely many arithmetic progressions in $\mathbb{N}$, all with the same common difference of successive members. When $\langle a, b\rangle = 1$ we can simplify the right side of (20).

Theorem 4.22. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. Then for each $b \in \mathbb{N}$ with $\langle a, b\rangle = 1$ we have*

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcap_{i=1}^{k} (\mathbb{N} \setminus M(p_i)). \tag{22}$$

Proof: Since $\langle a, b\rangle = 1$ we have $\langle p_i^{\alpha_i}, b\rangle = 1$ for every $i \in \{1, 2, \ldots, k\}$ so, by (2), $\mathbb{N} \cap P_F(p_i^{\alpha_i}, b) \subset \mathbb{N} \setminus M(p_i)$ and then

$$(\mathbb{N} \cap P_F(p_i^{\alpha_i}, b)) \cup (\mathbb{N} \setminus M(p_i)) = \mathbb{N} \setminus M(p_i). \tag{23}$$

Equality (22) follows from (20) and (23). $\square$

Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. For each $b \in \mathbb{N}$ let C_b be the set of $c \in \mathbb{N}$ so that $c \leq a$ and for each $i \in \{1, 2, \dots, k\}$ we have $\langle p_i, c \rangle = 1$. Reasoning as in the proof of Theorem 4.21, using (22) instead, we obtain the following result.

Theorem 4.23. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. Then for each $b \in \mathbb{N}$ with $\langle a, b \rangle = 1$ we have*

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \bigcup_{c \in C_b} P(a,c).$$

4.4 Totally Brown subsets of the Szczuka space. In this subsection we characterize the arithmetic progressions $P(a,b)$ that are totally Brown in $(\mathbb{N}, \tau_S)$. In [15, Theorem 3.4, page 878] it is shown that $P(a,b)$ is connected in $(\mathbb{N}, \tau_S)$ if and only if $\langle a, b \rangle = 1$. From the results that we have seen, the proof of 2) implies 3) in the next result is simpler than the one presented in [15, Theorem 3.4, page 878].

Theorem 4.24. *Let $a, b \in \mathbb{N}$. Then the following assertions are equivalent:*

1) *$P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_S)$;*
2) *$P(a,b)$ is Brown in $(\mathbb{N}, \tau_S)$;*
3) *$P(a,b)$ is connected in $(\mathbb{N}, \tau_S)$;*
4) *$\langle a, b \rangle = 1$.*

In particular, $P(a,b) \in \mathcal{B}_S$ is totally Brown in $(\mathbb{N}, \tau_S)$ if and only if $a = 1$.

Proof: We have seen that totally Brown spaces are Brown and that Brown spaces are connected, so 1) implies 2) and 2) implies 3). Now assume 3). If $\langle a, b \rangle \neq 1$ by Corollary 4.9, $P(a,b)$ is totally separated in $(\mathbb{N}, \tau_S)$. Hence $P(a,b)$ is not connected in $(\mathbb{N}, \tau_S)$. This shows that 3) implies 4). Now assume 4). Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \dots, O_n$ of $P(a,b)$. For each $i \in \{1, 2, \dots, n\}$ let U_i be an open subset of $(\mathbb{N}, \tau_S)$ so that $O_i = P(a,b) \cap U_i$ and take $b_i \in O_i$. Then there exists $a_i \in \mathbb{N}$ such that $\Theta(a_i) \subset \Theta(b_i)$ and $P(a_i, b_i) \subset U_i$. For every $i \in \{1, 2, \dots, n\}$, by (10),

$$\text{(24)} \qquad \mathbb{P} \setminus \Theta(a_i) \subset \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)) \qquad \text{and} \qquad P(a,b) \cap P(a_i, b_i) \neq \emptyset.$$

Since $A = \bigcup_{i=1}^{n} \Theta(a_i)$ is finite and by Theorem 3.8 the set $P(a,b) \cap \mathbb{P}$ is infinite, there exists $p \in (P(a,b) \cap \mathbb{P}) \setminus A$. Then, by (24),

$$p \in P(a,b) \cap \left(\bigcap_{i=1}^{n} (\mathbb{P} \setminus \Theta(a_i)) \right) \subset P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)) \right).$$

Hence, by (15),

$$\emptyset \neq P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(a_i,b_i)) \right) = P(a,b) \cap \left(\bigcap_{i=1}^{n} [\mathrm{cl}_{\mathbb{N}}(P(a,b)) \cap \mathrm{cl}_{\mathbb{N}}(P(a_i,b_i))] \right)$$
$$= P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(a,b) \cap P(a_i,b_i)) \right) \subset P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(O_i) \right).$$

This shows that $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_S)$. Therefore 4) implies 1). With this the equivalence between assertions 1)–4) is complete. Now assume that $P(a,b) \in \mathcal{B}_S$. Then $\Theta(a) \subset \Theta(b)$. If $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_S)$ then, by 1) implies 4), $\langle a,b \rangle = 1$ so by 2) of Proposition 3.9,

$$\emptyset = \Theta(\langle a,b \rangle) = \Theta(a) \cap \Theta(b) = \Theta(a).$$

Then $a = 1$. Conversely, if $a = 1$ then $\langle a,b \rangle = 1$ and by 4) implies 1) the set $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_S)$. □

Corollary 4.25. *For each $a, b \in \mathbb{N}$ the arithmetic progression $P(a,b)$ is either totally separated or totally Brown in $(\mathbb{N}, \tau_S)$.*

Proof: If $\langle a,b \rangle = 1$, by Theorem 4.24, $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_S)$. If $\langle a,b \rangle \neq 1$, by Corollary 4.9, $P(a,b)$ is totally separated in $(\mathbb{N}, \tau_S)$. □

By [1, Corollary 5.15] the same two possibilities mentioned in Corollary 4.25 are satisfied in $(\mathbb{N}, \tau_G)$. However, by [1, Theorem 6.9], each arithmetic progression $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_K)$.

It is worth to compare Corollary 4.12 with the comment previous to [15, Corollary 3.5, page 879] in which it is said that due to the equivalence between 3) and 4) of Theorem 4.24 "we can easily see that every base of the topology τ_S contains some disconnected arithmetic progression". And due to this in [15, Corollary 3.5, page 879] it is claimed that $(\mathbb{N}, \tau_S)$ is not locally connected.

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