Some isomorphic properties in projective tensor products

Ioana Ghenciu

Abstract. We give sufficient conditions implying that the projective tensor product of two Banach spaces X and Y has the p-sequentially Right and the p-L-limited properties, $1 \le p < \infty$.

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1. Introduction

For two Banach spaces X and Y, the projective tensor product space of X and Y will be denoted by $X \otimes_\pi Y$. In [10] it was studied whether $X \otimes_\pi Y$ has the sequentially Right (SR) property or the L-limited property, when X and Y have the respective property. In [21] we introduced the p-(SR) and the p-L-limited properties for $1 \le p < \infty$.

In this paper we use results about relative weak compactness in spaces of compact operators to study whether the p-(SR) and the p-L-limited properties lift from the Banach spaces X and Y to $X \otimes_\pi Y$.

2. Definitions and notation

Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, and X^* will denote the continuous linear dual of X. The space X embeds in Y (in symbols $X \hookrightarrow Y$) if X is isomorphic to a closed subspace of Y. An operator $T\colon X \to Y$ will be a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X,Y)$, $W(X,Y)$, and $K(X,Y)$.

A subset S of a Banach space X is said to be *weakly precompact* (or *weakly conditionally compact*) provided that every sequence from S has a weakly Cauchy subsequence. A Banach space X is called *weakly sequentially complete* if every

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weakly Cauchy sequence in X is weakly convergent. A Banach space X has the *Grothendieck property* if w*-convergent sequences in X^* are weakly convergent.

An operator $T\colon X \to Y$ is called *completely continuous* (or *Dunford–Pettis*) if T maps weakly convergent sequences to norm convergent sequences.

A Banach space X has the *Dunford–Pettis property* (DPP) if every weakly compact operator $T\colon X \to Y$ is completely continuous for any Banach space Y. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ whenever (x_n^*) is weakly null in X^* and (x_n) is weakly null in X, see [11, Theorem 1]. If X is a $C(K)$-space or an L_1-space, then X has the DPP. The reader can check [12] and [11] for results related to the DPP.

A bounded subset A of X is called a *Dunford–Pettis* (or DP) (*limited*, respectively) subset of X if each weakly null (w*-null, respectively) sequence (x_n^*) in X^* tends to 0 uniformly on A; i.e.

$$\sup_{x\in A} |x_n^*(x)| \to 0.$$

Every DP (limited, respectively) subset of X is weakly precompact, see [2, page 2], [28, page 377] ([6, Proposition], [32, Lemma 1.1.5, page 25], respectively).

A bounded subset A of X^* is called a *V-subset* of X^* provided that

$$\sup_{x^*\in A} |x^*(x_n)| \to 0$$

for each weakly unconditionally convergent series $\sum x_n$ in X.

A Banach space X has *property* (V) ((wV), respectively) if every V-subset of X^* is relatively weakly compact [25] (weakly precompact, respectively). A Banach space X has property (V) if and only if every unconditionally converging operator T from X to any Banach space Y is weakly compact, see [25, Proposition 1]. It is known that $C(K)$ spaces and reflexive spaces have property (V), see [25, Theorem 1, Proposition 7]).

For $1 \le p < \infty$, p^* denotes the conjugate of p. If $p = 1$, c_0 plays the role of l_{p^*}. The unit vector basis of l_p will be denoted by (e_n).

Let $1 \le p < \infty$. We denote by $l_p(X)$ the Banach space of all p-summable sequences with the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}.$$

Let $1 \le p < \infty$. A sequence (x_n) in X is called *weakly p-summable* if $(\langle x^*, x_n\rangle) \in l_p$ for each $x^* \in X^*$, see [13, page 32], [29, page 134]. Let $l_p^{\mathrm{w}}(X)$ denote the set of all weakly p-summable sequences in X. The space $l_p^{\mathrm{w}}(X)$ is a Banach space with the norm

$$\|(x_n)\|_p^{\mathrm{w}} = \sup\left\{ \left(\sum_{n=1}^{\infty} |\langle x^*, x_n\rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\}.$$

If $p = \infty$, then $l_\infty(X) = l_\infty^{\mathrm{w}}(X)$, see [13, page 33]; if (x_n) is a bounded sequence in X, then

$$\|(x_n)\|_\infty^{\mathrm{w}} = \sup_n \|x_n\| = \|(x_n)\|_\infty.$$

Let $c_0^{\mathrm{w}}(X)$ be the space of weakly null sequences in X. This is a Banach space with the norm

$$\|(x_n)\|_{c_0^{\mathrm{w}}} = \sup_{\|x^*\|\le 1} \|(\langle x^*, x_n\rangle)\|_{c_0},$$

and $c_0^{\mathrm{w}}(X) \simeq W(l_1, X)$.

For $p = \infty$, we consider the space $c_0^{\mathrm{w}}(X)$ instead of $l_\infty^{\mathrm{w}}(X) = l_\infty(X)$.

If $p < q$, then $l_p^{\mathrm{w}}(X) \subseteq l_q^{\mathrm{w}}(X)$. Further, the unit vector basis of l_{p^*} is weakly p-summable for all $1 < p < \infty$. The weakly 1-summable sequences are precisely the weakly unconditionally convergent series and the weakly ∞-summable sequences are precisely weakly null sequences.

We recall the following isometries: $L(l_{p^*}, X) \simeq l_p^{\mathrm{w}}(X)$ for $1 < p < \infty$ and $L(c_0, X) \simeq l_p^{\mathrm{w}}(X)$ for $p = 1$; $T \to (T(e_n))$, see [13, Proposition 2.2, page 36].

Let $1 \le p \le \infty$. An operator $T\colon X \to Y$ is called *p-convergent* if T maps weakly p-summable sequences into norm null sequences. The set of all p-convergent operators is denoted by $C_p(X, Y)$, see [8].

The 1-convergent operators are precisely the unconditionally converging operators and the ∞-convergent operators are precisely the completely continuous operators. If $p < q$, then $C_q(X, Y) \subseteq C_p(X, Y)$.

A sequence (x_n) in X is called *weakly p-convergent* to $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable, see [8]. The weakly ∞-convergent sequences are precisely the weakly convergent sequences.

Let $1 \le p \le \infty$. A bounded subset K of X is *relatively weakly p-compact* (*weakly p-compact*, respectively) if every sequence in K has a weakly p-convergent subsequence with limit in X (in K, respectively).

An operator $T\colon X \to Y$ is *weakly* p-compact if $T(B_X)$ is relatively weakly p-compact, see [8]. The set of weakly p-compact operators $T\colon X \to Y$ will be denoted by $W_p(X, Y)$. If $p < q$, then $W_p(X, Y) \subseteq W_q(X, Y)$.

Suppose that $1 \le p < \infty$. An operator $T\colon X \to Y$ is called *p-summing* (or *absolutely p-summing*) if there is a constant $c \ge 0$ such that for any $m \in \mathbb{N}$ and any $x_1, x_2, \ldots, x_m$ in X,

$$\left(\sum_{i=1}^{m} \|T(x_i)\|^p\right)^{1/p} \leq c \sup\left\{\left(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

The least c for which the previous inequality always holds is denoted by $\pi_p(T)$, see [13, page 31]. The set of all p-summing operators from X to Y is denoted by $\Pi_p(X,Y)$. The operator $T\colon X \to Y$ is *p-summing* if and only if $(Tx_n) \in l_p(Y)$ whenever $(x_n) \in l_p^{\mathrm{w}}(X)$, see [13, Proposition 2.1, page 34], [12, page 59].

A topological space S is called *dispersed* (or *scattered*) if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $l_1 \not\hookrightarrow C(K)$, see [26, Main Theorem].

The Banach–Mazur distance $d(E,F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infimum is taken over all isomorphisms T from E onto F. A Banach space E is called an *$\mathcal{L}_\infty$-space* (*$\mathcal{L}_1$-space*, respectively) if there is a $\lambda \geq 1$ so that every finite dimensional subspace of E is contained in another subspace N with $d(N, l_\infty^n) \leq \lambda$ ($d(N, l_1^n) \leq \lambda$, respectively) for some integer n. Complemented subspaces of $C(K)$ spaces ($L_1(\mu)$ spaces, respectively) are $\mathcal{L}_\infty$-spaces ($\mathcal{L}_1$-spaces, respectively), see [5, Proposition 1.26]. The dual of an $\mathcal{L}_1$- space ($\mathcal{L}_\infty$-space, respectively) is an $\mathcal{L}_\infty$-space ($\mathcal{L}_1$-space, respectively), see [5, Proposition 1.27].

The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP, see [5, Corollary 1.30].

3. The p-(SR) and p-L-limited properties in projective tensor products

The *Right topology* on a Banach space X is the restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X and it is also the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$ compact subsets of X^*, see [27]. Further, $\tau(X^{**}, X^*)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of X^*, see [24].

An operator $T\colon X \to Y$ is *pseudo weakly compact* (pwc) (or *Dunford–Pettis completely continuous* (DPcc)) if it takes weakly null DP sequences in X into norm null sequences in Y, see [19], [33].

A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and DP, see [19, Proposition 1].

A bounded subset K of X^* is called a *Right set* or R-set if

$$\sup_{x^* \in K} |x^*(x_n)| \to 0$$

for each Right null sequence (x_n) in X.

A Banach space X is *sequentially Right* (SR) (or X *has property* (SR)) if every pseudo weakly compact operator $T\colon X \to Y$ is weakly compact for any Banach space Y, see [27].

A Banach space X is sequentially Right if and only if every Right subset of X^* is relatively weakly compact, see [24, Theorem 3.25].

A Banach space X is *weak sequentially Right* (wSR) (or *has the* (wSR) *property*) if every Right subset of X^* is weakly precompact, see [19].

Let $1 \le p < \infty$. An operator $T\colon X \to Y$ is called *DP p-convergent* if it takes DP weakly p-summable sequences to norm null sequences, see [21].

Let $1 \le p \le \infty$. A bounded subset A of a dual space X^* is called a *p-Right set*, see [21], if for every DP weakly p-summable sequence (x_n) in X,

$$\sup_{x^*\in A} |x^*(x_n)| \to 0.$$

Let $1 \le p \le \infty$. A Banach space X has the *p-(SR) (p-(wSR), respectively) property* if every p-Right subset of X^* is relatively weakly compact (weakly precompact, respectively).

The ∞-Right subsets of X^* are precisely the Right subsets and the ∞-(SR) property coincides with the (SR) property. If $p < q$, then a q-Right set in X^* is a p-Right set, since $l_p^{\mathrm{w}}(X) \subseteq l_q^{\mathrm{w}}(X)$. If X has the p-(SR) property, then it has the q-(SR) property, if $p < q$.

If $1 \le p < \infty$ and X has the p-(SR) property, then X has the (SR) property, and thus X^* is weakly sequentially complete, see [21, Proposition 3.3].

A bounded subset A of X^* is called an *L-limited set*, see [31], if

$$\sup_{x^*\in A} |x^*(x_n)| \to 0$$

for each limited weakly null sequence (x_n) in X.

A Banach space X has the *L-limited property* (*wL-limited property*, respectively) if every L-limited subset of X^* is relatively weakly compact, see [31], (weakly precompact, respectively, see [19]).

An operator $T\colon X \to Y$ is called *limited completely continuous* (lcc) if T maps limited weakly null sequences to norm null sequences, see [30].

Let $1 \le p < \infty$. An operator $T\colon X \to Y$ is called *limited p-convergent* if it carries limited weakly p-summable sequences in X to norm null ones in Y, see [17].

Let $1 \le p \le \infty$. A bounded subset A of a dual space X^* is called a *p-L-limited set*, see [21], if for every limited weakly p-summable sequence (x_n) in X,

$$\sup_{x^*\in A} |x^*(x_n)| \to 0.$$

Let $1 \leq p \leq \infty$. A Banach space X has the *p-L-limited property*, see [21], (*p-wL-limited property*, respectively) if every p-L-limited subset of X^* is relatively weakly compact (weakly precompact, respectively).

The ∞-L-limited property coincides with the L-limited property. If X has the p-L-limited property, then X has the L-limited property. Consequently, X^* is weakly sequentially complete and X has the Grothendieck property, see [21, Proposition 3.3].

In the following we consider the p-(SR) and p-L-limited properties in the projective tensor product $X \otimes_\pi Y$.

If $H \subseteq L(X,Y)$, $x \in X$ and $y^* \in Y^*$, let $H(x) = \{T(x)\colon T \in H\}$ and $H^*(y^*) = \{T^*(y^*)\colon T \in H\}$.

In the proof of Theorem 3.3 we will need the following results. We include the proof of the first result for the convenience of the reader.

Lemma 3.1 ([20]). *Let $1 \leq p < \infty$. Suppose that $L(X,Y^*) = \Pi_p(X,Y^*)$. If (x_n) is weakly p-summable in X and (y_n) is bounded in Y, then $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$.*

Proof: Without loss of generality suppose $\|(x_n)\|_p^{\mathrm{w}} \leq 1$ and $\|y_n\| \leq 1$. Let $T \in (X \otimes_\pi Y)^* \simeq L(X,Y^*)$, see [14, page 230]. Then

$$\sum_n |\langle T, x_n \otimes y_n\rangle|^p \leq \sum_n \|T(x_n)\|^p \leq \pi_p(T)^p.$$

Thus $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$. □

Lemma 3.2 ([4, Lemma 2]). *Let (x_n) be a DP sequence in X weakly converging to $x \in X$ and (y_n) be a DP sequence in Y weakly converging to $y \in Y$. Then $(x_n \otimes y_n)$ is a DP sequence in $X \otimes_\pi Y$ that converges weakly to $x \otimes y$.*

Theorem 3.3. *Let $1 \leq p < \infty$. Suppose that $L(X,Y^*) = K(X,Y^*) = \Pi_p(X,Y^*)$. If X and Y have the p-(SR) property, then $X \otimes_\pi Y$ has the p-(SR) property.*

Proof: Let H be a p-Right subset of $L(X,Y^*) = K(X,Y^*) = \Pi_p(X,Y^*)$ and let (T_n) be a sequence in H. By [18, Theorem 3], it is enough to show that (i) $H(x)$ is relatively weakly compact for all $x \in X$ and (ii) $H^*(y^{**})$ is relatively weakly compact for all $y^{**} \in Y^{**}$. Let $x \in X$. We show that $\{T_n(x)\colon n \in \mathbb{N}\}$ is a p-Right subset of Y^*. Suppose (y_n) is a DP weakly p-summable sequence in Y. Let $T \in L(X,Y^*) \simeq (X \otimes_\pi Y)^*$, see [14, page 230]. Because T is weakly compact, $T^{**}(X^{**}) \subseteq Y^*$. If $x^{**} \in X^{**}$, then $\sum_n |\langle x^{**}, T^*(y_n)\rangle|^p = \sum_n |\langle T^{**}(x^{**}), y_n\rangle|^p < \infty$. Thus $(T^*(y_n))$ is weakly p-summable in X^*. Hence

$$\sum_n |\langle T, x \otimes y_n\rangle|^p = \sum_n |\langle x, T^*(y_n)\rangle|^p < \infty.$$

Thus $(x \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$. Let (A_n) be a weakly null sequence in $L(X, Y^*) \simeq (X \otimes_\pi Y)^*$. Then $(A_n(x))$ is weakly null in Y^* and

$$\langle A_n, x \otimes y_n \rangle = \langle A_n(x), y_n \rangle \to 0,$$

since (y_n) is a DP sequence in Y. Therefore $(x \otimes y_n)$ is a DP sequence in $X \otimes_\pi Y$. Since (T_n) is a p-Right set,

$$\langle T_n, x \otimes y_n \rangle = \langle T_n(x), y_n \rangle \to 0.$$

Therefore $\{T_n(x) \colon n \in \mathbb{N}\}$ is a p-Right subset of Y^*, hence relatively weakly compact.

Let $y^{**} \in Y^{**}$. We show that $\{T_n^*(y^{**}) \colon n \in \mathbb{N}\}$ is a p-Right subset of X^*. Suppose (x_n) is a DP weakly p-summable sequence in X. For $n \in \mathbb{N}$,

$$\langle T_n^*(y^{**}), x_n \rangle = \langle y^{**}, T_n(x_n) \rangle.$$

We show that $(T_n(x_n))$ is a p-Right subset of Y^*. Suppose that (y_n) is a DP weakly p-summable sequence in Y. By Lemma 3.1, $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$. By Lemma 3.2, $(x_n \otimes y_n)$ is a DP sequence in $X \otimes_\pi Y$. Since $\{T_n \colon n \in \mathbb{N}\}$ is a p-Right set,

$$\langle T_n, x_n \otimes y_n \rangle = \langle T_n(x_n), y_n \rangle \to 0.$$

Therefore $(T_n(x_n))$ is a p-Right subset of Y^*, and thus relatively weakly compact.

Let $y \in Y$. An argument similar to the one above shows that $(x_n \otimes y)$ is a DP weakly p-summable sequence in $X \otimes_\pi Y$. Note that

$$\langle T_n, x_n \otimes y \rangle = \langle T_n(x_n), y \rangle \to 0,$$

since (T_n) is a p-Right set. Thus $(T_n(x_n))$ is w^*-null. Therefore $(T_n(x_n))$ is weakly null. This implies that $\{T_n^*(y^{**}) \colon n \in \mathbb{N}\}$ is a p-Right subset of X^*, thus relatively weakly compact. Then H is relatively weakly compact by [18, Theorem 3]. □

Theorem 3.4. *Let $1 \le p < \infty$. Suppose that $L(X, Y^*) = K(X, Y^*) = \Pi_p(X, Y^*)$. If X and Y have the p-L-limited property, then $X \otimes_\pi Y$ has the p-L-limited property.*

PROOF: The proof is similar to the proof of Theorem 3.3 and uses [4, Lemma 4]. □

If $L(X, Y^*) = K(X, Y^*)$, X has the p-(SR) property and Y is reflexive, then $X \otimes_\pi Y$ has the p-(SR) property, see [1, Theorem 3.20]. We obtain a similar result for the p-L-limited property.

Theorem 3.5. *Let $1 \le p < \infty$. Suppose that $L(X, Y^*) = K(X, Y^*)$. If X has the p-L-limited property and Y is reflexive, then $X \otimes_\pi Y$ has the p-L-limited property.*

Proof: Let H be a p-L-limited subset of $L(X, Y^*) = K(X, Y^*)$ and let (T_n) be a sequence in H. Let $x \in X$. The set $\{T_n(x) \colon n \in \mathbb{N}\}$ is a bounded set in a reflexive space, so it is relatively weakly compact.

Let $y \in Y^{**} \simeq Y$. We show that $\{T_n^*(y) \colon n \in \mathbb{N}\}$ is a p-L-limited subset of X^*. Suppose (x_n) is a limited weakly p-summable sequence in X. The proof of Theorem 3.3 shows that $(x_n \otimes y)$ is weakly p-summable in $X \otimes_\pi Y$. Let (A_n) be a w*-null sequence in $L(X, Y^*) \simeq (X \otimes_\pi Y)^*$. Then $(A_n^*(y))$ is w*-null in X^* and

$$\langle A_n, x_n \otimes y\rangle = \langle A_n^*(y), x_n\rangle \to 0,$$

since (x_n) is a limited sequence in X. Therefore $(x_n \otimes y)$ is a limited sequence in $X \otimes_\pi Y$. Since (T_n) is a p-L-limited set,

$$\langle T_n, x_n \otimes y\rangle = \langle T_n^*(y), x_n\rangle \to 0.$$

Therefore $\{T_n^*(y) \colon n \in \mathbb{N}\}$ is a p-L-limited subset of X^*, and thus relatively weakly compact. Then H is relatively weakly compact by [18, Theorem 3]. $\square$

Corollary 3.6. *Let $1 \le p < \infty$. Suppose $L(X, Y^*) = \Pi_p(X, Y^*)$ and X and Y have the p-(SR) property. If $l_1 \not\hookrightarrow X$ (or Y^* has the Schur property), then $X \otimes_\pi Y$ has the p-(SR) property.*

Proof: Let $T \colon X \to Y^*$ be an operator. Since T is p-summing, it is weakly compact and completely continuous, see [13, Theorem 2.17].

Thus T is compact by a result of E. Odell in [28, page 377]. If Y^* has the Schur property, then T is compact (since it is also weakly compact). Then $L(X, Y^*) = K(X, Y^*)$. Apply Theorem 3.3. $\square$

Observation 1.

(i) Let $1 \le p \le 2$. If X is an $\mathcal{L}_\infty$-space and Y is an $\mathcal{L}_p$-space, then every operator $T \colon X \to Y$ is 2-summing, see [13, Theorem 3.7].

(ii) If X and Y are $\mathcal{L}_\infty$-spaces, then $L(X, Y^*) = \Pi_p(X, Y^*)$, $2 \le p < \infty$. Indeed, by (i), every operator $T \colon X \to Y^*$ is 2-summing, and thus p-summing, $2 \le p < \infty$.

(iii) If X and Y are infinite dimensional $\mathcal{L}_\infty$-spaces, then $L(X, Y^*) = CC(X, Y^*)$ by [13, Theorems 3.7 and 2.17].

Corollary 3.7. *Let $2 \le p < \infty$. Suppose X and Y are $\mathcal{L}_\infty$-spaces and $l_1 \not\hookrightarrow X$ (or $l_1 \not\hookrightarrow Y$). If X and Y have the p-(SR) property, then $X \otimes_\pi Y$ has the p-(SR) property.*

Proof: Suppose $l_1 \not\hookrightarrow X$. By Observation 1, $L(X, Y^*) = \Pi_p(X, Y^*)$. By Corollary 3.6, $X \otimes_\pi Y$ has the p-(SR) property. If $l_1 \not\hookrightarrow Y$, then the previous argument shows that $Y \otimes_\pi X$ has the p-(SR) property. Hence $X \otimes_\pi Y \simeq Y \otimes_\pi X$ has the p-(SR) property. □

Let $1 \leq p \leq \infty$. A Banach space X has the *Dunford–Pettis property of order p* (DPP_p) if every weakly compact operator $T\colon X \to Y$ is p-convergent for any Banach space Y, see [8].

If X has the DPP, then X has the DPP_p for all $1 < p < \infty$.

A Banach space X has the DP*-*property* (DP*P) if all weakly compact sets in X are limited, see [7].

The space X has the DP*P if and only if $L(X, c_0) = CC(X, c_0)$, see [7, Proposition 2.1], [23, Theorem 1]. If X has the DP*P, then it has the DPP. If X is a Schur space or if X has the DPP and the Grothendieck property, then X has the DP*P.

Let $1 \leq p \leq \infty$. A Banach space X has the DP*-*property of order p* ($\mathrm{DP^*P}_p$) if all weakly p-compact sets in X are limited, see [16].

If X has the DP*P, then X has the $\mathrm{DP^*P}_p$ for all $1 \leq p < \infty$. If X has the $\mathrm{DP^*P}_p$, then X has the DPP_p.

If X has property (V), then X has the (SR) property, see [10, page 247].

Proposition 3.8. *Let $1 \leq p < \infty$.*

(i) *If X has the DPP_p and property (V), then X has the p-(SR) property.*
(ii) *If X has the DP^*P_p and property (V), then X has the p-L-limited property.*
(iii) *If X is an $\mathcal{L}_\infty$-space, then X^{**} has the p-(SR) property and the p-L-limited property.*

Proof: (i) Let $T\colon X \to Y$ be a DP p-convergent operator. Then T is p-convergent, since X has the DPP_p, see [21, Theorem 3.18]. Since T is unconditionally convergent and X has property (V), T is weakly compact. Then X has the p-(SR) property, see [21, Theorem 3.10].

(ii) Let $T\colon X \to Y$ be a limited p-convergent operator. Then T is p-convergent, since X has the $\mathrm{DP^*P}_p$, see [21, Theorem 3.17]. As above, T is weakly compact, and thus X has the p-L-limited property, see [21, Theorem 3.10].

(iii) Since X is an $\mathcal{L}_\infty$-space, X^{**} is complemented in some $C(K)$ space, see [13, Theorem 3.2]. Moreover, $C(K)$ spaces have the p-(SR) property (by (i)). Thus X^{**} has the p-(SR) property and property (V) (since these properties are inherited by quotients). Further, X^{**} has the DP*P, see [23, Corollary 5], thus the $\mathrm{DP^*P}_p$. Then X^{**} has the p-L-limited property. □

Proposition 3.9. *Let $1 \leq p \leq \infty$. A Banach space X has the p-L-limited property if and only if it has the p-(SR) property and the Grothendieck property.*

PROOF: The case $p = \infty$ is [10, Proposition 24].

Let $1 \leq p < \infty$. Suppose X has the p-L-limited property. Then X has the p-(SR) property and the Grothendieck property, see [21, Proposition 3.3].

Conversely, suppose X has the p-(SR) property and the Grothendieck property. Since X has the Grothendieck property, any DP set in X is limited. Hence any DP weakly p-summable sequence in X is limited weakly p-summable. Then any p-L-limited set in X^* is a p-Right set, and thus relatively weakly compact. $\square$

Corollary 3.10. *Let $2 \leq p < \infty$. Let $X = C(K_1)$, $Y = C(K_2)$, where K_1 and K_2 are infinite compact Hausdorff spaces and K_1 (or K_2) is dispersed. Then $X \otimes_\pi Y$ has the p-(SR) property.*

PROOF: We have that $C(K)$ spaces are $\mathcal{L}_\infty$-spaces, see [13, Theorem 3.2], and have the p-(SR) property. If K_1 (or K_2) is dispersed, then $l_1 \not\hookrightarrow C(K_1)$ (or $l_1 \not\hookrightarrow C(K_2)$), see [26, Main Theorem]. Apply Corollary 3.7. $\square$

Corollary 3.11. *Let $2 \leq p < \infty$. Suppose X and Y are $\mathcal{L}_\infty$-spaces, $l_1 \not\hookrightarrow Y$, and Y has the p-(SR) property. Then $X^{**} \otimes_\pi Y$ has the p-(SR) property.*

PROOF: Since X is an $\mathcal{L}_\infty$-space, X^{**} has the p-(SR) property by Proposition 3.8. Apply Corollary 3.7. $\square$

Every $L_p(\mu)$ space is an $\mathcal{L}_p$-space, $1 \leq p \leq \infty$, see [13, Theorem 3.2].

Corollary 3.12. *Let $1 \leq p < \infty$. Let X be a $C(K)$ space and $Y = l_r$, $r > 2$. Then $X \otimes_\pi Y$ has the p-(SR) property.*

PROOF: Since X is a $C(K)$ space, it has the p-(SR) property. If q is the conjugate of r, then $1 < q < 2$. Every operator $T\colon C(K) \to l_q$, $1 < q < 2$, is compact [34, Lemma, page 100]. Apply [1, Theorem 3.20]. $\square$

A $C(K)$ space has the Grothendieck property if and only if it contains no complemented copy of c_0, see [9].

Corollary 3.13. *Let $1 \leq p < \infty$. Let X be a $C(K)$ space with the Grothendieck property and $Y = l_r$, $r > 2$. Then $X \otimes_\pi Y$ has the p-L-limited property.*

PROOF: Since X is a $C(K)$ space with the Grothendieck property, it has the DP*P, see [23, Corollary 5]. Further, X has property (V), see [25, Theorem 1]. By Proposition 3.8 (or 3.9), X has the p-L-limited property. The proof of Corollary 3.12 shows that $L(X, Y^*) = K(X, Y^*)$. Apply Theorem 3.5. $\square$

Lemma 3.14. *Let $1 \leq p < \infty$.*

(i) *If X is an infinite dimensional space with the Schur property, then X does not have the p-(wSR) (the p-wL-limited, respectively) property.*

(ii) *If X has the p-(wSR) (the p-wL-limited, respectively) property, then $l_1 \overset{c}{\not\hookrightarrow} X$ and $c_0 \not\hookrightarrow X^*$.*

PROOF: (i) If X is an infinite dimensional space with the Schur property, then X does not have the (wSR) (the wL-limited, respectively) property, see [19, Corollary 5]. Hence X does not have the p-(wSR) (the p-wL-limited, respectively) property.

(ii) By (i), l_1 does not have the p-(wSR) (the p-wL-limited, respectively) property. Since the p-(wSR) (the p-wL-limited, respectively) property is inherited by quotients, it follows that if X has the p-(wSR) (the p-wL-limited, respectively) property, then $l_1 \overset{c}{\not\hookrightarrow} X$, and $c_0 \not\hookrightarrow X^*$, see [3, Theorem 4]. $\square$

Theorem 3.15. *Let $1 \leq p < \infty$.*

(i) *If $X \otimes_\pi Y$ has the p-(SR) property, then X and Y have the p-(SR) property and at least one of them does not contain l_1.*

(ii) *If $X \otimes_\pi Y$ has the p-L-limited property, then X and Y have the p-L-limited property and at least one of them does not contain l_1.*

PROOF: We only prove (i). The other proof is similar. Suppose that $X \otimes_\pi Y$ has the p-(SR) property. Then X and Y have the p-(SR) property, since this property is inherited by quotients. We will show that $l_1 \not\hookrightarrow X$ or $l_1 \not\hookrightarrow Y$. Suppose that $l_1 \hookrightarrow X$ and $l_1 \hookrightarrow Y$. Hence $L_1 \hookrightarrow X^*$, see [12, page 212]. Also, the Rademacher functions span l_2 inside of L_1, and thus $l_2 \hookrightarrow X^*$. Similarly $l_2 \hookrightarrow Y^*$. Then $c_0 \hookrightarrow K(X, Y^*)$, see [15, page 334], [22, Corollary 24]. By Lemma 3.14 we have a contradiction that concludes the proof. $\square$

Corollary 3.16. *Let $1 \leq p < \infty$. Suppose that $L(X, Y^*) = K(X, Y^*) = \Pi_p(X, Y^*)$. The following statements are equivalent:*

1. (i) *X and Y have the p-(SR) property and at least one of them does not contain l_1.*
   (ii) *$X \otimes_\pi Y$ has the p-(SR) property.*
2. (i) *X and Y have the p-L-limited property and at least one of them does not contain l_1.*
   (ii) *$X \otimes_\pi Y$ has the p-L-limited property.*

PROOF: We only prove 1. The other proof is similar.

(i) $\Rightarrow$ (ii) by Theorem 3.3.

(ii) $\Rightarrow$ (i) by Theorem 3.15. $\square$

Corollary 3.17. *Let $1 \leq p < \infty$. Suppose that X and Y have the DPP and $L(X, Y^*) = \Pi_p(X, Y^*)$. The following statements are equivalent:*

(i) *X and Y have the p-(SR) property and at least one of them does not contain l_1.*

(ii) *$X \otimes_\pi Y$ has the p-(SR) property.*

Proof: (i) $\Rightarrow$ (ii) Suppose that X and Y have the DPP. Without loss of generality suppose that $l_1 \not\hookrightarrow X$. Then X^* has the Schur property, see [11, Theorem 3]. Apply Corollary 3.6.

(ii) $\Rightarrow$ (i) by Theorem 3.15. $\square$

By Corollary 3.17, the space $C(K_1) \otimes_\pi C(K_2)$ has the p-(SR) property if and only if either K_1 or K_2 is dispersed.

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I. Ghenciu:

Mathematics Department, University of Wisconsin-River Falls, 410 S 3rd St, River Falls, Wisconsin, 54022, U.S.A.

E-mail: ioana.ghenciu@uwrf.edu

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