Convex orderings for stochastic processes*

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Abstract. We consider partial orderings for stochastic processes induced by expectations of convex or increasing convex (concave or increasing concave) functionals. We prove that these orderings are implied by the analogous finite dimensional orderings.

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1. Let E be a Polish space and let $\mathcal{F}$ be a class of functionals defined on $E^{\mathbb{N}}$. We consider two stochastic processes $X = \{X_n \,|\, n \in \mathbb{N}\}$ and $Y = \{Y_n \,|\, n \in \mathbb{N}\}$ taking values in E. Relations such as

$$E\phi(X) \leq E\phi(Y) \qquad \forall\, \phi \in \mathcal{F} \tag{1}$$

are often used to define partial orders for stochastic processes which have many applications in probability, mathematical statistics, mathematical economics and operations research (see for example Stoyan (1983)).

Our goal is to find families $\mathcal{F}_n$ of functions defined on E^n, such that

$$E\psi(X_1, \dots, X_n) \leq E\psi(Y_1, \dots, Y_n) \qquad \forall\, \psi \in \mathcal{F}_n, \quad \forall\, n \in \mathbb{N}$$

implies (1).

Kamae, Krengel and O'Brien (1977) proved the implication when E is partially ordered and $\mathcal{F}$ and $\mathcal{F}_n$ are the classes of increasing functions. The problem is reported as open by Stoyan (1983) in the case of the classes of increasing convex and increasing concave functions defined on a linear space E. In this note, we give a solution for these two families and the families of convex and concave functions. It may be noted that a similar problem was studied by Lindqvist (1988), who showed that a stochastic process is associated, if all its finite dimensional distributions are associated.

2. We introduce the following classes of functions defined on a convex subset U of a partially ordered topological vector space.

$$\begin{aligned}
\mathcal{F}^{\mathrm{I}}(U) &= \{f : U \to \mathbb{R} \,|\, f \text{ is increasing }\},\\
\mathcal{F}^{\mathrm{V}}(U) &= \{f : U \to \mathbb{R} \,|\, f \text{ is convex }\},\\
\mathcal{F}^{\mathrm{IV}}(U) &= \mathcal{F}^{\mathrm{I}}(U) \cap \mathcal{F}^{\mathrm{V}}(U),\\
\mathcal{F}^{\mathrm{C}}(U) &= \{f : U \to \mathbb{R} \,|\, f \text{ is concave }\},\\
\mathcal{F}^{\mathrm{IC}}(U) &= \mathcal{F}^{\mathrm{I}}(U) \cap \mathcal{F}^{\mathrm{C}}(U).
\end{aligned}$$

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Let W be a convex subset of a partially ordered Polish space. For every set $A \subset \mathbb{N}$, we endow W^A with the product topology and of the componentwise ordering. All the integrals that appear in the sequel are assumed to exist.

Theorem. *Let $X = \{X_n \mid n \in \mathbb{N}\}$ and $Y = \{Y_n \mid n \in \mathbb{N}\}$ be two stochastic processes with values in W, and let $\mathcal{F}$ be any of the classes $\mathcal{F}^{\mathrm{V}}$, $\mathcal{F}^{\mathrm{IV}}$, $\mathcal{F}^{\mathrm{C}}$, $\mathcal{F}^{\mathrm{IC}}$. If*

$$E\phi(X_1, \ldots, X_n) \leq E\phi(Y_1, \ldots, Y_n) \qquad \forall\, n \in \mathbb{N}$$

for every measurable $\phi \in \mathcal{F}(W^n)$ such that the above expectations exist, then $Eg(X) \leq Eg(Y)$ for every continuous functional $g \in \mathcal{F}(W^{\mathbb{N}})$.

The proof is based upon the following lemma.

Lemma. *Let $E = E_1 \times E_2$ be a convex subset of a topological vector space, let $H : E \to \mathbb{R}$ be a convex function bounded from below, and let $h : E_1 \to \mathbb{R}$ be defined by the relation:*

$$h(x) = \inf_{y \in E_2} H(x, y).$$

Then h is convex.

Proof of Lemma: Given a function $f : B \to \mathbb{R}$, we define

$$\mathrm{epi}\,(f) = \{(x, z) \in B \times \mathbb{R} \mid f(x) < z\}.$$

Let also

$$A_y = \{(x, z) \in E_1 \times \mathbb{R} \mid (x, y, z) \in \mathrm{epi}\,(H)\}, \quad y \in E_2.$$

Let us prove now that $\mathrm{epi}\,(h) = \bigcup_{y \in E_2} A_y$. If $(x, z) \in \bigcup_{y \in E_2} A_y$, then there exists $y_0 \in E_2$ such that $(x, z) \in A_{y_0}$ and

$$h(x) = \inf_{y \in E_2} H(x, y) \leq H(x, y_0) < z;$$

thus $(x, z) \in \mathrm{epi}\,(h)$. Conversely, let $(x, z) \in \mathrm{epi}\,(h)$, i.e. $h(x) = \inf_{y \in E_2} H(x, y) < z$; we can choose $\overline{y} \in E_2$ such that $H(x, \overline{y}) < z$. Then $(x, z) \in A_{\overline{y}} \subset \bigcup_{y \in E_2} A_y$.

Since a function f is convex, if and only if $\mathrm{epi}\,(f)$ is a convex set, the claim will follow if we prove that $\bigcup_{y \in E_2} A_y$ is convex. Let $(x_1, z_1), (x_2, z_2) \in \bigcup_{y \in E_2} A_y$; then there exist y_1, y_2 such that $H(x_1, y_1) < z_1$ and $H(x_2, y_2) < z_2$. The inequalities above and the convexity of H imply

$$H\big(\alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2)\big) \leq \alpha H(x_1, y_1) + (1 - \alpha) H(x_2, y_2) < \alpha z_1 + (1 - \alpha) z_2,$$

i.e.

$$\big(\alpha x_1 + (1 - \alpha) x_2, \alpha z_1 + (1 - \alpha) z_2\big) \in A_{\alpha y_1 + (1 - \alpha) y_2} \subset \bigcup_{y \in E_2} A_y.$$

□

Proof of Theorem: First, we prove the result for the classes $\mathcal{F}^{\mathrm{V}}$ and $\mathcal{F}^{\mathrm{IV}}$. Let $g : W^{\mathbb{N}} \to \mathbb{R}$ be a function bounded from below; for every $n \in \mathbb{N}$, we define functions $g_n : W^{\mathbb{N}} \to \mathbb{R}$ and $\widetilde{g}_n : W^n \to \mathbb{R}$ by the following relation:

$$g_n(u_1, u_2, \dots) = \widetilde{g}_n(u_1, \dots, u_n) = \inf_{\substack{s_k \in W \\ k > n}} g(u_1, \dots, u_n, s_{n+1}, \dots).$$

If $g \in \mathcal{F}(W^{\mathbb{N}})$, then the Lemma implies that $\widetilde{g}_n \in \mathcal{F}(W^n)$. Thus

$$E\widetilde{g}_n(X_1, \dots, X_n) \le E\widetilde{g}_n(Y_1, \dots, Y_n)$$

or, equivalently,

$$Eg_n(X) \le Eg_n(Y).$$

It is clear that $\{g_n \,|\, n \in \mathbb{N}\}$ is an increasing sequence. We show now that it converges pointwise to g. For every $x \in W^{\mathbb{N}}$ and $n > 0$, we choose a sequence $s^{(n)}_{n+1}, s^{(n)}_{n+2}, \dots$ such that, if

$$x^{(n)} = (x_1, \dots, x_n, s^{(n)}_{n+1}, s^{(n)}_{n+2}, \dots),$$

one has

$$|\, g(x^{(n)}) - g_n(x)\,| < 2^{-n}.$$

The relation

$$|\, g(x) - g_n(x)\,| \le |\, g(x) - g(x^{(n)})\,| + |\, g(x^{(n)}) - g_n(x)\,|,$$

the continuity of g and the convergence of the sequence $\{x^{(n)} \,|\, n \in \mathbb{N}\}$ to x imply that $\lim_{n\to\infty} g_n = g$.

It follows from the monotone convergence theorem that $Eg(X) \le Eg(Y)$.

Consider now the case of a function $g \in \mathcal{F}$ not necessarily bounded from below. Let $g^+ = \max(g, 0)$, $g^- = \max(-g, 0)$ and $h_n = \max(g, -n)$. Then, for every $n \in \mathbb{N}$, we have that $h_n^+ = g^+$ and $h_n^- \uparrow g^-$. Since $h_n \in \mathcal{F}$ and h_n is bounded from below, it follows that

$$Eh_n(X) \le Eh_n(Y).$$

The monotone convergence theorem implies that

$$\lim_{n\to\infty} Eh_n^-(\cdot) = Eg^-(\cdot);$$

therefore

$$\lim_{n\to\infty} Eh_n(\cdot) = Eg(\cdot)$$

and the claim follows immediately.

The result for the classes $\mathcal{F}^{\mathrm{C}}$ and $\mathcal{F}^{\mathrm{IC}}$ can be easily proved now, since $f \in \mathcal{F}^{\mathrm{C}}$, if and only if $(-f) \in \mathcal{F}^{\mathrm{V}}$ and $f \in \mathcal{F}^{\mathrm{IC}}$, if and only if $h \in \mathcal{F}^{\mathrm{IV}}$, where $h(x) = -f(-x)$.

□

References

[1] Stoyan D., *Comparison Methods for Queues and Other Stochastic Models*, Wiley, New York, 1983.

[2] Kamae T., Krengel U., O'Brien G.L., *Stochastic inequalities on partially ordered spaces*, Annals of Probability **5** (1977), 899–912.

[3] Lindqvist B.H., *Association of probability measures on partially ordered spaces*, Journal of Multivariate Analysis **26** (1988), 111–132.

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