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# Completely regular spaces

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**Dedicated to the memory of Zdeněk Frolík**

*Abstract.* We conduct an investigation of the relationships which exist between various generalizations of complete regularity in the setting of merotopic spaces, with particular attention to filter spaces such as Cauchy spaces and convergence spaces. Our primary contribution consists in the presentation of several counterexamples establishing the divergence of various such generalizations of complete regularity. We give examples of: (1) a contigual zero space which is not weakly regular and is not a Cauchy space; (2) a separated filter space which is a $z$-regular space but not a nearness space; (3) a separated, Cauchy, zero space which is $z$-regular but not regular; (4) a separated, Cauchy, zero space which is $\mu$-regular but not regular and not $z$-regular; (5) a separated, Cauchy, zero space which is not weakly regular; (6) a topological space which is regular and $\mu$-regular but not $z$-regular; (7) a filter, zero space which is regular and $z$-regular but not completely regular; and, (8) a regular Hausdorff topological space which is $z$-regular but not completely regular.

*Keywords:* merotopic space, nearness space, Cauchy space, filter merotopic space, pretopological space, zero space, complete regularity, weak regularity, $z$-regularity, $\mu$-regularity

*Classification:* 54C30, 54C40, 54E17, 18B30

## Introduction

During the last three decades, topologists have studied several categories, e.g.,

- Conv, the category of convergence spaces [F59].
- Mer, the category of merotopic spaces [K63].
- Fil, the category of filter merotopic spaces [K65], which is isomorphic to the category of grill-determined nearness spaces[BHR76].
- Chy, the category of Cauchy spaces [Ke68], [KR74], [LC89].
- Near, the category of nearness spaces [H74a], [H74b], [H88], [P88].

All of these categories contain Top, the category of all topological spaces (sometimes assuming a very weak separation axiom), as a subcategory.

After these categories had been defined and their fundamental properties explicated, topologists began extending various interesting subcategories of Top to the above larger categories. One of the most interesting subcategories of Top is the category of completely regular spaces. Perhaps the strongest reason for the importance of completely regular spaces is the fact that they are closely related to the real number system, but a secondary reason is that there are many equivalent ways of characterizing these spaces. However, in the larger categories mentioned above, the various formulations of complete regularity may not remain equivalent.

Our principal objective in this paper is to examine how various generalizations of complete regularity are related. In carrying out this investigation we will operate within the category Mer, a category which (essentially) contains all of the above mentioned categories.

In order to keep the exposition as brief as possible, we assume familiarity with merotopic spaces and nearness spaces (see, e.g., [H83] or [H88]), with convergence spaces and Cauchy spaces (see, e.g., [LC89]), and with the relationships that exist between all these (see, e.g., [BHL86]). We also assume familiarity with completely regular nearness spaces [BHO89]. However, we do present some of the basic definitions and fundamental results.

Recall that a merotopic space can have its structure given in any of four main ways: by means of the **uniform covers**, by means of the **near collections**, by means of the **far collections**, or by means of the **micromeric** (or **Cauchy**) **collections**. One should recall here that Katětov used the concept of a collection being micromeric as a primitive in his theory of merotopic spaces [K63], [K65]. Herrlich [H74b] has shown that the structure of a space can be determined by defining either of the four: the uniform covers, the far collections, the near collections, or the micromeric collections. For merotopic spaces, Katětov had earlier shown that the structure can be given equivalently either with uniform covers or with micromeric collections.

These notions are related as follows: A collection $\mathcal{A}$ of subsets of a space $X$ is **far** in $X$ provided the collection

$$\{X \setminus A \mid A \in \mathcal{A}\}$$

is a uniform cover of $X$. A **near** collection is one which is not far. A collection is **micromeric** (or **Cauchy**) provided that $\sec \mathcal{A}$ is a near collection, where sec is defined by the equation:

$$\sec \mathcal{A} = \{B \subset X \mid B \cap A \neq \emptyset \quad \text{for all } A \in \mathcal{A}\}.$$

Every merotopic space has an underlying Čech closure space whose structure is determined by the closure operator $\mathrm{cl}_X$ defined by :

$$x \in \mathrm{cl}_X A \qquad \Longleftrightarrow \qquad \{\{x\}, A\} \text{ is near in } X.$$

In general, this closure operator fails to be idempotent. It is idempotent, and hence is a topological closure, provided the merotopic space is a nearness space, i.e., satisfies the axiom [H74a] of Herrlich:

$$\mathrm{cl}_X \mathcal{A} \text{ is near} \qquad \Longrightarrow \qquad \mathcal{A} \text{ is near},$$

where

$$\mathrm{cl}_X \mathcal{A} = \{\mathrm{cl}_X A \mid A \in \mathcal{A}\}.$$

We can also say that for a micromeric collection $\mathcal{A}$ and for $x \in X$, $\mathcal{A}$ **converges** to $x$ (and write $\mathcal{A} \to x$) provided that the collection $\mathcal{A} \vee \dot{x}$ is micromeric[1, 2]. This concept of convergence can be used to characterize the closure operator: We have that $x \in \mathrm{cl}_X A$ iff $\sec\{A, \{x\}\} \to x$.

There arises the functor $\mathsf{T} : \mathsf{Near} \to \mathsf{Top}$. Its image is not all of $\mathsf{Top}$, rather it is the subcategory $\mathsf{Top}_\mathsf{S}$ of all symmetric topological spaces[3], i.e., those which satisfy the axiom of Šanin [Š43]:

$$x \in \mathrm{cl}_X\{y\} \qquad \Longleftrightarrow \qquad y \in \mathrm{cl}_X\{x\}.$$

The functor $\mathsf{T} : \mathsf{Near} \to \mathsf{Top}_\mathsf{S}$ has a right inverse $\mathsf{Top}_\mathsf{S} \to \mathsf{Near}$ which turns out to be a full embedding of $\mathsf{Top}_\mathsf{S}$ as a bicoreflective subcategory of $\mathsf{Near}$; we shall assume this embedding is an inclusion, an assumption which is tantamount to assuming that a symmetric topological space has its structure given by the set of open covers, i.e., a symmetric topological space is a nearness space whose uniform covers are precisely those covers which are refined by some open cover.

In this paper we are interested only in those topological spaces which are symmetric. Therefore, we shall shorten the terminology and when we say "topological space", we always mean "symmetric topological space". Moreover, we are interested only in merotopic spaces: when we say "space", we mean "merotopic space". Whenever $X$ and $Y$ are spaces, we shall use the notation $\mathrm{Hom}\,(X, Y)$ to denote the set of all uniformly continuous maps $f : X \to Y$.

We shall need to consider the separated nearness spaces. These have been defined by Herrlich in [H74b] and their properties explored in [BH78b] and [BH79]. The definition of these spaces is as follows. First, we say that a collection $\mathcal{A}$ of subsets of a merotopic space $X$ is **concentrated** iff $\mathcal{A}$ is both near and micromeric. We then define a nearness space $X$ to be **separated** provided that for any concentrated collection $\mathcal{A}$, the collection

$$\{B \subset X \mid \{B\} \cup \mathcal{A} \ \text{ is near }\}$$

is near (and hence is the unique maximal near collection containing $\mathcal{A}$). Some motivation for this terminology lies in the fact that a $\mathrm{T}_1$ topological space is separated iff it is Hausdorff.

Recall the definition of regularity [H74a] for merotopic spaces: A space $X$ is said to be **regular** if for every uniform cover $\mathcal{A}$ of $X$, the collection[4]

$$\{B \subset X \mid B < A \ \text{ for some } A \in \mathcal{A}\}$$

is a uniform cover of $X$.

[1] Here, $\dot{x}$ denotes the principal filter generated by $\{x\}$.

[2] For collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of $X$, $\mathcal{A} \vee \mathcal{B}$ denotes the collection of all $A \cup B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, while $\mathcal{A} \wedge \mathcal{B}$ denotes the collection of all $A \cap B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

[3] These spaces have also been called $\mathrm{R}_0$ spaces and essentially $\mathrm{T}_1$ spaces.

[4] $B < A$ means $\{A, X \setminus B\}$ is a uniform cover of $X$ or, equivalently, for some uniform cover $\mathcal{G}$ of $X$, we have $\mathrm{star}\,(B, \mathcal{G}) \subset A$.

Every regular space is a separated nearness space. Our terminology is chosen with regard to the fact that a topological space is regular as a nearness space iff it is regular in the usual topological sense.

Regularity is a rather strong requirement on a nearness space. In many ways, regular spaces are as well-behaved as uniform spaces [BH79]. We now define a less stringent concept.

**Definition 1.** A space $X$ is said to be **weakly regular** provided it satisfies the condition: Whenever $\mathcal{A}$ is a micromeric collection on $X$ then so is $\mathrm{cl}_X \mathcal{A}$.

**Proposition 2.** *If $X$ is a nearness space then the following are equivalent:*

(1) *$X$ is weakly regular.*
(2) *Every uniform cover of $X$ is refined by a uniform cover consisting of closed sets (i.e., closed in the underlying topological space $\mathsf{T}X$).*
(3) *Every far collection in $X$ is corefined[5] by a far collection of open sets.*

Proof: (1) $\Longrightarrow$ (2): Assume that $X$ is weakly regular and let $\mathcal{G}$ be a uniform cover of $X$. It suffices to show that

$$\mathcal{H} = \{H \subset X \mid H \text{ is closed and } H \subset G \text{ for some } G \in \mathcal{G}\}$$

is a uniform cover of $X$. Assume that $\mathcal{H}$ is not a uniform cover of $X$. Then there exists a micromeric collection $\mathcal{A}$ such that for every $A \in \mathcal{A}$ and for every $H \in \mathcal{H}$ we have $A \not\subset H$. Since $X$ is weakly regular, $\mathrm{cl}_X \mathcal{A}$ is micromeric. Hence, for some $G \in \mathcal{G}$ and some $A \in \mathcal{A}$ we have $\mathrm{cl}_X A \subset G$. But then $\mathrm{cl}_X A \in \mathcal{H}$ and we have a contradiction.

(2) $\Longrightarrow$ (3): Let $\mathcal{A}$ be a far collection in $X$. Then $\mathcal{G} = \{X \setminus A \mid A \in \mathcal{A}\}$ is a uniform cover of $X$ and so, by (2), $\mathcal{G}$ is refined by some uniform cover $\mathcal{H}$ whose members are closed. Then the far collection $\mathcal{B} = \{X \setminus H \mid H \in \mathcal{H}\}$ has open sets as members and $\mathcal{B}$ corefines $\mathcal{A}$.

(3) $\Longrightarrow$ (1): Let $\mathcal{A}$ be micromeric and suppose that $\mathrm{cl}_X \mathcal{A}$ is not. Then $\sec \mathrm{cl}_X \mathcal{A}$ is far and so, by hypothesis, it is corefined by some far collection $\mathcal{B}$ of open sets. Then $\sec \mathcal{B}$ is not micromeric and therefore $\mathcal{A} \not\subset \sec \mathcal{B}$. There exists $A \in \mathcal{A}$ with $A \notin \sec \mathcal{B}$. It follows that there exists $B \in \mathcal{B}$ with $B \cap A = \emptyset$. Since $\mathcal{B}$ corefines $\sec \mathrm{cl}_X \mathcal{A}$, we have $B \in \sec \mathrm{cl}_X \mathcal{A}$. Therefore, $B \cap \mathrm{cl}_X A \neq \emptyset$. Since $B$ is open, we have a contradiction. $\square$

**Proposition 3.**

(1) *Every regular space is weakly regular.*
(2) *A topological space is regular iff it is weakly regular.*

Proof: (1): Observe first that in any nearness space $X$ we have

$$B < A \qquad \Longleftrightarrow \qquad \mathrm{cl}_X A < \mathrm{int}_X B.$$

[5] $\mathcal{A}$ **corefines** $\mathcal{B}$ iff each member of $\mathcal{A}$ contains a member of $\mathcal{B}$.

Let $X$ be a regular space. Then $X$ is a nearness space and we may establish the statement in Proposition 2 (2) to show that $X$ is weakly regular. Let $\mathcal{A}$ be a uniform cover of $X$ and let

$$\mathcal{B} = \{B \subset X \mid B < A \ \text{ for some } \ A \in \mathcal{A}\}.$$

Then $\mathcal{B}$ refines $\mathrm{cl}_X \mathcal{B}$ and $\mathrm{cl}_X \mathcal{B}$ refines $\mathcal{A}$. By regularity, $\mathcal{B}$ is a uniform cover of $X$ and the proof of (1) is complete.

(2): Observe first that in any topological space $X$ we have

$$B < A \qquad \Longleftrightarrow \qquad \mathrm{cl}_X A \subset \mathrm{int}_X B.$$

Let $X$ be a weakly regular topological space. Let $\mathcal{A}$ be a uniform cover of $X$. We must show that

$$\mathcal{B} = \{B \subset X \mid B < A \ \text{ for some } \ A \in \mathcal{A}\}$$

is a uniform cover of $X$. $X$, being a topological space, is a nearness space. Hence $\mathrm{int}_X \mathcal{A}$ is a uniform cover of $X$. By the statement in Proposition 2 (2), there exists a uniform cover $\mathcal{H}$ of $X$ which refines $\mathrm{int}_X \mathcal{A}$ with the members of $\mathcal{H}$ being closed sets. Then $\mathcal{H} \subset \mathcal{B}$ and it follows that $\mathcal{B}$ is a uniform cover of $X$. $\square$

## Filter merotopic spaces

Filter merotopic spaces were defined by Katětov [K65] and were extensively studied in [R75] and [BHR76]. Here we shorten "filter merotopic space" to "filter space".

A space $X$ is said to be a **filter space** provided every micromeric collection $\mathcal{M}$ is corefined by some Cauchy filter $\mathcal{F}$.

Every topological space is a filter space (even every subtopological[6] space is).

We denote the category of all filter spaces by $\mathsf{Fil}$. $\mathsf{Fil}$ is bicoreflective in $\mathsf{Mer}$, and $\mathsf{Fil}$ is cartesian closed. For a description of its function space structure see [K65] or [BHR76]. Katětov proved that the function space structure of $\mathsf{Fil}$ is the one of continuous convergence. We let $\mathrm{id} : \mathsf{F}X \to X$ denote the $\mathsf{Fil}$ coreflection of a space $X$.

Several interesting subcategories of $\mathsf{Fil}$ were investigated in [BHL86]. Two of the most useful of these are the categories $\mathsf{Conv}_{\mathsf{S}}$ (of symmetric convergence spaces) and $\mathsf{Chy}$ (of Cauchy spaces).

A **Cauchy space** is a filter space $X$ which satisfies: If $\mathcal{A}$ and $\mathcal{B}$ are micromeric and if $\emptyset \notin \mathcal{A} \wedge \mathcal{B}$ then $\mathcal{A} \vee \mathcal{B}$ is micromeric.

The category of all Cauchy spaces is denoted by $\mathsf{Chy}$; it is bicoreflective in $\mathsf{Fil}$. Cauchy spaces are precisely what Katětov called "Hausdorff filter merotopic spaces". Cauchy spaces are usually defined using Cauchy filters only [KR74] instead of the more general micromeric collections. Nevertheless, there is no essential difference between these two approaches since isomorphic categories result [BHL86].

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[6] A nearness space is said to be a **subtopological space** provided it is a subspace of some topological space.

A filter space $X$ is called a $\mathsf{C}$ space provided that whenever $\mathcal{A}$ and $\mathcal{B}$ are micromeric in $X$ and $x \in X$ then we have

$$\mathcal{A} \to x \text{ and } \mathcal{B} \to x \quad \Longrightarrow \quad \mathcal{A} \vee \mathcal{B} \text{ is micromeric.}$$

A **convergence space** is a $\mathsf{C}$ space which satisfies the condition: For every micromeric collection $\mathcal{A}$ in $X$ there exists $x \in X$ such that $\mathcal{A} \to x$. The subcategory of $\mathsf{Mer}$ whose objects are the convergence spaces is denoted by $\mathsf{Conv}_{\mathsf{S}}$.
We mentioned above that Cauchy spaces are defined usually in terms of axioms about filters. The same thing is true about convergence spaces. If $\mathsf{Conv}$ denotes the category of all convergence spaces in the sense of Fisher [F59], then $\mathsf{Conv}_{\mathsf{S}}$ is isomorphic to that full subcategory of $\mathsf{Conv}$ whose objects are those convergence spaces satisfying the following symmetry axiom:

$$\dot{x} \to y \quad \Longrightarrow \quad x \text{ and } y \text{ have the same convergent filters.}$$

Between convergence spaces, a map $f : X \to Y$ is said to be **continuous at** $x$ iff a filter $\mathcal{F}$ converges to $x$ in $X$ implies the filter[7] $\mathcal{G} = \operatorname{stack}\{f[F] \mid F \in \mathcal{F}\}$ converges to $f(x)$ in $Y$. It then follows that $f : X \to Y$ is uniformly continuous iff it is continuous at $x$ for every $x \in X$. Because of this fact, we often omit the word "uniformly" in the phrase "uniformly continuous" when we are dealing with convergence spaces.
In the counterexamples which appear near the end of this paper, we have found it useful to define some spaces directly in terms of "neighborhoods" of points. Such spaces always turn out to be convergence spaces, in fact, even more special than convergence spaces: They are always pretopological spaces [C48], [BHL86], [LC89].
A **pretopological space** is a set $X$ endowed with a "neighborhood filter system" $\mathcal{B}$, i.e., to each $x \in X$ is associated a filter $\mathcal{B}(x)$ such that the following two axioms are satisfied:
(N1) $x \in \cap\mathcal{B}(x)$ for each $x \in X$.
(N2) $\mathcal{B}(x)$ is a filter on $X$ for each $x \in X$.
A map $f : (X_1, \mathcal{B}_1) \to (X_2, \mathcal{B}_2)$ between pretopological spaces is said to be **continuous at** $x$ provided

$$\mathcal{B}_2\big(f(x)\big) \subset \operatorname{stack}\{f[A] \mid A \in \mathcal{B}_1(x)\}.$$

We then say that $f$ is **continuous** iff $f$ is continuous at $x$ for each $x \in X$.
Each pretopological space $(X, \mathcal{B})$ becomes a convergence space if, for a filter $\mathcal{F}$ on $X$ and for $x \in X$, we define $\mathcal{F} \to x$ iff $\mathcal{B}(x) \subset \mathcal{F}$. The resulting convergence spaces are precisely those which satisfy the convergence axiom:

$$\cap\{\mathcal{F} \mid \mathcal{F} \to x\} \to x.$$

(The above informal description is actually a concrete isomorphism of categories.)
One final remark about pretopological spaces: They are also isomorphic to the category of closure spaces (in the sense of Čech, i.e, without idempotency). For a proof of this isomorphism, see Section III.14.B of [Č66].

[7] stack $\mathcal{A}$ for a collection $\mathcal{A}$ denotes the collection of all supersets of members of $\mathcal{A}$.

## Completely regular spaces

Completely regular nearness spaces have been defined in a way that enables one to determine internally all completely regular extensions of topological spaces. The following definitions and results are taken from [BHO89].

The definition of complete regularity involves a slight modification of the definition of regularity.

A space $X$ is said to be **completely regular** if for every uniform cover $\mathcal{A}$ of $X$, the collection

$$\{B \subset X \mid B \text{ is completely within } A \text{ for some } A \in \mathcal{A}\}$$

is a uniform cover of $X$. That $B$ is **completely within** $A$ means that there exists a uniformly continuous map $f : X \to [0,1]$ with $f[B] \subset \{0\}$ and $f[X \setminus A] \subset \{1\}$. Here, $[0,1]$ is understood to carry its usual topological structure: the set of all covers refined by some open cover.

Every completely regular space is regular, and every uniform space is completely regular. Not every regular space is completely regular: An example is any regular topological space which is not completely regular. Not every completely regular space is uniform: An example is any completely regular topological space which is not paracompact.

The concept of complete regularity can be formulated in terms of the micromeric collections: A space $X$ is completely regular if and only if it satisfies the following condition: Whenever $\mathcal{A}$ is a micromeric collection in $X$, then so is the collection

$$\{B \subset X \mid A \text{ is completely within } B \text{ for some } A \in \mathcal{A}\}.$$

The underlying topological space $\mathsf{T}X$ of a completely regular space $X$ is also completely regular (in the usual topological sense). A topological space is completely regular (as a merotopic space) if and only if it is completely regular as a topological space, in the usual sense.

A useful characterization of those subtopological spaces which are completely regular is that they are precisely the subspaces (in $\mathsf{Mer}$) of the completely regular topological spaces [BHO89].

$\mathsf{Creg}$, the full subcategory of $\mathsf{Near}$ whose objects are the completely regular spaces, is bireflective in $\mathsf{Near}$.

## Zero spaces

A classical result is that a topological space $X$ is completely regular iff the set of zero sets of all real-valued continuous functions forms a base for the closed sets of $X$. In [BHO89] the analogous property for merotopic spaces was defined, arriving at a category called $\mathsf{Zero}$.

The real line[8] as a topological space with the usual topology will be denoted by $\mathbb{R}_t$. Recall that $\mathrm{Hom}\,(X, \mathbb{R}_t)$ denotes, for any space $X$, the set of all uniformly

[8]The space $\mathbb{R}_t$, as well as $\mathbb{R}$ with various other related nearness structures, was studied in [BH78a.]

continuous maps $f : X \to \mathbb{R}_t$. The set of all bounded members of $\mathrm{Hom}\,(X, \mathbb{R}_t)$ is a $\phi$-algebra (in the sense of [HJ61]). For any space $X$, we have

$$\{\mathrm{Z}(f) \mid f \in \mathrm{Hom}\,(X, \mathbb{R}_t)\} = \{\mathrm{Z}(f) \mid f \in \mathrm{Hom}\,(X, \mathbb{R}_t) \text{ and } f \text{ is bounded }\},$$

where we are using the usual notation:

$$\mathrm{Z}(f) = \{x \in X \mid f(x) = 0\}$$
$$\mathrm{coz}\,(f) = \{x \in X \mid f(x) \neq 0\}.$$

We shall use the customary notation:

$$\mathrm{Z}(X) = \{\mathrm{Z}(f) \mid f \in \mathrm{Hom}\,(X, \mathbb{R}_t)\}$$
$$\mathrm{coz}\,(X) = \{\mathrm{coz}\,(f) \mid f \in \mathrm{Hom}\,(X, \mathbb{R}_t)\}.$$

Members of $\mathrm{Z}(X)$ are called **zero sets** and members of $\mathrm{coz}\,(X)$ are called **cozero sets**.

A space $X$ is said to be a **zero space** provided that every far collection in $X$ is corefined by a far collection consisting of zero sets.

The concept of being a zero space can be formulated either in terms of uniform covers or in terms of micromeric collections.

**Proposition 4.** *For any space $X$, the following are equivalent:*

(1) *$X$ is a zero space.*
(2) *Every uniform cover of $X$ is refined by a uniform cover consisting of cozero sets of $X$.*
(3) *Every micromeric collection in $X$ is corefined by a micromeric collection consisting of cozero sets.*

Proof: The proof is analogous to the proof of Proposition 2. Use zero (cozero) sets where in the proof of Proposition 2 we used closed (open) sets. □

Every zero space is necessarily a nearness space and every completely regular space is a zero space. Not every zero space is completely regular. In fact, a zero space need not even be weakly regular (see Example 5). For topological spaces, however, recall that being a zero space is equivalent to being completely regular.

$\mathsf{Zero}$, the full subcategory of $\mathsf{Near}$ whose objects are the zero spaces, is bireflective in $\mathsf{Near}$. Additional information about zero spaces can be found in [BHO89].

If $X$ is any space, then we have defined above a closure operator $\mathrm{cl}_X$ on $X$. At this time we are interested in a different closure operator on $X$ which we shall denote by $\mathrm{cl}_{\mathbb{R}}$ (in spite of the fact that this notation seems contradictory). $\mathrm{cl}_{\mathbb{R}}$ denotes the closure operator on $X$ which corresponds to the initial topology on $X$ induced by the source

$$\big(f : X \to \mathbb{R}_t\big)_{f \in \mathrm{Hom}\,(X, \mathbb{R}_t)}.$$

**Proposition 5.** *Let $X$ be a set and let $\mathsf{F}$ be a sublattice of the lattice of all maps $X \to \mathbb{R}$ with pointwise operations. Assume that the condition:*

$$f \in \mathsf{F} \ \text{ and } \ g : \mathbb{R}_t \to \mathbb{R}_t \ \text{ continuous } \quad \Longrightarrow \quad g \circ f \in \mathsf{F}$$

*is satisfied. Then*

$$\{\mathsf{Z}(f) \mid f \in \mathsf{F}\}$$

*is a base for closed sets of the initial topology on $X$ induced by the source*

$$\big(f : X \to \mathbb{R}_t\big)_{f \in \mathsf{F}}\,.$$

Proof: The usual argument used in such theorems can be adapted. See, e.g., the proof of Proposition 1.5.8 of [E89], but write "min" where Engelking writes "max" since we have interchanged 0 and 1. □

**Corollary 6.** *If $X$ is a merotopic space then $\mathrm{Z}(X)$ is a base for closed sets for the topology corresponding to the closure operator* $\mathrm{cl}_{\mathbb{R}}$.

We remark that it follows from the above result that $\mathrm{cl}_X A \subset \mathrm{cl}_{\mathbb{R}} A$ for every subset $A$ of $X$. Indeed, the uniform continuity of a map $f : X \to \mathbb{R}_t$ implies the uniform continuity (=continuity since $\mathsf{T}X$ is topological) of $f : \mathsf{T}X \to \mathbb{R}_t$, and consequently $Z(f)$ is closed in $\mathsf{T}X$. Hence, every set which is closed with respect to the topology induced by the operator $\mathrm{cl}_{\mathbb{R}}$ is also closed in $\mathsf{T}X$. Therefore, $\mathrm{cl}_X A \subset \mathrm{cl}_{\mathbb{R}} A$.

**Proposition 7.** *If $X$ is a zero space, then $\mathrm{cl}_X = \mathrm{cl}_{\mathbb{R}}$, i.e., the usual Čech closure operator of $X$ as a merotopic space is actually a topological closure and it is the one corresponding to the initial topology on $X$ induced by the source*

$$\big(f : X \to \mathbb{R}_t\big)_{f \in \mathrm{Hom}\,(X, \mathbb{R}_t)}.$$

Proof: We have already mentioned above that for any subset $A$ of $X$, we have $\mathrm{cl}_X A \subset \mathrm{cl}_{\mathbb{R}} A$. To show the reverse inclusion, let $x \notin \mathrm{cl}_X A$. Then $\{\, A, \{x\}\,\}$ is far in $X$ so for some $\mathcal{B} \subset Z(X)$, $\mathcal{B}$ is far in $X$ and $\mathcal{B}$ corefines $\{\, A, \{x\}\,\}$. For some $B \in \mathcal{B}$, we have $x \notin B$. Therefore, $A \subset B$. For some uniformly continuous map $f : X \to \mathbb{R}_t$ we have $B = Z(f)$. $B$ is closed with respect to the topology corresponding to the operator $\mathrm{cl}_{\mathbb{R}}$ and therefore $\mathrm{cl}_{\mathbb{R}} A \subset B$. Hence, $x \notin \mathrm{cl}_{\mathbb{R}} A$ and the proof is complete. □

**Corollary 8.** *If $X$ is a zero space then its underlying topology $\mathsf{T}X$ is completely regular.*

## $Z$-regular spaces

In this section, we consider a property which is not directly related to complete regularity, but is a slight variation on the definition of zero spaces.

**Definition 9.** A space $X$ is said to be a **$z$-regular space** provided it satisfies the condition: Every far collection in $X$ is corefined by a far collection consisting of cozero sets.

**Proposition 10.** *For any space $X$, the following are equivalent:*

(1) *$X$ is $z$-regular.*
(2) *Every uniform cover of $X$ is refined by a uniform cover consisting of zero sets.*
(3) *Every micromeric collection in $X$ is corefined by a micromeric collection consisting of zero sets.*

PROOF: The proof is analogous to the proofs of Propositions 1 and 4. □

**Proposition 11.**

(1) *Every completely regular space is $z$-regular.*
(2) *Every $z$-regular space is weakly regular.*

PROOF: (1): The proof is trivial from the definitions.
(2): See Proposition 16 below. □

The proofs of Propositions 12 and 13 below are straightforward textbook exercises.

**Proposition 12.** *$z$-regularity is productive, hereditary, and summable in* Mer.

**Proposition 13.** *The category of $z$-regular spaces is bireflective in* Mer. *If $X$ is any merotopic space, then its $z$-regular reflection is given by* id $: X \to \mathsf{Z}X$, *the identity map on the underlying sets, where we define:*

$$\mathcal{A} \text{ is a uniform cover of } \mathsf{Z}X$$

*iff*

$$\mathcal{A} \text{ is refined by some uniform cover } \mathcal{B} \text{ of } X \text{ such that } \mathcal{B} \subset Z(X).$$

*Furthermore,* $\mathrm{Hom}\,(X, \mathbb{R}_t) = \mathrm{Hom}\,(\mathsf{Z}X, \mathbb{R}_t)$. *(Note that the notation is tricky: $Z(X)$ denotes the collection of all zero subsets of $X$ while $\mathsf{Z}X$ denotes the $z$-regular reflection of $X$.)*

We remark that a $z$-regular space may fail to be a nearness space (Example 2) and a $z$-regular nearness space may fail to be regular (Example 3). Being a topological space doesn't help much: a $z$-regular topological space may fail to be completely regular (Example 8)[9]. Also, a zero space may fail to be $z$-regular (Example 4), and regularity doesn't even help (J. Reiterman and J. Pelant have produced an example of a regular zero space which is not $z$-regular [Private communication - unpublished]).

## $\mu$-REGULAR SPACES

We are interested in a notion of complete regularity defined for filter spaces by Katětov [K65]. In order to avoid confusion of terminology, we use the phrase "$\mu$-regular space" in place of Katětov's "completely regular filter space".

Recall the closure operator $\mathrm{cl}_{\mathbb{R}}$ introduced in the above section on zero spaces.

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[9]In this connection, note that Exercise 14.C.2 of [W70] is a misprint.

**Definition 14.** We say that a space $X$ is $\mu$**-regular** iff the condition:

$$\mathcal{A} \text{ is micromeric in } X \quad \Longrightarrow \quad \mathrm{cl}_{\mathbb{R}}\, \mathcal{A} \text{ is micromeric in } X.$$

**Proposition 15.** *For any space $X$, the following are equivalent:*

(1) *$X$ is $\mu$-regular.*
(2) *Every micromeric collection in $X$ is corefined by a micromeric collection consisting of sets which are closed with respect to the topology corresponding to the closure operator $\mathrm{cl}_{\mathbb{R}}$.*
(3) *Every uniform cover of $X$ is refined by a uniform cover consisting of sets which are closed with respect to the topology corresponding to the closure operator $\mathrm{cl}_{\mathbb{R}}$.*
(4) *Every far collection in $X$ is corefined by a far collection consisting of sets which are open with respect to the topology corresponding to the closure operator $\mathrm{cl}_{\mathbb{R}}$.*

Proof: The proof is analogous to the proofs of Propositions 1, 4, and 10. □

**Proposition 16.**

(1) *Every $z$-regular space is $\mu$-regular.*
(2) *Every $\mu$-regular space is weakly regular.*

Proof: (1): Let $X$ be a $z$-regular space and let $\mathcal{A}$ be micromeric in $X$. By Proposition 10, (1) $\Longrightarrow$ (3), $\mathcal{A}$ is corefined by some micromeric collection $\mathcal{B}$ of zero sets. It is enough to show that $\mathcal{B}$ corefines $\mathrm{cl}_{\mathbb{R}}\, \mathcal{A}$. Every $B \in \mathcal{B}$ is the zero set of some uniformly continuous map $f : X \to \mathbb{R}_t$. Such an $f$ is continuous with respect to the topology corresponding to $\mathrm{cl}_{\mathbb{R}}$. Therefore, $B$ is closed in that topology and the desired result follows.
(2): The result is immediate from the fact (observed after Corollary 6 above) that $\mathrm{cl}_X\, A \subset \mathrm{cl}_{\mathbb{R}}\, A$ for every subset $A$ of $X$. □

**Proposition 17.** *If $X$ is a zero space, then*

$$X \text{ is } \mu\text{-regular} \quad \Longleftrightarrow \quad X \text{ is weakly regular.}$$

Proof: This result follows immediately from the relation $\mathrm{cl}_{\mathbb{R}} = \mathrm{cl}_X$ given in Proposition 7. □

The proofs of the following three propositions are straightforward exercises.

**Proposition 18.** *The category of $\mu$-regular spaces is bireflective in* $\mathsf{Mer}$. *If $X$ is any merotopic space, then its $\mu$-regular reflection is given by* $\mathrm{id} : X \to \mathsf{M}X$, *the identity map on the underlying sets, where we define:*

$$\mathcal{A} \text{ is micromeric in } \mathsf{M}X$$

*iff*

*for some micromeric collection $\mathcal{B}$ in $X$ we have* $\mathrm{cl}_{\mathbb{R}}\,\mathcal{B}$ *corefines* $\mathcal{A}$.

*Furthermore,* $\mathrm{Hom}\,(X, \mathbb{R}_t) = \mathrm{Hom}\,(\mathsf{M}X, \mathbb{R}_t)$.

**Proposition 19.** *The category of all $\mu$-regular filter spaces is bireflective in* Fil *with*id $: X \to \mathsf{FM}X$ *being the $\mu$-regular filter reflection of a filter space $X$.*

**Proposition 20.** *The* Fil *coreflection of a $\mu$-regular space is also $\mu$-regular.*

## Examples

**Example 1.** A contigual[10] zero space $X$ which is not weakly regular and is not a Cauchy space. This example is due to J. Reiterman [Private communication - unpublished]; it is the first known example of this type.

The underlying set of $X$ is the open segment $]0,1[$. The structure of $X$ is obtained by refining the metric uniformity of the open segment $]0,1[$ by adding a single cover

$$\mathcal{G} = \{\, G_1, G_2 \,\}$$

where

$$G_1 = \,]0,1[\, \setminus \left\{ \frac{1}{2n} \mid n \in \mathbb{N} \right\}$$
$$G_2 = \,]0,1[\, \setminus \left\{ \frac{1}{2n+1} \mid n \in \mathbb{N} \right\}.$$

Thus, basic uniform covers are of the form $\mathcal{C}_\epsilon \wedge \mathcal{G}$ where $\mathcal{C}_\epsilon$ is a finite cover of $]0,1[$ by open segments of length $\epsilon$. Members of these covers are cozero with respect to the metric uniformity; thus they are cozero with respect to the space $X$ for the latter has a structure which is (strictly) finer. It follows that $X$ is a zero space. Since the basic uniform covers of $X$ are open in the usual topology of the open segment $]0,1[$, the topology $\mathsf{T}X$ of $X$ is the same as the usual topology of the open segment. Further, for every open set $G$, we have

$$\mathrm{cl}\; G = \mathrm{cl}\,(G \cap G_i) \qquad\qquad (i = 1,2).$$

Hence $\{\, \mathrm{cl}\; G \mid g \in \mathcal{C}_\epsilon \wedge \mathcal{G} \,\}$ is refined by $\mathcal{C}_\epsilon$. It follows that $\mathcal{G}$ cannot be refined by any closed uniform cover of $X$; therefore, $X$ is not weakly regular.

Clearly, $X$ is contigual. The fact that every contigual space is subtopological (and hence is a filter space) follows from [H74a; Proposition 5.11]; for an short proof of this fact see [BH82; Proposition 2.4]. In the present example, $X$ is a subspace of the topological space $]0,1[\, \cup \{\xi_1, \xi_2\,\}$ where a neighborhood base of $\xi_1$ consists of the sets

$$\{\xi_1\} \,\cup\, \left( ]0,\epsilon[\, \setminus \left\{ \frac{1}{2n} \mid n \in \mathbb{N} \right\} \right) \qquad\qquad \epsilon > 0,$$

[10] A space is said to be **contigual** iff every uniform cover is refined by some finite uniform cover.

and where a neighborhood base of $\xi_2$ consists of the sets

$$\{\xi_2\} \cup \left( ]0,\epsilon[ \setminus \left\{ \frac{1}{2n+1} \mid n \in \mathbb{N} \right\} \right) \qquad \epsilon > 0.$$

To show that $X$ is not a Cauchy space, let

$$\mathcal{F} = \text{stack}\,\{F_\epsilon \mid \epsilon > 0\}$$
$$\mathcal{G} = \text{stack}\,\{G_\epsilon \mid \epsilon > 0\}$$

where

$$F_\epsilon = ]0,\epsilon[ \setminus \{\frac{1}{2n} \mid n \in \mathbb{N}\}$$
$$G_\epsilon = ]0,\epsilon[ \setminus \{\frac{1}{2n+1} \mid n \in \mathbb{N}\}.$$

Then $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters on $X$ with $\emptyset \notin \mathcal{F} \wedge \mathcal{G}$ but with $\mathcal{F} \vee \mathcal{G}$ not a Cauchy filter.

**Example 2.** Example 2 is the pretopological space $X$ described in Example 3 below. It is a separated filter space which is a $z$-regular space but not a nearness space.

**Example 3.** A separated, Cauchy, zero space $E$ which is $z$-regular but not regular. On the real line let $\alpha$ be an irrational number and let $(\alpha_j)_{j\in\mathbb{N}}$ be a strictly monotone decreasing sequence of irrationals converging to $\alpha$. For $j > 1$ choose a sequence $(q_n^j)_{n\in\mathbb{N}}$ of rationals in the interval $]\alpha_j, \alpha_{j-1}[$ converging to $\alpha_j$. For $j = 1$ choose a sequence $(q_n^1)_{n\in\mathbb{N}}$ of rationals in the interval $]\alpha_1, \infty[$ converging to $\alpha_1$. Let

$$X = \big([\alpha,\infty[ \,\cap\, \mathbf{Q}\big) \cup \{\alpha\} \cup \{\alpha_j \mid j \in \mathbb{N}\}\,.$$

(Here $\mathbf{Q}$ denotes the set of all rational numbers.) A pretopological structure $p$ is defined on $X$ by means of the following neighborhood filters $\mathcal{B}(x)$ for $x \in X$:

$$\begin{aligned}
\mathcal{B}(q) &= \dot{q} \qquad \text{for } q \in X \cap \mathbf{Q} \\
\mathcal{B}(\alpha) &= \text{stack}\,\{F_n \cup \{\alpha\} \cup \{\,\alpha_j \,|\, j \geq n\,\}, \,|\, n \in \mathbb{N}\} \\
&\qquad \text{where} \quad F_n = \bigcup_{j=n}^{\infty} \{q_k^j \mid k \in \mathbb{N}\} \\
\mathcal{B}(\alpha_j) &= \text{stack}\,\{]\alpha_j - \frac{1}{n}, \alpha_j + \frac{1}{n}[ \,\cap\, X \mid n \in \mathbb{N}\}.
\end{aligned}$$

Clearly $(X,p)$ is Hausdorff.
We define a space $(E,\gamma)$ by $E = [\alpha,\infty[ \,\cap\, \mathbf{Q}$ and $\gamma$ is the merotopic subspace structure of $(X,p)$. By the results in the paper [BHL86], it follows that $(E,\gamma)$ is

a separated Cauchy space. Since the minimal Cauchy filters have an open base, $(E,\gamma)$ is a nearness space.
For each $x \in X$ we let $\mathcal{M}(x)$ be the trace of $\mathcal{B}(x)$ on $E$. For every $n$ and every $m \geq n$ there exists $j$ such that

$$E \setminus F_m \notin \mathcal{M}(\alpha_j) \qquad \text{and} \qquad F_n \notin \mathcal{M}(\alpha_j).$$

It follows that

$$\mathcal{M}(\alpha) \neq \{B \subset E \mid A < B \ \text{ for some } \ A \in \mathcal{M}(\alpha)\}\,.$$

Hence, $(E,\gamma)$ is not regular.
Next we show that $(E,\gamma)$ is $z$-regular, and therefore also $\mu$-regular and weakly regular. Let $\tau$ be the trace on $X$ of the discrete rational extension topology of $\mathbb{R}$ (every rational point is open). $(X,\tau)$ is metrizable and $\tau$ is a coarser structure than $p$. Therefore $\tau$ closed sets are $\tau$ zero sets and hence also $p$ zero sets. So traces of $\tau$ closed sets are $(E,\gamma)$ zero sets. Hence, each $\mathcal{M}(x)$ for $x \in E$ has a base consisting of zero sets, and it follows that $(E,\gamma)$ is $z$-regular.
Finally, we show that $(E,\gamma)$ is a zero space. Since the neighborhood filters in $(X,\tau)$ of the rational points as well as every $\alpha_j$ have a base of $\tau$ cozero sets, then the neighborhood filters in $(X,p)$ of each of those types of points have a base of $p$ cozero sets, and it follows that the traces of the neighborhood filters in $(E,\gamma)$ of each of those types of points have a base of $\gamma$ cozero sets. So we need only show that $\mathcal{M}(\alpha)$ has a $\gamma$ cozero set base. Let $n \in \mathbb{N}$ and consider the corresponding $F_n \in \mathcal{M}(\alpha)$. A continuous map $\hat{f}_n : (X,p) \to \mathbb{R}_t$ is defined as follows:

$$\begin{aligned}
\hat{f}_n(q_j^n) &= \frac{1}{j} && \text{if } \ j \geq 1\\
\hat{f}_n(q_j^{n+k}) &= \frac{1}{j+k} && \text{if } \ j \geq 1 \ \text{ and } \ k \in \mathbb{N}\\
\hat{f}_n(x) &= 0 && \text{elsewhere.}
\end{aligned}$$

The restriction $f_n$ of $\hat{f}_n$ is uniformly continuous $(E,\gamma) \to \mathbb{R}_t$ and $F_n = f_n^{-1}]0,\infty[$. Therefore, $F_n$ is a cozero set.

**Example 4.** A separated, Cauchy, zero space $X$ which is $\mu$-regular but not regular and not $z$-regular.

We let $\omega$ denote the first infinite ordinal, $\Omega$ the first uncountable ordinal, $\omega\omega$ the ordinal product, and $Z = [0,\omega\omega] \times [0,\Omega]$ the set theoretic product of the sets $[0,\omega\omega]$ and $[0,\Omega]$. On $Z$ we place a pretopological structure by defining filterbases for the neighborhood filters as follows:

$$\mathcal{S}(\alpha,\beta) = \big\{\{x\}\big\} \ \text{ if } x = (\alpha,\beta), \text{ and either } \beta < \Omega \text{ or } \alpha < \omega\omega \text{ is a successor ordinal}$$

$$\mathcal{S}(\omega n,\Omega) = \{]\delta,\omega n] \times ]\gamma,\Omega] \mid \delta < \omega n \ \text{ and } \ \gamma < \Omega\} \quad \text{ for } \ n < \omega$$

$$\mathcal{S}(\omega\omega,\Omega) = \{]\delta,\omega\omega] \times \{\Omega\} \mid \delta < \omega\omega\}\,.$$

That is to say, the neighborhood filters are $\mathcal{B}(x) = \operatorname{stack}\mathcal{S}(x)$ for each $x \in Z$. We define $X$ as the merotopic subspace of $Z$ where

$$X = \{(\alpha,\beta) \in Z \mid \beta < \Omega \ \text{ or } \ \alpha \ \text{ is a successor ordinal }\}.$$

$Z$ is a strict completion of $X$ in the sense of [L89], and so it follows that a map $X \to \mathbb{R}_t$ is uniformly continuous iff it has a continuous extension $Z \to \mathbb{R}_t$.
In order to show that $X$ is not $z$-regular, consider the micromeric collection

$$\mathcal{G} = \big\{ \big( ]\delta,\omega\omega] \times \{\Omega\} \big) \cap X \mid \delta < \omega\omega \big\} ,$$

which is the trace on $X$ of the neighborhood filterbase $\mathcal{S}(\omega\omega,\Omega)$. Assuming $X$ is $z$-regular, $\mathcal{G}$ is corefined by a micromeric collection $\mathcal{P}$ whose elements are zero sets of $X$. Since $\mathcal{P}$ is micromeric in $X$, it is also micromeric in $Z$. Therefore, there exists a neighborhood filter of some point of $Z$ which corefines $\mathcal{P}$. It is not difficult to see that such a point must in fact be $(\omega\omega,\Omega)$. Therefore, we have that $\mathcal{B}(\omega\omega,\Omega)$ corefines $\mathcal{P}$, which in turn corefines $\mathcal{G}$. Let $B = ]0,\omega\omega] \times \{\Omega\}$. Then $B \in \mathcal{B}(\omega\omega,\Omega)$ and so there exist $P \in \mathcal{P}$ and $G \in \mathcal{G}$ with $G \subset P \subset B$. Since $P$ is a zero set of $X$, there exists a uniformly continuous map $f : X \to \mathbb{R}_t$ such that $f \geq 0$ and $f^{-1}[\{0\}] = P$. Let $\hat{f} : Z \to \mathbb{R}_t$ be the continuous extension of $f$. By definition of $\mathcal{G}$, there exists $\delta < \omega\omega$ such that

$$G = \big( ]\delta,\omega\omega] \times \{\Omega\} \big) \cap X.$$

There exists $k < \omega$ with $\delta < \omega k$. The sequence

$$\big(\omega k + 1, \Omega\big), \big(\omega k + 2, \Omega\big), \cdots$$

converges to $\big(\omega(k+1),\Omega\big)$ in $Z$. Since $\hat{f}(\alpha,\Omega) = 0$ whenever $\alpha$ is a successor ordinal between $\omega k$ and $\omega(k+1)$, it follows that $\hat{f}\big(\omega(k+1),\Omega\big) = 0$. The continuity of $\hat{f} : Z \to \mathbb{R}_t$ implies that

$$\big\{[-\frac{1}{m},\frac{1}{m}] \mid m < \omega\big\} \qquad \text{corefines} \qquad \big\{\hat{f}[H] \mid H \in \mathcal{B}\big(\omega(k+1),\Omega\big)\big\}.$$

Therefore we have

$$\forall\, m < \omega \quad \exists\, \delta_m < \omega(k+1) \quad \exists\, \gamma_m < \Omega \qquad \hat{f}\big( \, ]\delta_m, \omega(k+1)] \times \, ]\gamma_m,\Omega]\big) \subset [-\frac{1}{m},\frac{1}{m}].$$

Put $\gamma = \sup_n \gamma_n$. Then $\gamma < \Omega$. It follows that

$$\forall\, m < \omega \qquad \{\omega(k+1)\} \times \, ]\gamma,\Omega] \subset ]\delta_m,\omega(k+1)] \times \, ]\gamma_m,\Omega].$$

Therefore,

$$\forall\, m < \omega \qquad \hat{f}\big(\{\omega(k+1)\} \times \, ]\gamma,\Omega]\big) \subset [-\frac{1}{m},\frac{1}{m}].$$

So finally we can conclude that

$$\hat{f}\big(\{\omega(k+1)\} \times ]\gamma, \Omega]\big) = \{0\}.$$

But

$$\big(\{\omega(k+1)\} \times ]\gamma, \Omega]\big) \cap X \subset P \subset B = ]0, \omega\omega] \times \{\Omega\}.$$

Since $]\gamma, \Omega]$ is not a subset of $\{\Omega\}$, we have a contradiction.
To prove that $X$ is a zero space, it suffices to show that the trace on $X$ of each neighborhood filter of $Z$ has a base of cozero sets. Each of the filters

$$\begin{aligned} &\mathcal{B}(\alpha, \beta), && \beta < \Omega, \\ &\mathcal{B}(\alpha, \beta), && \alpha \text{ a successor ordinal}, \\ &\mathcal{B}(\omega n, \Omega), && n < \omega, \end{aligned}$$

clearly has a base of cozero sets of $Z$: In each case, the sets in the defining filterbase are cozero sets as can be shown by using the characteristic function of each such set (i.e., the function which is 1 on the set and 0 off it). It follows, in each of the above three cases, that the traces on $X$ have bases of cozero sets of $X$.
Note that

$$\big\{]\omega n, \omega\omega] \times \{\Omega\} \mid n < \omega\big\}$$

is a base for $\mathcal{B}(\omega\omega, \Omega)$. Let $n < \omega$ and let

$$M = \big(]\omega n, \omega\omega] \times \{\Omega\}\big) \cap X.$$

To show that $M$ is a cozero set of $X$, we define $h : Z \to \mathbb{R}_t$ such that $h$ is 0 at all points of $Z$ except those of the form $(\alpha, \Omega)$ where $\alpha$ is a successor ordinal with $\omega n < \alpha$. If $n < k < \omega$ and $0 < p < \omega$, we define

$$h\big(\omega k + p, \Omega\big) = \frac{1}{k^p}.$$

Clearly, $M = X \cap \operatorname{coz}(h)$, so the only thing left to check is the continuity of $h : Z \to \mathbb{R}_t$. At each discrete point, $h$ is trivially continuous. At $(\omega m, \Omega)$ with $m \leq n$, $h$ is 0 on every basic neighborhood. Consider a point $(\omega m, \Omega)$ with $n < m$. For $\epsilon > 0$ we choose $p < \omega$ such that

$$\frac{1}{(m-1)^p} < \epsilon$$

and we choose $\gamma < \Omega$ arbitrarily. Then

$$h\big(]\omega(m-1) + p, \omega m] \times ]\gamma, \Omega]\big) \subset [-\epsilon, \epsilon],$$

and it follows that $h$ is continuous at each point $(\omega m, \Omega)$ with $n < m$. Finally, consider the point $(\omega\omega, \Omega)$. For each $\epsilon > 0$ we choose $m$ with $n < m < \omega$ such that $\frac{1}{m} < \epsilon$. Then

$$h\big(]\omega m, \omega\omega] \times \{\Omega\}\big) \subset [-\epsilon, \epsilon],$$

and it follows that $h$ is continuous at $(\omega\omega, \Omega)$. This completes the proof that $X$ is a zero space.

Since $X$ is a zero space, it is a nearness space. Therefore, the underlying closure operator is topological and, by Proposition 7, $\mathrm{cl}_X = \mathrm{cl}_{\mathbb{R}}$. Clearly, the trace on $X$ of each basic neighborhood of points of $Z$ are closed in $X$. It follows that $X$ is a $\mu$-regular space.

It remains to prove only that $X$ is not a regular space. Let

$$\mathcal{M} = \{\big(]\delta, \omega\omega] \times \{\Omega\}\big) \cap X \mid \delta < \omega\omega\},$$

i.e., $\mathcal{M}$ is the trace on $X$ of the basic neighborhoods of $(\omega\omega, \Omega)$. Clearly, it suffices to show that

$$\{S \subset X \mid M < S \ \text{ for some } \ M \in \mathcal{M}\}$$

is not micromeric. For that purpose, it is sufficient to show that for every $k$ with $n < k < \omega$, we have

$$\big(]\omega k, \omega\omega] \times \{\Omega\}\big) \cap X \not< \big(]\omega n, \omega\omega] \times \{\Omega\}\big) \cap X.$$

Select any $l$ with $k < l < \omega$. Clearly $\big(]\omega n, \omega\omega] \times \{\Omega\}\big) \cap X$ does not belong to the trace of $\mathcal{B}(\omega l, \Omega)$ on $X$. On the other hand,

$$X \setminus \Big(\big(]\omega k, \omega\omega] \times \{\Omega\}\big) \cap X\Big)$$

does not belong to the trace of $\mathcal{B}(\omega l, \Omega)$ on $X$ either. It follows that

$$\Big\{ \big(]\omega n, \omega\omega] \times \{\Omega\}\big) \cap X\,,\ X \setminus \Big(\big(]\omega k, \omega\omega] \times \{\Omega\}\big) \cap X\Big) \Big\}$$

is not a uniform cover of $X$, and we are through.

**Example 5.** A separated, Cauchy, zero space $X$ which is not weakly regular.

Let $X = \mathbb{R}$ with the following merotopic structure: $\mathcal{A}$ is micromeric in $X$ iff either $\mathcal{A}$ is corefined by some point's usual neighborhood filter in $\mathbb{R}_t$ or $\mathcal{A}$ is corefined by the filterbase

$$\mathcal{F} = \{F_n \mid n \in \mathbb{N}\}$$

where

$$F_n = \bigcup_{k \geq n} \,]k - \frac{1}{2}, k + \frac{1}{2}[\,.$$

Since every point has an $\mathbb{R}_t$ neighborhood disjoint from some member of $\mathcal{F}$, $X$ is clearly a Cauchy space, and since every equivalence class of Cauchy filters on $X$ has a minimum (either an $\mathbb{R}_t$ neighborhood filter or the filter generated by $\mathcal{F}$) $X$ is separated. We remark that the underlying closure operator of $X$ as a merotopic space and the closure operator in the $\mathsf{Conv_S}$ coreflection of $X$ both coincide with

the closure operator of $\mathbb{R}_t$. Since $\{\mathrm{cl}_X F \mid F \in \mathcal{F}\}$ is not corefined by $\mathcal{F}$, the space $X$ is not weakly regular.
It remains to be shown that $X$ is a zero space. Let $\mathcal{W}(x)$ be the $\mathbb{R}_t$ neighborhood filter of a point $x$. $\mathcal{W}(x)$ has a base consisting of cozero sets of $\mathbb{R}_t$, and, moreover, if $f : \mathbb{R}_t \to [0,1]$ is continuous and coz $f \subset ]x-\epsilon, x+\epsilon[$ for some $\epsilon > 0$, then there exists $n_0$ such that $f$ is 0 on $F_n$ for every $n \geq n_0$. It follows that $\{f[F] \mid F \in \mathcal{F}\}$ generates the filter $\dot{0}$ and hence $f : X \to [0,1]$ is uniformly continuous. Thus, $\mathcal{W}(x)$ has a base consisting of cozero sets of $X$.
The proof will be complete if we can show that every member of $\mathcal{F}$ is a cozero set of $X$. Let $n \in \mathbb{N}$ and let $I_n = [n - \frac{1}{2}, n + \frac{1}{2}]$ with the usual topology, i.e., as a subspace of $\mathbb{R}_t$. The open interval $]n - \frac{1}{2}, n + \frac{1}{2}[$ is a cozero set in $I_n$. Let $f : I_n \to [0,1]$ be continuous such that coz $f = ]n - \frac{1}{2}, n + \frac{1}{2}[$. We extend $f$ to a continuous map $\tilde{f} : \mathbb{R} \to [0,1]$ in the following way:

$$\tilde{f} = \begin{cases} 0, & \text{if } x \leq n - \frac{1}{2} \\ \frac{f(x-l)}{l+1}, & \text{if } x \in I_{n+l}\,. \end{cases}$$

Clearly coz $\tilde{f} = F_n$. In order to show that $\{\tilde{f}[F] \mid F \in \mathcal{F}\}$ converges to 0, let $\epsilon > 0$ and choose $l_0$ such that $\frac{1}{l_0+1} < \epsilon$. Then $\tilde{f}[I_{n+l}] \subset [0,\epsilon]$ for every $l \geq l_0$. Hence $\{\tilde{f}[F] \mid F \in \mathcal{F}\}$ is corefined by the usual neighborhood filter of 0. It follows that $\tilde{f} : X \to [0,1]$ is uniformly continuous and therefore $F_n$ is a cozero set of $X$. This completes the proof that $X$ is a zero space.

**Example 6.** A topological space $Z$ which is regular and $\mu$-regular but not $z$-regular.
This example is based on Example 17 of van Est and Freudenthal [vEF51]. Let $Z = \mathbb{R} \cup \{\omega\}$, where $\omega \notin \mathbb{R}$. Let $\mathbb{P}$ denote the space of irrational numbers with the usual topology. Using Lemma 10 of [vEF51]] we can express $\mathbb{P}$ as a pairwise disjoint union of a family $(Y_n)$ of subspaces of $\mathbb{P}$ where each $Y_n$ is of the second category in every open set of $\mathbb{R}$. We establish the notation: If $p_i$ denotes the $i^{\text{th}}$ prime number, $\rho > 0$, and $\xi \in Y_i$, then we let

$$V_{\varrho p_i}(\xi) = \{\frac{s}{t} \mid s \in \mathbf{Z},\, t = p_i^\alpha \text{ for some } \alpha \in \mathbb{N}, \text{ and } |\xi - \frac{s}{t}| < \frac{1}{t} < \varrho\}\,.$$

We are using $\mathbb{N}$ to denote the set of positive natural numbers, $\mathbf{Z}$ to denote the set of all integers, and $\mathbf{Q}$ to denote the set of all rational numbers. We make $Z$ into a topological space by introducing the following neighborhoods:

(1) If $q \in \mathbf{Q}$, then we take

$$\{A \subset Z \mid q \in A\}$$

as a neighborhood base at $q$, i.e., every rational point is to be a discrete point.

(2) If $\xi \in \mathbb{P}$ with $\xi \in Y_i$, then we take as a neighborhood base at $\xi$ the collection:

$$\{V_{\varrho p_i}(\xi) \cup V_{\varrho p_{i+1}}(\xi) \cup \{\xi\} \mid \varrho > 0\}\,.$$

(3) For each natural number $n$, we let $H_n$ denote the set of all real numbers strictly greater than $n$ and we define

$$W_n = \{\omega\} \cup \left( H_n \cap \left[ \left( \bigcup_{i=n}^{\infty} Y_i \right) \cup \bigcup \left\{ V_{1\,p_i}(\xi) \mid i \geq n \text{ and } \xi \in Y_i \right\} \right] \right).$$

Then we let

$$\{W_n \mid n \geq 1\}$$

be a neighborhood base at $\omega$.

The only non triviality arising in the proof that $Z$ is a topological space is the demonstration that each $W_n$ contains a neighborhood of each of its irrational points. Let $\xi_0 \in Y_i$ for some $i \geq n$. Then

$$V_{1\,p_i}(\xi_0) \cup V_{1\,p_{i+1}}(\xi_0) \subset \bigcup \{V_{1\,p_j}(\xi) j \geq n \text{ and } \xi \in Y_j\}.$$

Indeed, let

$$v = \frac{s}{p_{i+1}^{\alpha}} \in V_{1\,p_{i+1}}(\xi_0).$$

Using the density of $Y_{i+1}$, choose $\xi \in Y_{i+1}$ between $v$ and $\xi_0$. It follows that

$$v \in V_{1\,p_{i+1}}(\xi).$$

In order to show that $Z$ is regular and $\mathrm{T}_1$, we consider each kind of point in turn. For each rational point $q$, the set $\{\{q\}\}$ is a neighborhood base of closed sets. Consider next a point $\xi \in \mathbb{P}$. If $\xi \in Y_i$, then the set

$$A = V_{\varrho\,p_i}(\xi) \cup V_{\varrho\,p_{i+1}}(\xi) \cup \{\xi\}$$

is open and closed in $Z$. It is clear that $\omega$, any rational in $Z \setminus A$, or any $\xi' \in Y_j$ for $j \notin \{\, i-1, i, i+1 \,\}$ all have a neighborhood contained in $Z \setminus A$. Suppose $\xi' \in Y_j$ with $j \in \{\, i-1, i, i+1 \,\}$ and with $\xi' \neq \xi$. Let $\eta = |\xi' - \xi|$. Then

$$A \cap \left] \xi' - \frac{\eta}{2}, \xi' + \frac{\eta}{2} \right[$$

is finite (perhaps empty). So there exists $\delta > 0$ such that

$$V_{\delta\,p_j}(\xi') \cup V_{\delta\,p_{j+1}}(\xi') \cup \{\xi'\} \;\subset\; Z \setminus A.$$

Thus we have established that every irrational point of $Z$ has a neighborhood base of closed sets.

Finally, we consider the point $\omega$. For every $n$ there exists $k$ such that $W_k \subset \mathrm{cl}_X W_k \subset W_n$. Fix $n$ and choose such a $k$ with $k \geq n+1$. If $z \in \mathrm{cl}_X W_k$ and $z \notin W_k$ then $z$ has to be an irrational point $z = \xi$ with $\xi \in Y_j$ for some $j < k$. Moreover, for all $\varrho$, we have

$$\big(V_{\varrho\,p_j}(\xi) \cup V_{\varrho\,p_{j+1}}(\xi) \cup \{\xi\}\big) \cap W_k \;\neq\; \emptyset.$$

It follows that $j \geq k-1$. Hence $j \geq n$. We have that $\xi \in \mathrm{cl}_X W_k$. Hence $\xi$ belongs to the usual closure of $H_k$ and therefore $\xi \in H_n$. Finally we can conclude that $\xi \in W_n$. The proof that $Z$ is regular and $\mathrm{T}_1$ is finished.

We next prove that $Z$ is $\mu$-regular. Note that since $Z$ is a topological space, $\mu$-regularity means the following: Let $Z'$ be the completely regular reflection of $Z$. Then $Z$ is $\mu$-regular if and only if the following condition is satisfied:

For every $z \in Z$ and for every $V$ a neighborhood of $z$ in $Z$, there is a neighborhood $U$ of $z$ in $Z$ such that $\mathrm{cl}_{Z'} U \subset V$.

First, let $z = q \in \mathbf{Q}$. Since $\{q\}$ is open and closed in $Z$, the characteristic function $\chi_{\{q\}} : Z \to \mathbb{R}_t$ is continuous. Hence $\{q\}$ is closed in $Z'$.

Secondly, let $z = \xi \in \mathbb{P}$ with say $\xi \in Y_i$. We have already proved above that

$$A = V_{\varrho\, p_i}(\xi) \cup V_{\varrho\, p_{i+1}}(\xi) \cup \{\xi\}$$

is open and closed in $Z$. So $\chi_A : Z \to \mathbb{R}_t$ is continuous and hence $A$ is closed in $Z'$.

Thirdly, let $z = \omega$ and let $W_k$ be a basic neighborhood of $z$. Since we already proved that $Z$ is regular, it suffices to show that $\mathrm{cl}_{Z'} W_k = \mathrm{cl}_Z W_k$. Let $z \in \mathrm{cl}_{Z'} W_k$ and suppose that $z \notin W_k$. Then $z \in \mathbb{R}$, i.e., $z \neq \omega$. In order to prove that $z \in \mathrm{cl}_Z W_k$, we show that the neighborhood filters of $z$ in $Z$ and $Z'$ coincide. As before, note that for $q \in \mathbf{Q}$ we have that $\chi_{\{q\}} : Z \to \mathbb{R}_t$ is continuous and hence $\{q\}$ is open in $Z'$, while for $\xi \in \mathbb{P}$ and $A$ as above, $\chi_A : Z \to \mathbb{R}_t$ is continuous and hence $A$ is open in $Z'$. This concludes the proof that $Z$ is $\mu$-regular.

We show that $Z$ is not $z$-regular. If $n \geq 2$ and $W_n$ is a basic neighborhood of $\omega$, then there is no continuous function $f : Z \to \mathbb{R}_t$ and $k \geq n$ such that $W_k \subset Z(f) \subset W_n$. Indeed, it follows from the results in [vEF51] that the fact that $f$ equals zero on $Y_k \cap H_k$ implies that it also takes the value zero on some points outside $W_n$. (Note: The fact that $Z$ is not completely regular has been proved on page 366 of [vEF51]: $Y_1$ is closed and $Z$ cannot be separated from $\omega$ by a continuous function.

**Example 7.** A filter, zero space $X$ which is regular, and $z$-regular but not completely regular.

Let $Z$ be the topological space defined in Example 5 above and let $X$ be the rational numbers as a nearness subspace of $Z$. Clearly, $X$ is a filter nearness space which is regular. Also $X$ is not completely regular.

In order to show that $X$ is a zero space, it suffices to show that the trace on $X$ of any neighborhood filter of $Z$ has a base consisting of cozero sets. Since for rational points $q$ the set $\{\, q \,\}$ is open and closed, the characteristic function $\chi_{\{q\}} : Z \to \mathbb{R}_t$ is continuous. Hence the restriction $f = \left(\chi_{\{q\}} | X\right) : X \to \mathbb{R}_t$ is uniformly continuous and we have $\{\, q \,\} = \mathrm{coz}\, f$.

The same argument is used for the trace of the neighborhood filter of an irrational point $\xi$: Any set

$$A = V_{\varrho\, p_i}(\xi) \cup V_{\varrho\, p_{i+1}}(\xi) \cup \{\xi\}$$

is open and closed in $Z$ and hence the characteristic function $\chi_A : Z \to \mathbb{R}_t$ is continuous. Therefore the restriction $f = (\chi_A | X) : X \to \mathbb{R}_t$ is uniformly continuous and we have $A = \mathrm{coz}\, f$.

Consider next the trace of the neighborhood filter of $\omega$. It has as a base

$$\{W_n \cap X \mid n \in \mathbb{N}\}.$$

We show that for every $n$, the set $W_n \cap X$ is a cozero set in $X$. Notice that

$$W_n \cap X = H_n \cap \bigcup \{V_{1\,p_i}(\xi) \mid i \geq n \ \text{ and } \ \xi \in Y_i\}\,.$$

Let $n$ be fixed. Let $h : Z \to \mathbb{R}_t$ be the function defined as follows: $h$ takes the value 0 everywhere except on $W_n \cap X$. For $q \in W_n \cap X$ with $q = \frac{s}{t}$ ($\frac{s}{t}$ in irreducible form), and with $t = p_i^\alpha$ for some $\alpha$ and for some $i \geq n$, we define $h(q) = \frac{1}{t}$.
We show that $h$ is continuous. Clearly, $h$ is continuous at every rational point. Let $\xi$ be irrational and select $i$ so that $\xi \in Y_i$. Let $\epsilon > 0$. Choose $\varrho < \epsilon$ and consider the neighborhood

$$A = V_{\varrho\,p_i}(\xi) \cup V_{\varrho\,p_{i+1}}(\xi) \cup \{\xi\}.$$

Let $\frac{s}{t} \in A$ ($\frac{s}{t}$ in irreducible form).If $\frac{s}{t} \notin W_n$ then $h(\frac{s}{t}) = 0$. If $\frac{s}{t} \in W_n$ then $h(\frac{s}{t}) = \frac{1}{t}$, and since $\frac{s}{t} \in A$ it follows that $|\xi - \frac{s}{t}| < \frac{1}{t} < \varrho$, and hence $h(\frac{s}{t}) < \epsilon$. So we can conclude that $h[A] \subset [-\epsilon, \epsilon[$, therefore $h$ is continuous at each irrational point. In order to show that $h$ is continuous at $\omega$, let $\epsilon > 0$ and choose $m$ such that $m \geq n$ and $\frac{1}{p_m} < \epsilon$. It suffices to show that $h[W_m) \subset [-\epsilon, \epsilon[$. Let $q$ be rational, $q \in W_m$. Then there exist $i \geq m$ and $\xi \in Y_i$ such that $q \in V_{1\,p_i}(\xi)$, and consequently $q = \frac{s}{t}$ (in irreducible form) where $s \in \mathbf{Z}$ and $t = p_i^\alpha$ for some $\alpha \in \mathbb{N}$ with $|\xi - \frac{s}{t}| < \frac{1}{t}$. Hence we have

$$h(\frac{s}{t}) = \frac{1}{t} = \frac{1}{p_i^\alpha} \leq \frac{1}{p_m^\alpha} < \frac{1}{p_m} < \epsilon.$$

This completes the proof that $h$ is continuous.
It follows that $f = h\,|\,X : X \to \mathbb{R}_t$ is uniformly continuous. Moreover, $W_n \cap X = \operatorname{coz} f$. The proof that $X$ is a zero space is complete.
In order to show that $X$ is a $z$-regular space, we show that the trace on $X$ of the neighborhood filters of each point of $Z$ has a base of zero sets. For rational $q \in Z$ and for irrational $\xi \in Z$, a proof analogous to what we did above can be given. So consider the point $\omega \in Z$. Fix $n$. Define a function $h : Z \to \mathbb{R}_t$ by letting $h$ take the value 0 everywhere except at points in the set $X \setminus (W_n \cap X)$. For any point $\frac{s}{t} \in X \setminus (W_n \cap X)$, ($\frac{s}{t}$ in irreducible form) let $h(\frac{s}{t}) = \frac{1}{t}$. As before, it can be shown that $h$ is continuous at each rational and at each irrational point. For the continuity at $\omega$, note that if $\epsilon > 0$ and $m \geq n$ then $h|W_n = 0$ and therefore $h[W_m] \subset [-\epsilon, \epsilon[$. Finally, $W_n \cap X = Z(h|X)$.

**Example 8.**
A regular Hausdorff topological space $X$ which is $z$-regular but not completely regular. This example (the first known of its type, we believe) is due to M.E. Rudin [Private communication - unpublished].

Let $\mathbb{P}$ denote the set of all irrational numbers in the open segment $]0, 1[$. Let $K$ be the intersection of $\mathbb{P}$ and the standard Cantor set, let $H = \mathbb{P} \setminus K$, and let $h : K \to H$ be a homeomorphism. Since $K$ is order isomorphic to $\mathbb{P}$ and $H$ is the union of countably many disjoint intervals from $\mathbb{P}$, we can assume each of the $H$ intervals is the image under $h$ of a $K$ interval having a rational endpoint. Let

$$D = ]0,1[^2 \ \setminus \ \{(x,x) \mid x \ \text{ is rational}\}$$

and topologize $D$ by:

(1) Each $(x,y)$ with $x \neq y$ is isolated.
(2) A basic neighborhood of $(x,x)$ for $x \in H$ is of the form $[(x,x)\,,\,(x+\epsilon,x)[$ for some $\epsilon > 0$.
(3) A basic neighborhood of $(x,x)$ for $x \in K$ is of the form $[(x,x)\,,\,(x,x-\epsilon)[$ for some $\epsilon > 0$.

For each $n \in \mathbb{N}$, let $D_n$ be a copy of $D$ (with the topology described above). Let $Y$ be the quotient space of the disjoint union of the $D_n$'s with each $(x,x)$ from the $K$ of $D_n$ identified with $\big(h(x),h(x)\big)$ from the $H$ of $D_{n+1}$.
Our space is $X = Y \cup \{p\}$, $p$ being just an extra point we throw in. A basic open neighborhood of $p$ in $X$ is of the form

$$\{p\} \cup \Big(D_n \setminus \ \text{diagonal of } \ D_n\Big) \cup \bigcup_{m>n} D_m$$

for some $n \in \mathbb{N}$. Pictorially:

$$\Big| H_1 \quad D_1 \quad K_1 \Big|\Big| H_2 \quad D_2 \quad K_2 \Big|\Big| H_3 \quad D_3 \quad K_3 \Big|\Big| H_4 \quad D_4 \qquad \cdot \quad \cdot \quad \cdot \qquad p.$$

Note that $Y$ is an open subspace of $X$. It is not difficult to see that $X$ is a regular Hausdorff space with every point except $p$ having a clopen neighborhood base and with $p$ having a zero set neighborhood base. Hence $X$ is $z$-regular. But since $p$ cannot be separated from $H_1$ by any continuous real-valued map, $X$ is not completely regular.

The following table summarizes the properties of our counterexamples.

| | compl. reg. | zero | reg. | $z$-reg. | $\mu$-reg. | weak. reg. | separ. | fil. | near. |
|---|---|---|---|---|---|---|---|---|---|
| Ex. 1 | − | + | − | − | − | − | + | + | + |
| Ex. 2 | − | − | − | + | + | + | + | + | − |
| Ex. 3 | − | + | − | + | + | + | + | + | + |
| Ex. 4 | − | + | − | − | + | + | + | + | + |
| Ex. 5 | − | + | − | − | − | − | + | + | + |
| Ex. 6 | − | − | + | − | + | + | + | + | + |
| Ex. 7 | − | + | + | + | + | + | + | + | + |
| Ex. 8 | − | − | + | + | + | + | + | + | + |

Figure 1 Table of properties of counterexamples
+ means has the property; − means does not have it

The following diagrams make visible the relationships which exist between the properties we have been considering. These diagrams exhibit all relationships which hold, except those obtained by transitivity.

The first diagram exhibits the relationships which hold when we assume that we are dealing only with topological spaces.
Remarks for the topological case:

- Regular does not imply $\mu$-regular. See Example 2 of [BM76].
- $\mu$-regular does not imply $z$-regular. See Example 1 of [BM76].
- $z$-regular does not imply completely regular. See Example 8 above, due to M.E. Rudin.

$$
\begin{tikzcd}
\text{Completely regular} & \longleftrightarrow \arrow[d] & \text{zero space} \\
 & z\text{-regular} \arrow[d] & \\
 & \mu\text{-regular} \arrow[d] & \\
\text{regular} & \longleftrightarrow & \text{weakly regular}
\end{tikzcd}
$$

Diagram 1. The topological case.

The second diagram is for general merotopic spaces (not assuming the space to be a nearness space).
Remarks:

- If a space is either completely regular, a zero space, or regular then it is necessarily a nearness space.
- a $z$-regular space, and therefore also a $\mu$-regular space or a weakly regular space, need not be a nearness space (our Example 2 is such a space).

$$
\begin{tikzcd}
 & \text{Completely regular} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{regular} \arrow[ddr] & \text{zero space} & z\text{-regular} \arrow[d] \\
 & & \mu\text{-regular} \arrow[dl] \\
 & \text{weakly regular} &
\end{tikzcd}
$$

Diagram 2. The general merotopic space case.

In the third diagram we assume that the spaces under consideration are weakly regular.

$$\begin{tikzcd}
 & \text{Completely regular} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{regular} & \text{zero space} \arrow[dr] & z\text{-regular} \arrow[d] \\
 & & \mu\text{-regular}
\end{tikzcd}$$

Diagram 3. The weakly regular case.

In our last diagram, we assume that the spaces under consideration are regular.
Remarks for the regular case:

- Zero does not imply $z$-regular. J. Reiterman and J. Pelant have an example [Private communication - unpublished].
- $z$-regular does not imply zero (see Example 8 above due to M.E. Rudin).

$$\begin{tikzcd}
\text{Completely regular} \arrow[d] \arrow[dr] & \\
\text{zero space} \arrow[dr] & z\text{-regular} \arrow[d] \\
 & \mu\text{-regular}
\end{tikzcd}$$

Diagram 4. The regular case.

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