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Multiplication of nonadditive cuts in AST

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Abstract. Three complete characteristics of couples of nonadditive cuts such that $\underline{J \times K} \neq \overline{J \times K}$ are given. The equality $\overline{J \times K} = J\,!\,K$ is proved for all couples of nonadditive cuts. Some examples of nonadditive cuts are described.

Keywords: alternative set theory, cuts of natural numbers, inner and outer cut of a class, inner and outer product of two cuts, logarithmical cut

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Introduction.

Cuts (initial segments) of natural numbers play in AST (alternative set theory) a role similar to that one played by cardinal numbers in the classical set theory. To every class X, the inner cut $\underline{X}$ (of natural numbers being numbers of elements of subsets of X) and the outer cut $\overline{X}$ (of numbers which cannot be numbers of elements of supersets of X) are defined. $\underline{X}$ and $\overline{X}$ may be different and they may have arbitrary values provided that they are proper cuts and $\underline{X} \leq \overline{X}$ (see [Tz 1987]). If we restrict ourselves on Borel classes or on real ones, then these possibilities are much more restricted. The exact list of them may be found in [KZ 1989]. An obvious mathematical task is to extend arithmetical operations on cuts. But there are different possibilities how to do this (see [KZ 1988]). In the mentioned paper, there are examples demonstrating that these extensions may differ. One such example for the operation of product is the couple of cuts $\alpha + \mathrm{FN}$ and $\alpha - \mathrm{FN}$ for $\alpha > \mathrm{FN}$. Here the equalities $\alpha + \mathrm{FN} = \alpha^2/(\alpha - \mathrm{FN})$ and $\alpha - \mathrm{FN} = \alpha^2/(\alpha + \mathrm{FN})$ are remarkable (instead of FN any additive cut can be used there). In this paper, we proceed in the opposite direction; we prove that in the case of the product of nonadditive cuts the mentioned difference is relatively infrequent. We give three complete characterizations of couples of nonadditive cuts for which the inner product differs from the outer one. We prove also that for nonadditive cuts J, K, the equality $\overline{J \times K} = J\,!\,K$ holds. Two of the mentioned characteristics are connected with the given example. Namely, we prove (in Sect. 1) that for nonadditive cuts $\underline{J \times K} \neq \overline{J \times K}$, iff $K = \gamma/J$ for a suitable natural number γ and (in Sect. 2) that if $\underline{J \times K} \neq \overline{J \times K}$, then in almost all cases $K = r(\alpha - (J - \alpha))$ for a suitable rational number r and a suitable $\alpha \in J$, moreover, the cut $J - \alpha$ is "very small" with respect to α. In Sect. 2 there are investigated products of some typical couples of nonadditive cuts. In Sect. 3 there are some examples of cuts illustrating the matter investigated in the sections 1 and 2. New definitions and the notation are introduced in Sect. 0.

0. Preliminaries.

We use small Greek letters for natural numbers (also infinite ones), except π and σ used in the notions π-class and σ-class (the intersection, the union, resp., of a countable system of set-theoretically definable classes). For finite natural numbers we use $m, n, k, \dots$. Latin capitals are used for classes and J, K are reserved for cuts (parts of the class of natural numbers containing with every element also smaller numbers). For sets we use small Latin letters. For rationals (also infinite) we use $r, s, t \dots$.

Definition 0.1. (1) $\llcorner r \lrcorner = \max\{\alpha \in N; \alpha \leq r\}$ – the lower integer part.

(2) $\ulcorner r \urcorner = \min\{\alpha \in N; \alpha \geq r\}$ – the high integer part.

We extend the ordering of N also for cuts.

Definition 0.2. Suppose that J, K are proper cuts (not equal to any natural number). Then $J \leq K \equiv J \subseteq K, r \leq J \equiv \llcorner r \lrcorner \in J$. Other cases and the extension of $<$, we left to the reader.

Definition 0.3. For proper cuts J, K and a rational number r, we define

(1) $r \cdot J = \{\alpha; (\exists\, \beta \in J)(\alpha \leq \llcorner r \cdot \beta \lrcorner\})$,
(2) $\gamma/J = \{\alpha; (\forall\, \beta < J)(\alpha \cdot \beta < \gamma)\}$,
(3) $J \cdot K = \{\gamma; (\exists\, \alpha < J)(\exists\, \beta < K)(\gamma < \alpha \cdot \beta)\}$ – the inner product,
(4) $J \,!\, K = \{\gamma; (\forall\, \alpha > J)(\forall\, \beta > K)(\gamma < \alpha \cdot \beta)\}$ (hence $J \,!\, N = N$) – the outer product.

Remarks: (1) All the definitions are in accordance with [KZ 1988], we only differ in the notation $J \cdot K$ which is used also in the case $J \cdot K \neq J \,!\, K$.

(2) It is possible to define also "mixed" products (for $\alpha < J$ and $\beta > K$ and vice versa). Both these products are estimated by $J \cdot K$ from below and by $J \,!\, K$ from above and hence we do not study them here, as we prove that the difference between the inner and outer product is relatively infrequent. Nevertheless, e.g. Theorem 1.4 is applicable for these products, too.

Definition 0.4. $J^+ = \{\alpha; (\forall\, \beta \in J)(\beta + \alpha + 1 \in J)\}$ is called the additive part of J (see [S 1988]).

Remember that $|u|$ denotes the number of elements of the set u.

Definition 0.5. (1) $\underline{X} = \{\alpha; (\exists\, u \subsetneqq X)(\alpha = |u|)\}$ (the inner cut of X).

(2) $\overline{X} = \{\alpha; (\forall\, u \supseteq X)(\alpha < |u|)\}$ (the outer cut).

(3) If $\underline{X} = \overline{X}$ we put $|X| = \underline{X}$, if $\underline{X} \neq \overline{X}$, we say that X has no cut.

(See [Tz 1987], [KZ 1988].)

A cut J is called additive, iff $(\forall\, \alpha, \beta \in J)(\alpha + \beta \in J)$. Remember that J^+ is additive and a cut J is additive, iff $J = J^+$. Remember also that $J \cdot K = \underline{J \times K} \leq \overline{J \times K} \leq J \,!\, K$ (see [KZ 1988]). There are examples given in the mentioned paper demonstrating the fact that equalities need not hold in the above inequalities.

1. Products of the form $J \times (\gamma/J)$.

In the first section, we prove that for nonadditive proper cuts J and K, the cartesian product $J \times K$ has no cut, iff there is γ such that $K = \gamma/J$. Hence we generalize the fact that $\underline{(\alpha + \mathrm{FN}) \times (\alpha^2/(\alpha + \mathrm{FN}))} = \alpha^2 - \alpha \cdot \mathrm{FN}$ and $\overline{(\alpha + \mathrm{FN}) \times (\alpha^2/(\alpha + \mathrm{FN}))} = \alpha^2 + \alpha \cdot \mathrm{FN}$. Moreover, we prove (except other interesting facts) the fact that for nonadditive cuts J, K we have $\overline{J \times K} = J \,!\, K$.

We prove, at first, a special (but substantial) case of one implication of the main theorem.

Theorem 1.1. *If J is a nonadditive proper cut and if $\alpha < J < 2\alpha$, then $K = 2\alpha^2/J$ is a nonadditive proper cut such that $K^+ = J^+, K \cdot J = 2\alpha^2 - \alpha \cdot J^+$ and $\overline{K \times J} = 2\alpha^2 + \alpha J^+$. Hence $K \times J$ has no cut, as $\underline{K \times J} = K \cdot J = 2\alpha^2 - \alpha J^+$.*

Proof: The case $J = \alpha + J^+$, i.e. $K = 2\alpha - J^+$ is easy and can be left to the reader. Analogously for $J = \alpha + \beta - J^+$. The proof of $J \cdot K = 2\alpha^2 - \alpha \cdot J^+$ is easy, too. Hence we concentrate on the case $(\forall\, \beta)(J \neq \beta + J^+ \;\&\; J \neq \beta - J^+)$. We prove that there is no set m of the cardinality $2\alpha^2$ such that $m \supset J \times K$. Then the rest of the proof is easy and it is left to the reader. We prove this by contradiction. We define the function $f(\beta) = \min(\mathrm{rng}(((2\alpha \times 2\alpha) - m) \restriction (\beta + 1))) - 1$. The function $f(\beta)$ is decreasing and we have $m \supset \{\langle \gamma, \delta \rangle; \gamma \leq f(\delta)\} \supset J \times K$. Choose $\varepsilon > J^+$ such that $f(\alpha) - 2\varepsilon > J$ (this is possible, as $f(\alpha) > J$, J^+ is additive and $J \neq f(\alpha) - J^+$ by the assumption). Then there is $\gamma \in J$ such that $\gamma > \alpha \,\&\, \gamma + \varepsilon > J$, hence $\ulcorner 2\alpha^2/(\gamma + \varepsilon) \urcorner \in K$. Now we estimate (from below) the cardinality of m. We have $|m| > \ulcorner 2\alpha^2/(\gamma + \varepsilon) \urcorner \cdot \gamma + \alpha \cdot 2\varepsilon \geq 2\alpha^2 - \varepsilon \ulcorner 2\alpha^2/(\gamma + \varepsilon) \urcorner + \varepsilon \cdot 2\alpha \geq 2\alpha^2$, a contradiction. □

Note that the assumption that $K = \gamma/J$ is a proper cut is equivalent to the property that γ is sufficiently large. This assumption is expressed exactly in the following lemma.

Lemma 1.2. *Let J be a nonadditive proper cut and $\alpha < J < 2\alpha$. γ/J is a proper cut, iff for every $\beta > J^+$ we have $\gamma \cdot \beta > \alpha^2$. In this case we have $(\gamma/J)^+ = (\gamma/\alpha^2) \cdot J^+$.*

Proof: Remember that $\beta > J^+$, iff $(\exists\, \alpha_1 \in N)(\alpha_1 < J < \alpha_1 + \beta)$. Now it suffices to realize the equality $\gamma/\alpha_1 - \gamma/(\alpha_1 + \beta) = \gamma \cdot \beta/(\alpha_1 \cdot (\alpha_1 + \beta))$ and use the facts that J^+ is additive and that for sufficiently small $\beta > J^+$, the inequality $\alpha < \alpha_1 < 2\alpha$ follows from $\alpha_1 < J < \alpha_1 + \beta$. □

Note: The proof of the general case of the considered implication (for γ such that $(\forall\, \beta > J^+)(\gamma \cdot \beta > \alpha^2)$, where $\alpha < J < 2\alpha$, it holds that $K = \gamma/J$ is a proper cut such that $J \times K$ has no cut) can be obtained now by a modification of the proof of Theorem 1.1. We leave this modification to the reader.

It can be noticed that in the investigated case ($J \times K$ has no cut) the following equality holds $\beta \cdot J^+ = \alpha \cdot K^+$ where $\alpha < J < 2\alpha$ and $\beta < K < 2\beta$. The importance of this equality is proved by the following theorem.

Theorem 1.3. *Let J and K be nonadditive cuts such that $\alpha < J < 2\alpha$ and $\beta < K < 2\beta$. If $\beta \cdot J^+ > \alpha \cdot K^+$, then there are $\beta_0, \beta_1 \in N$ such that $\beta_0 < K < \beta_1$ and $|J \times K| = \beta_0 \cdot J = \beta_1 \cdot J$.*

Proof: Let us choose $\beta_0 < K < \beta_1$ such that $(\beta_1 - \beta_0) \cdot \alpha < \beta \cdot J^+$. Moreover, $\beta < \beta_0$ may be supposed. We prove that $\beta_1 \cdot J$ is the inner cut of $J \times K$.

$\beta_1 \cdot J \geq \underline{J \times K}$ is obvious. For the proof of the opposite inequality, the property $(\forall\ \alpha_0 \in J)(\exists\ \alpha_1 \in J)(\alpha_0 \cdot \beta_1 < \beta_0 \cdot \alpha_1)$ suffices. Choose $\alpha_1 \in J$ such that $\beta(\alpha_1 - \alpha_0) > 2(\beta_1 - \beta_0) \cdot \alpha$. It is possible, as $\ulcorner(\beta_1 - \beta_0) \cdot \alpha / \beta\urcorner \in J^+$ and J^+ is additive. Then we have $\beta_0 \cdot (\alpha_1 - \alpha_0) > \beta \cdot (\alpha_1 - \alpha_0) > 2(\beta_1 - \beta_0) \cdot \alpha > (\beta_1 - \beta_0) \cdot \alpha_0$. We obtain the needed inequality by the substraction of $\beta_0 \cdot \alpha_0$ from the both sides.

Now we have to prove that $\beta_0 \cdot J$ is the outer cut of $J \times K$. $\overline{J \times K} \geq \beta_0 \cdot J$ is obvious. To prove the opposite inequality, it suffices to justify $(\forall\ \alpha_1 > J)(\exists\ \alpha_0 > J)(\alpha_0 \cdot \beta_1 < \beta_0 \cdot \alpha_1)$. We proceed analogously as above.

The theorem proves the necessity of the condition $\beta \cdot J^+ = \alpha \cdot K^+$ for $\underline{J \times K} \neq \overline{J \times K}$. Later (the remark after Theorem 2.3) we prove that the condition is not sufficient even in the connection with another necessary condition of the opposite set cofinality of the cuts J and K.

By the following theorem we obtain $(J \cdot K)^+$. Moreover, the equality $(J \cdot K)^+ = (\overline{J \times K})^+ = (\overline{J\,!\,K})^+$ holds and another relation for the three cuts $J \cdot K, \overline{J \times K}$ and $J\,!\,K$ is proved. □

Theorem 1.4. *If J, K are nonadditive cuts and $\alpha < J < 2\alpha, \beta < K < 2\beta$, then $(J \cdot K)^+ = (\overline{J \times K})^+ = (J\,!\,K)^+ = \max(\alpha \cdot K^+, \beta \cdot J^+)$. Moreover, we have $(\forall\ \gamma > (J \cdot K)^+)(\exists\ \delta)(\delta < J \cdot K \leq \overline{J \times K} \leq J\,!\,K < \delta + \gamma)$.*

Proof: Let us suppose $\max(\alpha \cdot K^+, \beta \cdot J^+) = \alpha \cdot K^+$. We prove at first the additional property. If $\gamma > \alpha \cdot K^+$, then $\llcorner\gamma/5\lrcorner > \alpha \cdot K^+$ (due to the additivity of $\alpha \cdot K^+$). Let us put $\gamma_0 = \llcorner\gamma/5\lrcorner, \gamma_a = \min(\gamma_0/\beta, \alpha)$ and $\gamma_b = \llcorner\gamma_0/\alpha\lrcorner$. Then we have $\gamma_b > K^+, \gamma_a > J^+$ and hence there are $\alpha_1 < J$ and $\beta_1 < K$ such that $\alpha < \alpha_1 < J < \alpha_1 + \gamma_a$ and $K < \gamma_b + \beta_1$. Thus we have $J\,!\,K < (\alpha_1 + \gamma_a)(\beta_1 + \gamma_b) = \alpha_1 \cdot \beta_1 + \alpha_1 \cdot \gamma_b + \beta_1 \cdot \gamma_a + \gamma_a \cdot \gamma_b \leq \alpha_1 \cdot \beta_1 + 2\alpha \cdot \gamma_b + 2\beta \cdot \gamma_a + \alpha \cdot \gamma_b \leq \alpha_1 \cdot \beta_1 + 5\gamma_0 \leq \alpha_1 \cdot \beta_1 + \gamma$.

From the additional property we have $(J \cdot K)^+ \leq \alpha \cdot K^+, (\overline{J \times K})^+ \leq \alpha \cdot K^+, (J\,!\,K)^+ \leq \alpha K^+$.

Now we prove $(J \cdot K)^+ \geq \alpha \cdot K^+$. Let $\gamma < \alpha \cdot K^+$ and $\gamma_0 < K^+$ be such that $\alpha \cdot \gamma_0 > \gamma$. If $\alpha \leq \alpha_1 < J, \beta_1 < K$, then $\alpha_1 \cdot \beta_1 + \gamma \leq \alpha_1(\beta_1 + \gamma_0)$ and $\beta_1 + \gamma_0 < K$ (by the definition of K^+). We have proved that for $\gamma < \alpha \cdot K^+$ we have $\gamma \in (J \cdot K)^+$. The proof of $(J\,!\,K)^+ \geq \alpha \cdot K^+$ is quite analogous and it is left to the reader.

To prove $(\overline{J \times K})^+ \geq \alpha \cdot K^+$ it suffices for every $m \supseteq J \times K$ and every $\gamma < \alpha \cdot K^+$ to find $\widetilde{m} \supseteq J \times K$ such that it holds $|m| > |\widetilde{m}| + \gamma$. Without loss of generality we may assume "convexity of m" $(\forall\ \langle\alpha, \beta\rangle)((\exists \langle\alpha_1, \beta_1\rangle \in m)(\alpha < \alpha_1\ \&\ \beta < \beta_1) \Longrightarrow \langle\alpha, \beta\rangle \in m))$, hence $(\forall\ \delta \in \alpha)(m''\{\delta\} \in N\ \&\ m''\{\delta\} > K)$. Let us put $\gamma_0 = \ulcorner\gamma/\alpha\urcorner$, hence $\gamma_0 \in K^+$. Now it suffices to put $\widetilde{m}''\{\delta\} = m''\{\delta\} - \gamma_0$, as we have $m''\{\delta\} - \gamma_0 \supset K$. □

Remark: Note that in [KZ 1989] an example is given of a real semiset X such that $\underline{X}$ and $\overline{X}$ are both nonadditive and $(\underline{X})^+ \neq (\overline{X})^+$.

Lemma 1.5. (a) *Let J and K be cuts such that for no $\gamma \in N$ the equality $J = \gamma - J^+$ holds. If $J \leq K$ and $(\forall\ \delta > J^+)(\exists\ \alpha)(\alpha \leq J \leq K \leq \alpha + \delta)$, then $J = K$.*

(b) *Let J and K be cuts such that $J^+ = K^+$ and there is $\gamma \in N$ with $J = \gamma - J^+$. If $J \leq K$ and $(\forall\ \delta > J^+)(\exists\ \alpha)(\alpha \leq J \leq K \leq \alpha + \beta)$, then $J = K$ or $K = \gamma + J^+$.*

Proof: (a) By contradiction: If $J < \gamma < K$, then there is $\delta > J^+$ such that $\gamma - \delta > J$ (otherwise $J = \gamma - J^+$ contradicting our assumption). But in this case we have $(\forall\, \alpha \in J)(\alpha + \delta \in \gamma < K)$ which is in contradiction to our assumption.

(b) Obviously $K \leq \gamma + J^+$. If $\gamma < K$, then $K \geq \gamma + J^+$ due to $K^+ = J^+$. If $\gamma > K$, then $K \leq \gamma - J^+$ since $K^+ = J^+$. □

Corollary 1.6. *If we prove, for nonadditive cuts J, K, that for no γ the equality $J \cdot K = \gamma - (J \cdot K)^+$ holds, then $|J \times K| = J \cdot K = J \,!\, K$.*

In the following theorem, we prove the opposite implication of the main assertion.

Theorem 1.7. *If J and K are nonadditive cuts such that $\underline{J \times K} \neq \overline{J \times K}$, then there is $\gamma \in N$ such that $K = \gamma / J$.*

Proof: Let α, β be such that $\alpha < J < 2\alpha$, $\beta < K < 2\beta$. Due to Theorem 1.3 we know that $\beta \cdot J^+ = \alpha \cdot K^+$. We also know (Corollary 1.6) that there is γ such that $J \cdot K = \gamma - \beta \cdot J^+$. We prove that $\overline{J \times K} = \gamma + \beta \cdot J^+$. By contradiction, we prove $\gamma < \overline{J \times K}$. $\gamma > \overline{J \times K}$ implies namely $\gamma - \beta \cdot J^+ \geq \overline{J \times K}$ (by Theorem 1.4 $(\overline{J \times K})^+ = \beta \cdot J^+$), hence $\gamma - \beta \cdot J^+ = \underline{J \times K} < \overline{J \times K} \leq \gamma - \beta \cdot J^+$, a contradiction. We prove now $K \leq \gamma / J$. Let us fix $\beta_1 \in K$. We have to prove $(\forall\, \alpha_1 \in J)(\beta_1 < \gamma / \alpha_1)$, but we know that $\alpha_1 \cdot \beta_1 < \gamma$, as $\alpha_1 \cdot \beta_1 < \gamma - \beta \cdot J^+$. It suffices to prove $\gamma / J \leq K$. Let us fix $\delta \in \gamma / J$. Hence there is $\alpha_1 > J$ such that $\delta < \gamma / \alpha_1$. Now it suffices to prove that $(\forall\, \beta_1 > K)(\delta < \beta_1)$. But we have $\alpha_1 \cdot \beta_1 > J \,!\, K \geq \overline{J \times K} = \gamma + \beta \cdot J^+ > \gamma$ and hence $\delta < \gamma / \alpha_1 < \beta_1$. □

Our result can be now collected to the promised assertion.

Corollary 1.8. *The cartesian product of two proper nonadditive cuts J, K has no cut, iff there is γ such that $K = \gamma / J$.*

Remark: Due to Theorem 1.4 we know that $(J \cdot K)^+ = \alpha \cdot K^+$ (where $\alpha < J < 2\alpha$). If we use an arbitrary $\widetilde{\gamma}$ such that $|\widetilde{\gamma} - \gamma| < \alpha \cdot K^+$ instead of γ, then the equality $K = \widetilde{\gamma} / J$ holds, too.

The following theorem proves that the necessary conditions $\beta \cdot J^+ = \alpha \cdot K^+$ and $J \cdot K = \gamma - \alpha \cdot K^+$ for the nonexistence of the cut of $J \times K$ have a position different from that one of the couple of conditions $\beta \cdot J^+ = \alpha \cdot K^+$ and J, K has the opposite set cofinality.

Theorem 1.9. *Let J, K be nonadditive cuts and let α, β be such that $\alpha < J < 2\alpha$ and $\beta < K < 2\beta$. If $\beta \cdot J^+ = \alpha \cdot K^+$ and there is γ such that $J \cdot K = \gamma - \alpha \cdot K^+$, then either there are α_1, β_1 such that $J = \alpha_1 - J^+$ and $K = \beta_1 - K^+$ or $J \times K$ has no cut.* (*Note that in the first case $|J \times K| = \gamma - \alpha \cdot K^+$.*)

Proof: We have to prove that if the first case does not hold, then $K = \gamma / J$. $K \leq \gamma / J$ is obvious, as $J \cdot K < \gamma$. We prove $K \geq \gamma / J$. Let us suppose $\alpha = \beta$ hence $J^+ = K^+$. The general case is only an obvious modification.

(a) If $J = \alpha_1 - J^+$, then $\gamma / J = \ulcorner \gamma / \alpha_1 \urcorner + J^+$ and we suppose (without loss of generality) $\gamma = \alpha_1^2$. We have to prove $\alpha_1 < K$. If $\alpha_1 > K$, then $\alpha_1 - K^+ > K$ (the equality leads to the first case of the conclusion of the theorem). Hence there is $\varepsilon > K^+$ such that $\alpha_1 - \varepsilon > K$ and $\alpha_1 \cdot (\alpha_1 - \varepsilon) = \gamma - \varepsilon \cdot \alpha_1 > J \cdot K = \gamma - \alpha \cdot J^+$, a contradiction.

(b) If $J = \alpha_1 + J^+$, then $\gamma/J = \beta_1 - J^+$ for $\beta_1 = \llcorner \gamma/\alpha_1 \lrcorner$. If $K < \gamma/J$, then there is $\varepsilon > J^+$ such that $\beta_1 - \varepsilon > K$. But we have $(\beta_1 - \varepsilon) \cdot (\alpha_1 + J^+) < \gamma - \alpha \cdot J^+$, a contradiction.

(c) We have neither $J = \alpha_1 + J^+$ nor $J = \alpha_1 - J$ for any α_1. We want to prove $(\forall\, \beta_1 < \gamma/J)(\beta_1 < K)$. If there is β_1 such that $\beta_1 < \gamma/J$ and $\beta_1 > K$, then there are $\alpha_1 > J$ such that $\beta_1 < \gamma/\alpha_1$ and $\varepsilon > J^+$ such that $\alpha_1 - \varepsilon > J$. But then we have $(\alpha_1 - \varepsilon) \cdot \beta_1 < \gamma - \alpha \cdot J^+ = J \cdot K$, a contradiction. $\square$

Remark: Note that the triple of necessary conditions for $J \times K$ not having a cut, namely $J \cdot K = \gamma - \alpha \cdot K^+, \alpha \cdot K^+ = \beta \cdot J^+$ and J, K has opposite set cofinalities, is also sufficient, as the first case of the conclusion in the last theorem is excluded by the third condition. This triple of conditions is the third complete characterization of couples of nonadditive cuts such that their cartesian product has no cut.

The following corollary proves that for nonadditive cuts J, K, the equality $\overline{J \times K} = J\,!\,K$ holds. (Note that the equality $J \cdot K = \underline{J \times K}$ is obvious for all cuts.)

Corollary 1.10. *For every couple of nonadditive cuts J, K, we have $\overline{J \times K} = J\,!\,K$.*

Proof: Let α, β be such that $\alpha < J < 2\alpha, \beta < K < 2\beta$.

(a) If there is no γ such that $J \cdot K = \gamma - \max(\alpha \cdot K^+, \beta \cdot J^+)$, then by Theorem 1.4 and Lemma 1.5(a) we have $J \cdot K = \underline{J \times K} = \overline{J \times K} = J\,!\,K$.

(b) If $J \cdot K = \gamma - \max(\alpha \cdot K^+, \beta \cdot J^+)$ and $J \cdot K < \overline{J \times K}$, then $\alpha \cdot K^+ = \beta \cdot J^+$ and $\overline{J \times K} = J\,!\,K = \gamma + \alpha \cdot K^+$ by Theorem 1.4.

(c) If $J \cdot K = \gamma - \max(\alpha \cdot K^+, \beta \cdot J^+), J \cdot K = \overline{J \times K}$ and $\alpha \cdot K^+ = \beta \cdot J^+$, then by Theorem 1.8 we have $J = \alpha_1 - J^+, K = \beta_1 - K^+$ for suitable α_1, β_1 and hence $J\,!\,K = \gamma - \alpha \cdot K^+$.

(d) If $\alpha \cdot K^+ < \beta \cdot J^+$, then by Theorem 1.3 there are β_1, β_2 such that $\beta_1 < K < \beta_2$ and $\beta_1 \cdot J = \beta_2 \cdot J (= J \cdot K = \overline{J \times K})$. Obviously $\beta_1 \cdot J \le J \cdot K \le \overline{J \times K} \le J\,!\,K \le \beta_2 \cdot J$. $\square$

Remarks: (1) The assertion of the last corollary can be reformulated as follows: If one of products of two cuts J, K is not additive, then $\overline{J \times K} = J\,!\,K$.

(2) In the following paper concerning the products of additive cuts, we shall prove that only in the case that both J and K are additive, the inequality $\overline{J \times K} < J\,!\,K$ is possible. An example is remembered in Sect. 3.

Corollary 1.11. *If X, Y have nonadditive cuts and if $|X|, |Y|$ has a cut, then $|X \times Y| = |X| \cdot |Y|$.*

Proof: $|\underline{X}| \cdot |\underline{Y}| = \underline{X} \cdot \underline{Y} \le \underline{X \times Y} \le \overline{X \times Y} \le X\,!\,Y = |X| \cdot |Y|$. $\square$

The dual equality $\overline{J \times K} = J\,!\,K$ to the obvious one $J \times K = J \cdot K$ leads to dual versions of Corollary 1.6 and Theorem 1.9 where we change $J \cdot K$ by $J\,!\,K$ and $-$ by $+$. We can restate the dual version of Theorem 1.9: Let J, K be nonadditive cuts and let α, β be such that $\alpha < J < 2\alpha$ and $\beta < K < 2\beta$. If $\beta \cdot J^+ = \alpha \cdot K^+$ and there is γ such that $J\,!\,K = \gamma + \alpha \cdot K^+$, then either there are α_1, β_1 such that

$J = \alpha_1 + J^+$ and $K = \beta_1 + K^+$, or $J \times K$ has no cut. (Note that in the first case $|J \times K| = \gamma + \alpha \cdot K^+$.) In the same manner we can restate in the dual form the third characterization of couples of nonadditive cuts such that their cartesian product has no cut. We do not give here these dual versions precisely and their proofs are left to the reader.

2. Product of the form $(\alpha + J) \times (\alpha - J)$.

In this section, we find product of some special cuts. By this we prove that if $J \times K$ has no cut and J, K are nonadditive cuts, then in "almost all cases" $J = \alpha + \widetilde{J}$ and $K = \beta - (\beta/\alpha) \cdot \widetilde{J}$ where $\widetilde{J}$ is "small with respect to α" (and $(\beta/\alpha) \cdot \widetilde{J}$ is "small w.r.t. β"). Hence we converse, in a sense, the fact that $\underline{(\alpha + \mathrm{FN}) \cdot (\alpha - \mathrm{FN})} = \alpha^2 - \alpha \cdot \mathrm{FN}$ and $\overline{(\alpha + \mathrm{FN}) \times (\alpha - \mathrm{FN})} = \alpha^2 + \alpha \cdot \mathrm{FN}$.

Theorem 2.1. *Let J and K be additive cuts, $\alpha > J$ and $\beta > K$. If $\beta \cdot J = \alpha \cdot K$, then $\underline{(\alpha + J) \cdot (\beta - K)} = \alpha \cdot \beta - \alpha \cdot K$ and $\overline{(\alpha + J) \times (\beta - K)} = \alpha \cdot \beta + \alpha \cdot K$.*

PROOF: Let us begin with the inner cut. If $\gamma < J$ and $2\delta > K$, then $(\alpha + \gamma)(\beta - 2\delta) = \alpha \cdot \beta - 2\delta \cdot \alpha + \gamma \cdot \beta - \gamma \cdot 2\delta < \alpha \cdot \beta - \delta \cdot \alpha$, as $\gamma \cdot \beta < \beta \cdot J = \alpha \cdot K < \delta \cdot \alpha$. The direct proof of the opposite inequality is left to the diligent reader, as it follows from Theorem 1.4 and the following step of the proof.

To prove the second equality (outer cut), it suffices to prove that for every set m such that $m \supset (\alpha + J) \times (\beta - K)$, we have $m > \alpha \cdot \beta + \alpha \cdot K$ and to use the first part of the proof and Theorem 1.4 (remember that $(\alpha + J)^+ = J$). Without loss of generality we can suppose that m is "convex" ($(\langle \gamma_1, \delta_1 \rangle \in m \,\&\, \gamma \leq \gamma_1 \,\&\, \delta \leq \delta_1 \Rightarrow \langle \gamma, \delta \rangle \in m$). Let us put $\gamma_1 \max \{\gamma; \langle \alpha + \gamma, \ulcorner \beta/2 \urcorner \rangle \in m\}$, especially $\gamma_1 > J$. There is $\gamma > J$ such that $4\gamma < \gamma_1$ (due to the additivity of J). Now we put $\delta = \llcorner \beta \cdot \gamma/\alpha \lrcorner$, hence $\delta > K$ (remember $\beta \cdot J = \alpha \cdot K$). Now we have $|m| > (\beta - \delta) \cdot \alpha + (\beta/2) \cdot 4\gamma \geq \alpha \cdot \beta + \beta \cdot \gamma$. We have proved $\overline{(\alpha + J) \times (\beta - K)} \geq \alpha \cdot \beta + \alpha \cdot K \supset \alpha \cdot \beta - \alpha \cdot K \geq \underline{(\alpha + J) \cdot (\beta - K)}$. The needed two equalities are obtained by Theorem 1.4. $\square$

Remarks: (1) If $\beta \cdot J > \alpha \cdot K$, then $|(\alpha + J) \times (\beta - K)| = \alpha \cdot \beta + \beta \cdot J$ and if $\alpha \cdot K > \beta \cdot J$, then $|(\alpha + J) \times (\beta - K)| = \alpha \cdot \beta - \alpha \cdot K$ (which the reader can easily prove using Theorem 1.3).

(2) Note that, for additive cuts J, K, we have the following assertion: if $\beta \cdot J = \alpha \cdot K, \alpha > J$ and $\beta > K$, then $\alpha + J = \alpha \cdot \beta/(\beta - K)$ and $\beta - K = \alpha \cdot \beta/(\alpha + J)$ (use Theorem 1.7).

The following theorem is only an appendix to the interesting case solved by the previous theorem.

Theorem 2.2. *If J, K are additive cuts such that $\alpha > J$ and $\beta > K$, then $|(\alpha + J) \times (\beta + K)| = \alpha \cdot \beta + \max(\alpha \cdot K, \beta \cdot J)$ and $|(\alpha - J) \times (\beta - K)| = \alpha \cdot \beta - \max(\alpha \cdot K, \beta \cdot J)$.*

PROOF: In the case $\beta \cdot J \neq \alpha \cdot K$, we use Theorem 1.3. If $\beta \cdot J = \alpha \cdot K$, then the existence of the cut of the cartesian product follows by the set cofinality $\alpha + J$ and $\beta + K$ ($\alpha - J$ and $\beta - K$, respectively). The inner cut of the products can be found using Theorem 1.4. $\square$

Other typical cases of products are solved by the following two theorems. The first one asserts that if β is "relatively large w.r.t. (with respect to) α and J", then $(\alpha+\beta+J)\times(\alpha-\beta-J)$ has a cut; the second one asserts that if β is "relatively small w.r.t. α and J", then $(\alpha+\beta+J)\times(\alpha-\beta-J)$ has no cut.

Theorem 2.3. *Let J be a nonadditive cut such that there is no γ with $J=\gamma+J^+$ or $J=\gamma-J^+$ and let $\alpha > J$. If $\omega\in N$ has the property $\alpha/\omega > J \,\&\, (\forall\, \delta\in J^+)(\omega\cdot\delta\in J^+)$, then $(\forall\ \beta>\alpha/\omega)(|(\alpha+\beta+J)\times(\alpha-\beta-J)| = (\alpha+\beta+J)\cdot(\alpha-\beta-J))$. Moreover, there is no γ such that $(\alpha+\beta+J)\cdot(\alpha-\beta-J) = \gamma-\alpha\cdot J^+$ or $(\alpha+\beta+J)\cdot(\alpha-\beta-J) = \gamma+\alpha\cdot J^+$. (Obviously we suppose $\alpha-\beta > J$.)*

Proof: To prove that the product has a cut it is sufficient, by Corollary 1.6, to check that the inner cut has not the form $\gamma-\alpha\cdot J^+$. We prove this by contradiction. Obviously, we can suppose $\gamma<\alpha^2$. Let us put $\beta_0 = \llcorner\sqrt{\alpha^2-\gamma}\lrcorner$, i.e. $\alpha^2-(\beta_0+1)^2\le\gamma\le\alpha^2-\beta_0^2$.

(1) If $\beta_0 > \beta+J$, then there are $\beta_1 > \beta+J$ and $\delta > J^+$ such that $\beta_1 < \beta_0, \beta_1-\delta > \beta+J$ and $\beta_1-\delta-\llcorner\delta/\omega\lrcorner\in\beta+J$. Now we have $(\alpha+\beta+J)\cdot(\alpha-\beta-J) > (\alpha+\beta_1-\delta-\llcorner\delta/\omega\lrcorner)\cdot(\alpha-\beta_1+\delta)\ge\alpha^2-\beta_1^2+2\delta\cdot\beta_1-\delta^2-\llcorner\delta/\omega\lrcorner\cdot\alpha\ge\gamma$, as $\alpha^2-\beta_1^2\ge\alpha^2-\beta_0^2>\gamma, \delta\cdot\beta_1>\delta^2$ and $\delta\cdot\beta_1>\llcorner\delta/\omega\lrcorner\cdot\alpha$ (as $\beta_1>\beta>\alpha/\omega$).

(2) If $\beta_0 < \beta+J$, then we prove that there is $\varepsilon > J^+$ such that $\gamma-\varepsilon\cdot\alpha > (\alpha+\beta+J)\cdot(\alpha-\beta-J)$. Let $\beta_1>\beta_0$ and $\beta_1>\beta$ be such that $\beta_1\in\beta+J$ and there is $\delta>J^+$ such that $\beta_1+\delta<\beta+J$ and $\beta_1+\delta+\llcorner\delta/\omega\lrcorner>\beta+J$. Then we have $(\alpha+\beta_1+\delta+\llcorner\delta/\omega\lrcorner)\cdot(\alpha-\beta_1-\delta)\le\alpha^2-\beta_1^2-2\beta_1\cdot\delta-\delta^2+\llcorner\delta/\omega\lrcorner\cdot\alpha\le\alpha^2-(\beta_0+1)^2-\llcorner\delta/\omega\lrcorner\cdot\alpha$ as $\beta_1\cdot\delta>\delta\cdot(\alpha/\omega)$. Now it suffices to put $\varepsilon=\llcorner\delta/\omega\lrcorner$.

The proof that $(\alpha+\beta+J)\cdot(\alpha-\beta-J)\ne\gamma+\alpha\cdot J^+$ is easy. We proceed once more by contradiction. If for some γ we have the equality, then there are $\kappa_1\in\alpha+\beta+J$ and $\kappa_2\in\alpha-\beta-J$ such that $\kappa_1>\alpha$ and $\kappa_1\cdot\kappa_2>\gamma$. By the assumption $J\ne\beta_1-J^+$ there is $\delta>J^+$ such that $\kappa_2+\delta\in\alpha-\beta-J$, hence $\kappa_1\cdot(\kappa_2+\delta)>\gamma+\delta\cdot\alpha$, a contradiction. □

Remark: In [S 1988] there is an example of a cut $\widetilde{J}$ such that $\widetilde{J}^+\ne\widetilde{J}\,\&\,\widetilde{J}^+=\mathrm{FN}$ and for any γ we have neither $\widetilde{J}=\gamma+\mathrm{FN}$ nor $\widetilde{J}=\gamma-\mathrm{FN}$. Let δ be such that $\delta<\widetilde{J}<2\delta$ and let us put $J=12\delta+\widetilde{J}$ and $K=6\delta-\widetilde{J}$. If we put $\alpha=9\delta,\beta=3\delta$ and $\omega=4$, then by Theorem 2.3 we have that $J\times K$ has a cut. If we put $\alpha=10\delta$ and $\beta=5\delta$, then $\beta\cdot J^+=\alpha\cdot K^+$ and J,K have the opposite set cofinality. Hence we have the example promised after Theorem 1.3.

Corollary 2.4. *Let J be nonadditive cut such that there is no β with $J=\beta+J^+$ or $J=\beta-J^+$ and let $\alpha<J<2\alpha$ and $(\forall\ \omega\in N)(\alpha/\omega>J^+\Rightarrow\alpha/\omega^2>J^+)$; then $(\forall\ \beta\in J$ such that $2\beta>J)(|J\times(\beta-(J-\beta))| = J\cdot(\beta-(J-\beta))$ and there is no γ with $J\cdot(\beta-(J-\beta))=\gamma-\alpha\cdot J^+$ or $J\cdot(\beta-(J-\beta))=\gamma+\alpha\cdot J^+$.*

Remark: Note that the cuts from the example in the remark after Theorem 2.3 do not have the property required in Corollary 2.4. An example of such cuts is given in Sect. 3.

Theorem 2.5. *Let J be a nonadditive cut. If $(\exists\,\omega)(\exists\,\delta)(\alpha/\omega>J\,\&\,\delta>J^+\,\&\,\delta/\omega<J^+)$, then for every $\beta<\alpha/\omega$ there is $\beta_1\in J$ such that $\underline{(\alpha+\beta+J)\times(\alpha-\beta-J)}=$*

$\alpha^2 - (\beta+\beta_1)^2 - \alpha\cdot J^+$ *and* $\overline{(\alpha+\beta+J)\times(\alpha-\beta-J)} = \alpha^2 - (\beta+\beta_1)^2 + \alpha J^+$

Proof: Due to the assumption $J \neq J^+$ we may suppose $\delta < J$. We prove, at first, that there is $\beta_1 \in J$ such that $(\alpha+\beta+J)\cdot(\alpha-\beta-J) = \alpha^2-(\beta+\beta_1)^2-\alpha\cdot J^+$. Let us fix δ having the property from the assumptions ($\delta < J \,\&\, \delta > J^+ \,\&\, \delta/\omega < J^+$). There is $\beta_1 < J$ such that $\beta_1+\delta > J$. The product of a couple of numbers from the investigated cuts may be expressed in the form $(\alpha+\beta+\beta_1+\varepsilon)\cdot(\alpha-(\beta+\beta_1+\varepsilon+\varepsilon_1)) = \alpha^2-(\beta+\beta_1)^2-2\varepsilon\cdot(\beta+\beta_1+\varepsilon/2)-\varepsilon_1(\alpha+\beta+\beta_1+\varepsilon)$. Hence we know that the product is less than $\alpha^2-(\beta+\beta_1)^2$. By Theorem 1.4, we know that $((\alpha+\beta+J)\cdot(\alpha-\beta-J))^+ = \alpha\cdot J^+$ and hence $(\alpha+\beta+J)\cdot(\alpha-\beta-J) \leq \alpha^2-(\beta+\beta_1)^2-\alpha\cdot J^+$. To prove the opposite inequality, let us choose an arbitrary $\gamma > J^+$. We prove that there is a couple of numbers of the investigated cuts such that the product is larger than $\alpha^2-(\beta+\beta_1)^2-\gamma\cdot\alpha$. Let us choose ε_1 such that $J^+ < \varepsilon_1 < \gamma/3$. To this ε_1 there is ε such that $\beta_1+\varepsilon < J \,\&\, \beta_1+\varepsilon+\varepsilon_1 > J$. Then we can estimate $\beta < \alpha/\omega, \beta_1+\varepsilon < \alpha/\omega$ (as $\beta_1+\varepsilon < J < \alpha/\omega$). Hence $2\cdot(\beta+\beta_1+\varepsilon/2)\cdot\varepsilon < 2\cdot(\alpha/\omega+\alpha/\omega)\cdot\varepsilon \leq 4\alpha\cdot(\delta/\omega)$. Now we have $\delta/\omega < J^+$ (by the assumption) and thus $4\cdot(\delta/\omega) < J^+$ (additivity of J^+), hence $\alpha\cdot 4\cdot(\delta/\omega) < \alpha\cdot(\gamma/3)$. The estimation $\beta+\beta_1+\varepsilon < \alpha$ is obvious and thus the inequality is proved.

To prove $\overline{(\alpha+\beta+J)\times(\alpha-\beta-J)} = \alpha^2-(\beta+\beta_1)^2+\alpha J^+$, it suffices to show that for every set $m \supset (\alpha+\beta+J)\times(\alpha-\beta-J)$ we have $|m| > \alpha^2-(\beta+\beta_1)^2$. We may assume that m is "convex" $((\forall\,\alpha,\beta)((\exists\,\langle\alpha_1,\beta_1\rangle \in m)(\alpha \leq \alpha_1 \,\&\, \beta \leq \beta_1) \Rightarrow \langle\alpha,\beta\rangle \in m))$. Let γ_1 be the largest natural number such that $\langle\alpha+\beta+\beta_1+\gamma_1, \llcorner\alpha/2\lrcorner\rangle \in m$. Let us suppose that $J \neq \beta_1+\gamma_1-J^+$. Then there is $\gamma > J^+$ such that $\beta_1+\gamma_1-\gamma > J$. In the same way as in the first part of the proof, we can find $\varepsilon_1, \varepsilon$ such that $\beta_1+\varepsilon < J, \beta_1+\varepsilon+\varepsilon_1 > J, (\alpha+\beta+\beta_1+\varepsilon)\cdot(\alpha-(\beta+\beta_1+\varepsilon+\varepsilon_1)) > \alpha^2-(\beta+\beta_1)^2-\gamma\cdot\alpha/2$. But the set m contains more than $\gamma\cdot\alpha/2$ elements out of the cartesian product of the mentioned couple of numbers. In the case that $J = \beta_1+\gamma_1-J^+$, we proceed analogously using the second coordinate, as $\alpha-\beta-J = \alpha-\beta-\beta_1+\gamma_1+J^+$. □

Remark: Note that if we have even $\alpha/\omega < \delta$ (hence $\alpha/\omega^2 < J^+$), then we obtain the following more pleasant expressions $\underline{(\alpha+\beta+J)\times(\alpha-\beta-J)} = \alpha^2-\alpha J^+$ and $\overline{(\alpha+\beta+J)\times(\alpha-\beta-J)} = \alpha^2+\alpha\cdot J^+$. In Sect. 3 we give an example in which both β, β_1 play a substantial role.

The following theorem solves further two cases which enter the mind after last two theorems.

Theorem 2.6. *Let J be a nonadditive cut such that there is no β with $J = \beta+J^+$ or $J = \beta-J^+$. If $\alpha > J$, then $(\alpha+J^+)\cdot(\alpha-J) = \alpha\cdot(\alpha-J)$ and $(\alpha+J)(\alpha-J^+) = (\alpha+J)\cdot\alpha$.*

Proof: $(\alpha+J^+)\cdot(\alpha-J) \geq \alpha\cdot(\alpha-J)$ is obvious. We prove the opposite inequality. If $\gamma < \alpha-J$, then there is $\varepsilon > J^+$ such that $\gamma+\varepsilon < \alpha-J$. For $\delta \in J^+$ we know that $(\alpha+\delta)\cdot\gamma < \alpha\cdot(\gamma+\varepsilon)$ as $\delta\cdot\gamma < \alpha\cdot\varepsilon$.

$(\alpha+J)\cdot(\alpha-J^+) \leq (\alpha+J)\cdot\alpha$ is obvious. To prove the opposite inequality, we choose for $\gamma \in J$ a number δ such that $\delta > J^+ \,\&\, \gamma+3\delta < J \,\&\, 3\delta < \alpha$. Now we have $(\alpha+\gamma)\cdot\alpha < (\alpha+\gamma+3\delta)\cdot(\alpha-\delta) = (\alpha+\gamma)\cdot\alpha+2\delta\cdot\alpha-(\alpha+\gamma)\cdot\delta+\delta\cdot\alpha-3\delta^2$. □

Corollary 2.7. *With the same assumptions as in Theorem* 2.6, *we have* $|(\alpha+J^+)\times(\alpha-J)| = \alpha\cdot(\alpha-J)$ *and* $|(\alpha+J)\times(\alpha-J^+)| = (\alpha+J)\cdot\alpha$.

The following theorem is in a sense the converse of Theorem 2.5. But at first we find two equivalents of the property $(\exists\,\alpha)(\exists\,\delta)(\exists\,\omega)(\alpha < J < \alpha+\alpha/\omega \,\&\, \delta/\omega < J^+ < \delta)$ known from Theorem 2.5. Another (complicated) equivalent proving how infrequent the negation of the property is, will be described later.

Lemma 2.8. *The following three properties are equivalent.*

(1) $(\exists\,\alpha)(\exists\,\delta)(\exists\,\omega)(\alpha < J < \alpha+\alpha/\omega \,\&\, \delta/\omega < J^+ < \delta)$;
(2) $(\exists\,\alpha)(\exists\omega)(\alpha < J < \alpha+\alpha/\omega \,\&\, \alpha/\omega^2 < J^+ < \alpha/\omega)$;
(3) *J is nonadditive proper cut such that* $(\forall\,\alpha, \alpha < J < 2\alpha)(\exists\,\omega)(\alpha/\omega^2 < J^+ < \alpha/\omega)$.

Proof: (3) ⇒ (2) Let us choose α_1 such that $\alpha < \alpha_1 < J < \alpha_1 + \alpha/\omega$. Then we have $\alpha < \alpha_1 < 2\alpha$ and due to additivity of J^+, the property $\alpha_1/\omega^2 < J^+ < \alpha_1/\omega$ is preserved.

(2) ⇒ (1) It suffices to put $\delta = \ulcorner\alpha/\omega\urcorner$.

(1) ⇒ (3) Let us fix an arbitrary triple $\alpha_1, \delta_1, \omega_1$ satisfying (1) and let α be such that $\alpha < J < 2\alpha$. If $\alpha/\omega_1 \le \delta_1$, then it suffices to put $\omega = \omega_1$ and the property $J^+ < \alpha/\omega$ holds due to additivity of J^+ (analogous to (3) ⇒ (2)). If $\alpha/\omega_1 > \delta_1$, then ω be the least number such that $\alpha/\omega < \delta_1$. □

Theorem 2.9. *Let J and K be nonadditive cuts such that $J\times K$ has no cut. If one of the cuts* (e.g. J) *has the property* $(\exists\,\alpha)(\exists\,\delta)(\exists\,\omega)(\alpha < J < \alpha+\alpha/\omega \,\&\, \delta/\omega < J^+ < \delta)$, *then the second cut* (e.g. K) *has this property, too. In this case we have, moreover: There are α, β such that $\alpha < J$ and $K = \beta - (\beta/\alpha)\cdot(J-\alpha)$.* (*For* $\alpha = \beta$, *we obtain exactly the product described in Theorem* 2.5.)

Proof: From the fact that $J\times K$ has no cut, it follows that there is γ such that $J\cdot K = \gamma - \widetilde{\beta}\cdot J^+$ (where $\widetilde{\beta} < K < 2\widetilde{\beta}$, see Theorem 1.4). By Theorem 1.7 we know that $K = \gamma/J$. Let us put $\beta = \ulcorner\gamma/\alpha\urcorner$. Now it suffices to check that $K = \beta - (\beta/\alpha)\cdot\widetilde{J}$, where $\widetilde{J} = \{\delta; \alpha+\delta < J\}$. The proof that K has the property can be left to the reader. For the sake of simplicity we suppose $\gamma = \alpha^2$ (the general case is only somewhat more incomprehensive). If δ is such that $\alpha+\delta < J$, then $\alpha^2/(\alpha+\delta) = (\alpha-\delta)\cdot\alpha^2/(\alpha^2-\delta^2) > \alpha - \widetilde{J}$ – hence $K \ge \alpha - \widetilde{J}$. Let δ be such that $\alpha+\delta < J$. We want to prove that $\alpha-\delta > K$. We have $\alpha-\delta = (\alpha^2-\delta^2)/(\alpha+\delta)$ and to prove $\alpha-\delta > K$, it hence suffices to check $\delta^2 < \alpha\cdot J^+$ (see the remark after Theorem 1.7). We have $\alpha < \alpha+\delta < J < \alpha+\alpha/\omega$, hence $\delta < \alpha/\omega$ and $\delta^2 < \alpha\cdot(\alpha/\omega^2)$. But we know that $\alpha/\omega^2 < J^+$ by the assumption. □

Remark: In Sect. 3, we give both examples: a cut having the property and a cut not having the property.

Next theorem describes another equivalent of the property $(\exists\,\alpha)(\exists\,\omega)(\alpha < J < \alpha+\alpha/\omega \,\&\, \alpha/\omega^2 < J^+)$ proving how infrequent the negation of the property is. But we give a definition before, which is useful when investigating additive cuts.

Definition 2.10. For a cut J we define $\lg(J) = \{\alpha; 2^\alpha < J\}$.

We left to the reader the proofs of the following easy assertions: $\lg(J)$ is a cut, $J \leq K \Rightarrow \lg(J) \leq \lg(K)$ and $\lg(J)$ is a proper cut, iff J is additive.

Theorem 2.11. *The following two properties are equivalent.*

(1) $(\exists\, \alpha)(\exists\, \omega)(\alpha < J < \alpha + \alpha/\omega \,\&\, \alpha/\omega^2 < J^+)$;

(2) *J is a nonadditive proper cut such that if* $\lg(J^+)$ *is nonadditive, then* $\lg(J^+) \neq \gamma - (\lg(J^+))^+$ *where γ is such that $2^\gamma < J < 2^{\gamma+1}$ (there is exactly one such γ).*

Proof: By Lemma 2.8, we have that (1) is equivalent to J is nonadditive proper cut such that $(\forall\, \alpha, \alpha < J < 2\alpha)(\exists\, \omega)(\alpha/\omega^2 < J^+ < \alpha/\omega)$. Due to additivity of J^+, this is equivalent to $(\exists\, \alpha, \alpha < J < 2\alpha)(\exists\, \omega)(\alpha/\omega^2 < J^+ < \alpha/\omega)$. The last inequality can be equivalently rewritten to $\lg(\alpha) - 2 \cdot \lg(\omega) < \lg(J^+) < \lg(\alpha) - \lg(\omega)$. If $\lg(J^+)$ is additive, then we have $\lg(J^+) < \lg(\alpha)$ (as $J^+ < \alpha$) and $\lg(J^+) < \llcorner \lg(\alpha)/2 \lrcorner$. It suffices now to choose ω such that $\lg(\omega) = \llcorner \lg(\alpha)/2 \lrcorner - 2$. If $\lg(J^+)$ is nonadditive and $\lg(J^+) \neq \gamma - (\lg(J^+))^+$ (in this case we have $\lg(\alpha) = \gamma$ or $\lg(\alpha) + 1 = \gamma$), then $\lg(J^+) < \gamma - (\lg(J^+))^+$. Hence there is $\widetilde{\omega} > (\lg(J^+))^+$ such that $\lg(\alpha) - 2\widetilde{\omega} < \lg(J^+) < \lg(\alpha) - \widetilde{\omega}$. Now it suffices to choose ω such that $\lg(\omega) = \widetilde{\omega}$. If $\lg(J^+) = \gamma - (\lg(J^+))^+$ (now we have $\lg(\alpha) = \gamma$ or $\lg(\alpha) + 1 = \gamma$), then $\lg(J^+) < \lg(\alpha) - \lg(\omega)$ implies $\lg(\omega) < (\lg(J^+))^+$. Hence $2 \cdot \lg(\omega) < (\lg(J^+))^+$ and $\lg(J^+) < \lg(\alpha) - 2 \cdot \lg(\omega)$. Thus the inequality does not hold for any ω. □

The following theorem solves the question which immediately arises after the last theorem.

Theorem 2.12. *Let J be a nonadditive cut such that for no δ we have $J = \delta + J^+$ or $J = \delta - J^+$. If γ is such that $2^\gamma < J < 2^{\gamma+1}$ and $\lg(J^+) = \gamma - (\lg(J^+))^+$, then for every α such that $2^\gamma < \alpha < J$, the product $J \times (\alpha - (J - \alpha))$ has a cut.*

Proof: By Theorem 2.3 it suffices to prove that for every α, having the properties from the assumption, there is $\beta < J - \alpha$ such that β is "relatively large w.r.t. α and $J - (\alpha + \beta)$". Hence we are to find ω such that $\beta > \alpha/\omega > J - (\alpha + \beta)$ and $(\forall\, \delta < J^+)(\omega \cdot \delta < J^+)$. Let us choose β such that $\alpha + \beta < J < \alpha + \beta + \beta/4$ which is possible as J is not of the form $\delta + J^+$. Let us put $\omega = \ulcorner \alpha/\beta \urcorner + 1$. The conditions $\beta > \alpha/\omega$ and $\alpha/\omega > \beta/4 > J - (\alpha + \beta)$ are fulfilled. It suffices to prove the property $(\forall\, \delta < J^+)(\omega \cdot \delta < J^+)$ and due to the additivity of J^+, the inequality is equivalent to $\lg(\omega) + \lg(\delta) < \lg(J^+)$. As $\alpha/\omega > J^+$ we know that $\lg(\alpha) - \lg(\omega) > \lg(J^+) = \gamma - (\lg(J^+))^+ = \lg(\alpha) - (\lg(J^+))^+$ and hence $\lg(\omega) < (\lg(J^+))^+$. From this we know that for every δ such that $\lg(\delta) < \lg(J^+) = \gamma - (\lg(J^+))^+$, we have $\lg(\delta) + \lg(\omega) < \lg(J^+)$. □

Remark: The results of Theorem 2.9 and Theorem 2.12 can be collected to the following fact. If $J \times K$ has no cut, then K can be expressed "almost surely" in the form $K = (\beta/\alpha) \cdot (\alpha - (J - \alpha))$ for suitable α, β. Moreover, α is "a good approximation" of J (e.g. $\forall\, k \in \mathrm{FN})(\alpha < J < \alpha/k)$ holds). Only in the case that one (and hence also the second) of the cuts has neither the form $J = \delta + J^+$ nor $J = \delta - J^+$ (hence J is not real – see [S 1988]) and simultaneously $\lg(J^+) = \gamma - (\lg(J^+))^+$

where $2^\gamma < J < 2^{\gamma+1}$, we have that for no α, β, the equality $K = (\beta/\alpha)\cdot(\alpha-(J-\alpha))$ holds.

3. Examples.

In this section we give examples promised in the first and second ones.

Remember that in Theorem 1.1 it is proved that $J \times (2\alpha^2/J)$ has no cut for J nonadditive and $\alpha < J < 2\alpha$. This fact is also preserved if the cuts are multiplied by sufficiently large rational numbers ($r \cdot J$ must be a proper cut). Remember, moreover, that in [Tz 1988] there is noticed that a necessary condition for $J \times K$ having no cut is the opposite set cofinality of J and K. Another necessary condition is that $\beta \cdot J^+ = \alpha \cdot K^+$ for $\alpha < J < 2\alpha$ and $\beta < K < 2\beta$ (by Theorem 1.3). In the remark after Theorem 2.3, an example of cuts J, K, having the two conditions and still $\underline{J \times K} = \overline{J \times K}$, is given.

By Theorem 1.8, an example of cuts J, K (e.g. $J = K = \alpha - \mathrm{FN}$) such that $\beta \cdot J^+ = \alpha \cdot K^+, J \cdot K = \gamma - \alpha \cdot K^+$ and $\underline{J \times K} = \overline{J \times K}$, can be obtained. Hence the second couple of necessary conditions does not suffice.

The example that the last couple of conditions (the opposite set cofinality and $J \cdot K = \gamma - \max(\alpha \cdot K^+, \beta \cdot J^+)$ is not sufficient, we obtain if we put $J = \alpha - \mathrm{FN}$ and $K = \alpha^2 + \mathrm{FN}$. In this case we have $|(\alpha - \mathrm{FN}) \times (\alpha^2 + \mathrm{FN})| = \alpha^3 - \alpha^2 \cdot \mathrm{FN}$ by Theorem 1.3.

Remember from [KZ 1988] that $\alpha/\,\mathrm{FN} = |(\alpha/\,\mathrm{FN})\times \mathrm{FN}\,| < (\alpha/\,\mathrm{FN})\,!\,\mathrm{FN} = \alpha\cdot\mathrm{FN}$ is an example of two cuts such that $\overline{J \times K} < J\,!\,K$. (Both cuts must be additive.)

Now we give an example of a nonadditive cut J such that $(\forall\, \beta \in N)(J \neq \beta + J^+ \,\&\, J \neq \beta - J^+)$ and for α such that $\alpha < J < 2\alpha$ we have $(\forall\, \omega)(\alpha/\omega > J^+ \Rightarrow \alpha/\omega^2 > J^+)$. We promised this example in the remark following Corollary 2.4. This example can serve also as one case of cuts noticed in the remark after Theorem 2.9. As we have mentioned, an example of a cut J such that $J^+ = \mathrm{FN}$ and $(\forall\, \beta)(J \neq \beta + \mathrm{FN} \,\&\, J \neq \beta - \mathrm{FN})$ is given in [S 1988]. Generally, this holds for any additive cut not being π (i.e. $K \neq N$ and there is no descending sequence $\{\alpha_n; \alpha_n \in N \,\&\, n \in \mathrm{FN}\}$ such that $K = \bigcap_{n\in\mathrm{FN}} \alpha_n$). Such a cut can be constructed as the union of an increasing transfinite sequence $\{y_\beta; \beta \in \Omega\}$. We start with a transfinite decreasing sequence $\{x_\beta; \beta \in \Omega\}$ such that $\bigcap\{x_\beta; \beta \in \Omega\} = K$ and $\beta_1 > \beta_2 \Rightarrow 3x_{\beta_1} < x_{\beta_2}$ and on the β-th step we guarantee that $y_\beta + x_\beta < J < y_\beta + 2x_\beta$. Let $^*\,\mathrm{FN}$ be any revealment (standard extension) of FN (see [SV 1979] or [SV 1980] for the definitions). If $\alpha > {}^*\,\mathrm{FN}$, then $\alpha/{}^*\,\mathrm{FN}$ is an additive cut not being π and hence there is a cut J such that $\alpha < J < 2\alpha, J^+ = \alpha/{}^*\,\mathrm{FN}$ and $(\forall\, \beta)(J \neq \beta + J^+ \,\&\, J \neq \beta - J^+)$. This cut has the property $\alpha/\omega > J^+ \Rightarrow \alpha/\omega^2 > J^+$, as $\alpha/\omega > J^+ \Rightarrow \omega < {}^*\,\mathrm{FN}$.

Remark: Note that for $\widetilde{\alpha}$ such that $2\widetilde{\alpha} < J < 3\widetilde{\alpha}$ the couple $J, 2\widetilde{\alpha} - (J - 2\widetilde{\alpha})$ is a new (more complicated but more expressive) example of a couple of cuts such that $\beta \cdot J^+ = \alpha \cdot K^+, J, K$, have the opposite set cofinality, and $J \times K$ has a cut.

In the remark following Theorem 2.5 there is a circumstance which gives us a more pleasant expression of the product. The following example describes a case in which β, β_1 play a substantial role in the description of the product. We also give, by this example, the second case of cuts promised in the remark following Theorem 2.9.

If $\gamma > \mathrm{FN}$, then $2^\gamma \cdot \mathrm{FN}$ is an additive cut not being a π class and hence we are able to construct a cut J such that $J^+ = 2^\gamma \cdot \mathrm{FN}, 2^{3\gamma} < J < 2 \cdot 2^{3\gamma}$ and $(\forall\ \beta)(J \neq \beta + J^+ \,\&\, J \neq \beta - J^+)$. Using the notation of Theorem 2.5, let us put $\alpha = 2^{5\gamma}, \beta = 2^{4\gamma}, \omega = 2^{\gamma-1}, \delta = 2^{2\gamma}$. The assumptions of Theorem 2.5 are fulfilled and we prove that $\alpha^2 - \beta^2 > (\alpha + \beta + J)\,!\,(\alpha - \beta - J) = \alpha^2 - (\beta + \beta_1)^2 + \alpha J^+$ for a suitable $\beta_1 < J$. If ε is such that $\mathrm{FN} < \varepsilon < \gamma/2$, then there is β_2 such that $\beta_2 < J < \beta_2 + 2^{\gamma+\varepsilon}$ and $\beta_2 > 2^{3\gamma}$. Now we prove the inequality $(\alpha+\beta+\beta_2+2^{\gamma+\varepsilon}) \cdot (\alpha-\beta-\beta_2) < \alpha^2 - \beta^2$. We have namely $(2^{5\gamma} + 2^{4\gamma} + \beta_2 + 2^{\gamma+\varepsilon}) \cdot (2^{5\gamma} - 2^{4\gamma} - \beta_2) < (2^{5\gamma})^2 - (2^{4\gamma})^2 + 2^{6\gamma+\varepsilon} - 2\beta_2 \cdot 2^{4\gamma} < (2^{5\gamma})^2 - (2^{4\gamma})^2$. For the first inequality, let us note that only some negative numbers are omitted on the righthand side and for the second inequality, notice that $\beta_2 > 2^{3\gamma}$.

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