On FU(p)-spaces and p-sequential spaces

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Abstract. Following Kombarov we say that X is p-sequential, for $p \in \alpha^*$, if for every non-closed subset A of X there is $f \in {}^{\alpha}X$ such that $f(\alpha) \subseteq A$ and $\bar{f}(p) \in X \backslash A$. This suggests the following definition due to Comfort and Savchenko, independently: X is a FU(p)-space if for every $A \subseteq X$ and every $x \in A^-$ there is a function $f \in {}^{\alpha}A$ such that $\bar{f}(p) = x$. It is not hard to see that $p \leq_{\mathrm{RK}} q$ ($\leq_{\mathrm{RK}}$ denotes the Rudin–Keisler order) $\Leftrightarrow$ every p-sequential space is q-sequential $\Leftrightarrow$ every FU(p)-space is a FU(q)-space. We generalize the spaces S_n to construct examples of p-sequential (for $p \in U(\alpha)$) spaces which are not FU(p)-spaces. We slightly improve a result of Boldjiev and Malykhin by proving that every p-sequential (Tychonoff) space is a FU(q)-space $\Leftrightarrow$ $\forall\, \nu < \omega_1$ ($p^{\nu} \leq_{\mathrm{RK}} q$), for $p, q \in \omega^*$; and S_n is a FU(p)-space for $p \in \omega^*$ and $1 < n < \omega$ $\Leftrightarrow$ every sequential space X with $\sigma(X) \leq n$ is a FU(p)-space $\Leftrightarrow$ $\exists\, \{p_{n-2}, \dots, p_1\} \subseteq \omega^*$ ($p_{n-2} <_{\mathrm{RK}} \cdots <_{\mathrm{RK}} p_1 <_l p$); hence, it is independent with ZFC that S_3 is a FU(p)-space for all $p \in \omega^*$. It is also shown that $|\beta(\alpha) \setminus U(\alpha)| \leq 2^{\alpha}$ $\Leftrightarrow$ every space X with $t(X) < \alpha$ is p-sequential for some $p \in U(\alpha)$ $\Leftrightarrow$ every space X with $t(X) < \alpha$ is a FU(p)-space for some $p \in U(\alpha)$; if $t(X) \leq \alpha$ and $|X| \leq 2^{\alpha}$, then $\exists\, p \in U(\alpha)$ (X is a FU(p)-space).

Keywords: ultrafilter, Rudin–Frolík order, Rudin–Keisler order, p-compact, quasi M-compact, strongly M-sequential, weakly M-sequential, p-sequential, FU(p)-space, sequential, P-point

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0. Introduction.

The concept of "p-limit" (for $p \in \omega^*$) introduced by Bernstein [Be] is a very natural generalization of a convergent sequence. This notion motivates generalizations of some topological properties defined in terms of convergent sequences. In this paper, we study p-sequential/ity and FU(p)-spaces, for $p \in U(\alpha)$, which extend the concepts of sequentiality and Fréchet–Urysohn spaces, respectively. In particular, we give (in Section 2) an alternative definition of Rudin–Keisler order in terms of p-sequential/ spaces and FU(p)-spaces, for $p \in U(p)$. In Section 3, we generalize the sequential spaces $S_n (n < \omega)$ given by Arhangel'skii and Franklin [AF], and improve a result due to Boldjiev and Malykhin [BM].

1. Preliminaries.

All spaces mentioned here will be presumed to be completely regular Hausdorff (Tychonoff). If $A \subseteq X$, the closure of A will be denoted by $\mathrm{Cl}_X(A)$ or A^-. The Stone extension of a continuous function $f : X \to Y$ is denoted by $\bar{f} : \beta(X) \to \beta(Y)$, and the remainder of $\beta(X)$ by $X^* = \beta(X) \setminus X$. If $A \subseteq \beta(X)$ then $A_* = A \cap X^*$.

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The Greek letter α will stand for an infinite cardinal number, and α will also denote the discrete space whose underlying set is α. For $A \subseteq \alpha$, we write $\hat{A}$ for $\mathrm{Cl}_{\beta(\alpha)}(A)$. The Rudin–Frolík order on ω^* is defined by $p \leq_{\mathrm{RF}} q$, if there is an embedding $e \in {}^{\omega}\omega^*$ such that $\bar{e}(p) = q$ for $p, q \in \omega^*$, and Rudin–Keisler order on α^* is defined by $p \leq_{\mathrm{RK}} q$, if $\exists\, f \in {}^{\alpha}\alpha(\bar{f}(q) = p)$ for $p, q \in \alpha^*$. Notice that $\leq_{\mathrm{RF}} \subseteq \leq_{\mathrm{RK}}$. For $p, q \in \alpha^*$, we say $p \approx q$, if there is a permutation σ of α with $\bar{\sigma}(p) = q$; equivalently, $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$. The type of $p \in \alpha^*$ is $T(p) = \{q \in \omega^* : p \approx q\}$. If $M \subseteq \alpha^*$, we let $P_{\mathrm{RK}}(M) = \{q \in \alpha^* : \exists\, p \in M\,(q \leq_{\mathrm{RK}} p)\}$. The set of uniform ultrafilters on α is denoted by $U(\alpha) = \{p \in \alpha^* : \forall\, A \in p(|A| = \alpha)\}$ and $N(\alpha) = \beta(\alpha) \setminus U(\alpha)$. For $p, q \in \alpha^*$, their tensor product is defined by

$$p \otimes q = \{A \subseteq \alpha \times \alpha : \{\xi < \alpha : \{\zeta < \alpha : (\xi, \zeta) \in A\} \in q\} \in p\}$$

(for background and historical notes on tensor products see [CN2]). For $p, q \in \alpha^*$, $p \otimes q$ is an ultrafilter on $\alpha \times \alpha$ which can be viewed as an ultrafilter on α via any bijection between α and $\alpha \times \alpha$. Notice that $\otimes$ is not an associative operation on α^*. Nevertheless, Booth [Bo] pointed out that $\otimes$ induces a semigroup structure on the set of types of α^*. Thus, if $p \in \alpha^*$ and $1 \leq n < \omega$, we let p^n stand for a point in $T(p)^n$. In [Bo], the author also defined the power $T(p)^{\nu}$ for $\nu < \omega_1$ and $p \in \omega^*$ as follows:

For each $\omega \leq \nu < \omega_1$ fix an increasing sequence $\{\nu(n)\}_{n<\omega}$of ordinals in ω_1 so that

(1) $\omega(n) = n$ for $n < \omega$;
(2) if ν is a limit ordinal, then $\nu(n) \nearrow \nu$;
(3) if $\nu = \mu + m$ where μ is a limit ordinal and $m < \omega$, then $\nu(n) = \mu(n) + m$ for $n < \omega$. Let $p \in \omega^*$ and $\omega \leq \nu < \omega_1$.

Assume that $T(p)^{\mu}$ has been defined for all $\mu < \nu$. If ν is a limit ordinal, then we define $T(p)^{\nu} = T(\bar{f}_{\nu}(p))$, where $f_{\nu} \in {}^{\omega}\omega^*$ is an embedding with $f_{\nu}(n) \in T(p)^{\nu(n)}$ for $n < \omega$. If $\nu = \mu + 1$, then $T(p)^{\nu} = T(p)^{\mu} \otimes T(p)$. As above, p^{ν} stands for any point in $T(p)^{\nu}$ for $p \in \omega^*$ and for $\nu < \omega_1$. The basic properties of these powers are summarized in the following lemma.

Lemma 1.1.

(1) (*Blass* [CN2, 16.5]) *If $f \in {}^{\alpha}T(p)$ is a function such that $f(\xi) \in A^*_{\xi}$ for $\xi < \alpha$ and $\{A_{\xi} : \xi < \alpha\}$ is a partition of α, then $\bar{f}(q) \approx p \otimes q$ for $q \in \alpha^*$.*
(2) (*Booth* [Bo]) *If $\nu < \mu < \omega_1$ and $p \in \omega^*$, then $p^{\nu} <_{\mathrm{RF}} p^{\mu}$.*
(3) (*S. Garcia-Ferreira* [G1], [G2]) *If $\nu < \omega_1$ is a limit ordinal and $\omega \leq \mu < \nu$, then $p \otimes p^{\mu} \leq_{\mathrm{RF}} p^{\nu}$ for $p \in \omega^*$.*
(4) (*Booth* [Bo]) *Let $\nu < \omega_1$ and $p \in \omega^*$. If $f_{\nu} \in {}^{\omega}\omega^*$ is an embedding such that $\forall\, n < \omega(f_{\nu}(n) \approx p^{\nu(n)})$, then $\bar{f}(p) \approx p^{\nu}$.*

Bernstein [Be], in connection with problems on non-standard analysis, introduced the concept of "p-limit" ($x = p\text{-}\lim x_n \Leftrightarrow$ for each neighborhood V of x, $\{n < \omega : x_n \in V\} \in p$) and the notion of p-compactness (X is p-compact, if $\forall\, f \in {}^{\omega}X(\bar{f}(p) \in X)$) for $p \in \omega^*$. Bernstein's concepts were used by Kombarov [K1], [K2] to define topological properties which include, as particular cases, sequential and p-compact spaces. Savchenko [Sa] generalized Kombarov's concepts for arbitrary cardinals:

Definition 1.2 (Kombarov–Savchenko)**.** Let $\emptyset \neq M \subseteq \alpha^*$ and X a space. Then

(1) X is quasi M-compact, if $\forall\, f \in {}^{\alpha}X\ \exists\, p \in M(\bar{f}(p) \in X)$;

(2) X is strongly M-sequential, if for every non-closed subset A of X

$$\exists\, f \in {}^{\alpha}X\ \exists\, x \in X \setminus A\ \forall\, p \in M(\bar{f}(p) = x \wedge f(\alpha) \subseteq A);\text{ and}$$

(3) X is weakly M-sequential, if for every non-closed subset A of X

$$\exists\, f \in {}^{\alpha}X\ \exists\, p \in M(\bar{f}(p) \in X \setminus A f(\alpha) \subseteq A).$$

If $p \in \alpha^*$ and $M = \{p\}$, then strong M-sequentiality and weak M-sequentiality coincide and we say p-sequential/ity.

Let $\emptyset \neq M \subseteq \alpha^*$, X a strongly M-sequential space and $Y \subseteq X$. We define $Y_0 = Y, Y_{\eta+1} = \{x \in X :\ \exists\, f \in {}^{\alpha}Y_\eta\ \forall\, p \in M(\bar{f}(p) = x)\}$, for $\eta < \alpha^+$, and $Y_\eta = \bigcup_{\xi<\eta} Y_\xi$ if $\eta < \alpha^+$ is a limit ordinal. Notice that $\mathrm{Cl}_X(Y) = \bigcup_{\eta<\alpha^+} Y_\eta$. Denote by $\sigma_M(x)$ the infimum of the ordinals η with the property that $\mathrm{Cl}_X(Y) = Y_\eta$ for all $Y \subseteq X$. For $M = \omega^*$ we simply write $\sigma(X) : \sigma$ was first considered and studied in [AF].

Kombarov [K2] also introduced the weakly M-compact spaces for $\emptyset \neq M \subseteq \alpha^* : X$ is weakly M-compact, if for each α-sequence $\{x_\xi : \xi < \alpha\}$ there is $x \in X$ such that for all neighborhood V of x $\exists\, p \in M(\{\xi < \alpha : x_\xi \in V\} \in p)$. Next, we show that weak M-compactness coincides with quasi M^--compactness for $\emptyset \neq M \subseteq \alpha^*$. We need the following result due to Kombarov [K2].

Lemma 1.3 (Kombarov)**.** *If $\emptyset \neq M \subseteq \alpha^*$ is compact, then a space X is weakly M-compact, iff X is quasi M-compact.*

Theorem 1.4. *Let $\emptyset \neq M \subseteq \alpha^*$. A space X is weakly M-compact, if and only if X is quasi M^--compact.*

PROOF: Clearly, every weakly M-compact space is weakly M^--compact. By Lemma 1.3, we have that every weakly M-compact space is quasi M^--compact. It remains to show that every quasi M^--compact space is weakly M-compact. In fact, Let X be a quasi M^--compact space and $\{x_\xi\}_{\xi<\alpha}$ an α-sequence in X. Define $f \in {}^{\alpha}X$ by $f(\xi) = x_\xi$ for $\xi < \alpha$. Then $\exists\, p \in M^-(\bar{f}(p) = x \in X)$. For an open neighborhood V of x, we write $H(V)$ for $\{\xi < \alpha : x_\xi \in V\}$. Then $\bar{f}^{-1}(x) = \bigcap\{H(V)\hat{}\ : V \text{ is a neighborhood of } x\}$. If W is a neighborhood of x, there is $q \in M \cap H(W)\hat{}$ (because $p \in M^-$); hence, $H(W) = \{\xi < \alpha : x_\xi \in W\} \in q$. □

2. On p-sequential/ spaces.

In [K1], the author considered the following examples: For $p \in \alpha^*$, the subspace $\xi(p) = \alpha \cup \{p\}$ of $\beta(\alpha)$ is p-sequential/; and if $\emptyset \neq M \subseteq \alpha^*$, the space $\Xi(M) = \sum_{p\in M} \xi(p)$ (the disjoint union of the spaces $\xi(p)$ for $p \in M$) is weakly M-sequential. As a direct consequence of the following theorem, we obtain an alternative definition (Corollary 2.2) of Rudin–Keisler order.

Theorem 2.1. *For $M, N \subseteq \alpha^*$, the following statements are equivalent.*

(1) *weak M-sequentiality implies weak N-sequentiality:*
(2) *$\Xi(M)$ is weakly N-sequential.*
(3) *$M \subseteq P_{\mathrm{RK}}(N)$.*

PROOF: (1) $\Rightarrow$ (2). This is evident.

(2) $\Rightarrow$ (3). By hypothesis $\Xi(M) = \sum_{r \in M} \Xi(r)$ is weakly N-sequential. Fix $p \in M$ and $A \in p$. Then A is not a closed subset of $\xi(p) \subseteq \Xi(M)$. Since $\Xi(p)$ is weakly N-sequential, $\exists\, f \in {}^{\alpha}A\ \exists q \in N(\bar{f}(q) \in \xi(p) \setminus A \subseteq \Xi(M))$. Hence, $\bar{f}(q) = p$, that is, $p \leq_{\mathrm{RK}} q$. Thus, $M \subseteq P_{\mathrm{RK}}(N)$.

(3) $\Rightarrow$ (1). Let X be a weakly M-sequential space and A a non-closed subset of X. Then $\exists f \in {}^{\alpha}X\ \exists\, q \in M(f(\alpha) \subseteq A \wedge \bar{f}(q) \in X \setminus A)$. Choose $p \in N$ and $g \in {}^{\alpha}\alpha$ such that $\bar{g}(p) = q$. Define $h = f \circ g$. Then $h \in {}^{\alpha}X, h(\alpha) \subseteq A$ and $\bar{h}(p) = \bar{f}(\bar{g}(p)) = \bar{f}(q) \in X \setminus A$. Therefore, X is weakly N-sequential. $\square$

Corollary 2.2. *For $p, q \in \alpha^*$, the following conditions are equivalent.*

(1) $p \leq_{\mathrm{RK}} q$*;*
(2) *every p-sequential/ space is q-sequential;*
(3) *$\xi(p)$ is q-sequential; and*
(4) *every* FU(p)*-space is a* FU(q)*-space.*

The cardinality of a p-sequential/ space can be estimated as follows:

Theorem 2.3. *Let $p \in U(\alpha)$. If X is p-sequential/ and $A \subseteq X$, then $|A^-| \leq |A|^\alpha$. In particular, we have that $|X| \leq d(X)^\alpha$.*

PROOF: Let X be p-sequential/ and $A \subseteq X$. Set $A_p = \bigcup_{\eta < \alpha^+} A_\eta$. Since X is p-sequential/, then $A^- = A_p$ for $A \subseteq X$. By transfinite induction, we have that $|A_\eta| \leq |\bigcup_{\xi < \eta} A_\xi|^\alpha \leq (\sum_{\xi < \eta} |A_\xi|^\alpha)^\alpha \leq |A|^\alpha$ for all $\eta < \alpha^+$. Thus

$$|A^-| = |A_p| \leq \sum_{\eta < \alpha^+} |A_\eta| \leq |A|^\alpha$$

$\square$

3. On FU(p)-spaces.

A very natural generalization of Fréchet–Urysohn spaces by using the Bernstein's notion of "p-limit" is suggested in the next definition due to Comfort [G1] and Savchenko (see [BM]), independently.

Definition 3.1 (Comfort–Savchenko)**.** Let $\emptyset \neq M \subseteq \alpha^*$. A space X is a FU(M)-space, if for each $A \subseteq X$ and $x \in A^-$, $\exists\, f \in {}^{\alpha}A\ \forall\, p \in M(\bar{f}(p) = x)$.

Observe that the class of Fréchet–Urysohn spaces coincides with the class of FU(ω^*)-spaces. For $p \in U(\alpha)$, we write FU(p)-space for FU$(\{p\})$-space. In this section, we are principally concerned with FU(p)-spaces (for $p \in U(\alpha)$). It is evident that every Fréchet–Urysohn space is a FU(p)-space for all $p \in \omega^*$, and every FU(p)-space is p-sequential/ for $p \in U(\alpha)$. Our first aim is to show that there is a p-sequential/ space which is not a FU(p)-space. We slightly modify the construction of the sequential spaces S_n for $n<\omega$ (see [AF]). It suffices to give the generalization of the Arens space S_2, since for $2<n<\omega$ the definition should be clear. Indeed, for $p\in U(\alpha)$ we define $S_2(p)$ by

(1) $S_2(p)=\{x\}\cup\{x_\xi:\xi<\alpha\}\cup\{x_{\xi,\zeta}:\xi,\zeta<\alpha\}$;
(2) $x=p\text{-}\lim x_\xi$;
(3) $x_\xi=p\text{-}\lim x_{\xi,\zeta}$ for $\xi<\alpha$;
(4) $\{x_{\xi,\zeta}:\xi,\zeta<\alpha\}$ is discrete;
(5) for a base of neighborhoods at x : if $r=(A_\xi)_{\xi<\alpha}$ is an α-sequence such that $A_\xi\in p$ for $\xi<\alpha$ and $A\in p$, we define

$$V(r,A)=\{x\}\cup\{x_{\xi,\zeta}:\xi\in A\text{ and }\zeta\in A_\xi\}\cup\{x_\xi:\xi\in A\}.$$

It is not hard to prove that $S_2(p)$ is p-sequential/ for each $p\in\alpha^*$. For $2<n<\omega$, we may define $S_n(p)$ in a manner similar to the definition of S_n in [AF]. Observe that $S_n(p)\subseteq S_{n+1}(p)$ for $n<\omega$ and $p\in U(\alpha)$.

We can embed $S_n(p)$ in $\beta(\alpha)$. In fact, fix $n<\omega$ and $p\in U(\alpha)$. For $\xi_1<\alpha$ let $\{A(\xi_1):\xi_1<\alpha\}$ be a partition of α in subsets of cardinality α. For $k<n$ and for $(\xi_1,\dots,\xi_k)\in {}^k\alpha$, let $\{A(\xi_1,\dots,\xi_{k+1}):\xi_{k+1}\in\alpha\}$ be a partition of $A(\xi_1,\dots,\xi_k)$ in subsets of cardinality α. For $(\xi_1,\dots,\xi_n)\in {}^n\alpha$, take $p(\xi_1,\dots,\xi_{n-1})\in A(\xi_1,\dots,\xi_n)^*$ to be the p-limit of the α-sequence $A(\xi_1,\dots,\xi_n)$. Inductively, we define $p(\xi_1,\dots,\xi_{k-1})=p\text{-}\lim p(\xi_1,\dots,\xi_k)$, which is in $A(\xi_1,\dots,\xi_{k-1})^*$, for $1<k<\omega$ and $(\xi_1,\dots,\xi_{k-1})\in {}^{k-1}\alpha$. Now, we define $s=p\text{-}\lim p(\xi_1)$. It is then evident that $S_n(p)$ is homeomorphic to $\{s\}\cup\{p(\xi_1,\dots,\xi_k):(\xi_1,\dots,\xi_k)\in {}^k\alpha\text{ and }0<k<n\}\cup\alpha$ with the topology inherited from $\beta(\alpha)$. By Lemma 1.1 (1), $p(\xi_1,\dots,\xi_k)\in T(p)^{n-k}$ for $0<k<n$ and for $(\xi_1,\dots,\xi_k)\in {}^k\alpha$. Thus, we may assume that $s=p^n$.

Theorem 3.2. *Let $p,q\in U(\alpha)$ and $0<n<\omega$. Then $S_n(p)$ is a p-sequential/ space, and $S_n(p)$ is a* FU*(q)-space, if and only if $p^n\le_{\rm RK} q$.*

Proof: The first assertion follows directly from the definition. Assume that $S_n(p)=\{p^n\}\cup\{p(\xi_1,\dots,\xi_k):(\xi_1,\dots,\xi_k)\in {}^k\alpha\}\cup\alpha$ as above and $S_n(p)$ is a FU(q)-space. Since $p^n\in \mathrm{Cl}_{S_n(p)}(\alpha)$, then there is $f\in {}^\alpha\alpha$ such that $\bar f(q)=p^n$, that is, $p^n\le_{\rm RK} q$. Conversely, suppose that $p^n\le_{\rm RK} q$. For $0<k<n$, we let $\Gamma_0=\alpha$ and $\Gamma_k=\{p(\xi_1,\dots,\xi_k):(\xi_1,\dots,\xi_k)\in {}^k\alpha\}$. Notice that Γ_k is a strongly discrete subset of $U(\alpha)$ and that $\Gamma_k\subseteq T(p)^{n-k}$ ($T(p)^0=\alpha$) for $0<k\le n$. Let $A\subseteq S_n(p)$ and $x\in A^-\setminus A$. Choose $0<j\le n$ so that $x\in(A\cap\Gamma_j)^-$ (this is possible, since $S_n(p)=\{p^n\}\cup\bigcup_{0<k\le n}\Gamma_k$). Let $f\in {}^\alpha[A\cap\Gamma_j]$ be a bijection and let $r\in\alpha^*$ such that $\bar f(r)=x$. Since Γ_j is a strongly discrete subset of α^*, then $r\le_{\rm RK} x$. By hypothesis, we have that $x\le_{\rm RK} q$ and so $r\le_{\rm RK} q$. Let $g\in {}^\alpha\alpha$ with $\bar g(q)=r$ and define $h=f\circ g$. Then, $h(\alpha)\subseteq A$ and $\bar h(q)=x$. $\square$

The spaces S_n's can be generalized for each $p\in\omega^*$ and for each $\nu<\omega_1$ as follows: Assume that $S_\nu(p)$ has been defined for $\nu<\mu<\omega_1$ and $p\in\omega^*$ so that $\omega\subseteq S_\nu(p)\subseteq\beta(\omega)$. Let $\{A_n:n<\omega\}$ be a partition of ω with $|A_n|=\omega$ for $n<\omega$. Without loss of generality, we may suppose that $A_n\subseteq S_{\nu(n)}(p)$ and $p^{\nu(n)}\in S_{\nu(n)}(p)\subseteq\hat A_n$ for $n<\omega$, where $\{\nu(n)\}_{n<\omega}$ is the sequence of ordinals as in the definition of p^ν. By Lemma 1.1 (4), we have that $p^\nu \approx p\text{-}\lim p^{\nu(n)}$. Thus, we define $S_\nu(p) = \{p^\nu\} \cup \bigcup_{n<\omega} S_{\nu(n)}(p)$ with the subspace topology from $\beta(\omega)$. It is not hard to see that $S_\nu(p)$ is p-sequential/. Moreover, we have the following theorem. Before stating it, we proved a lemma.

Lemma 3.3. *Let $p \in \omega^*$ and $\nu < \omega_1$. If $X \subseteq S_\nu(p)$ and $p^\nu \in X^-$, then there is a discrete subset Y of ω^* such that $Y \subseteq X$ and $p^\nu \in Y^-$.*

Proof: We proceed by transfinite induction. In the proof of Theorem 3.2, we showed that the conclusion holds for each $n < \omega$. Assume that the lemma is true for all $\mu < \nu < \omega_1$. Let $\{A_n : n < \omega\}$ be a partition of ω in infinite subsets. We may suppose that $A_n \subseteq S_{\nu(n)}(p) \subseteq \hat{A}_n$ for $n < \omega$. Let $X \subseteq S_\nu(p)$ such that $p_\nu \in X^- \setminus X$. Notice that if $A \in p, B_n \in p^{\nu(n)}$ and $B_n \subseteq A_n$ for $n < \omega$, then $\bigcup_{n \in A} B_n \in p^{\nu(n)}$. Hence $D = \{n < \omega : p^{\nu(n)} \in [S_{\nu(n)}(p) \cap X]^-\} \in p^\nu$. By induction hypothesis, for each $n \in D$ there is $X_n \in S_{\nu(n)} \cap X$ discrete (in ω^*) such that $p^{\nu(n)} \in X_n^-$. It is then evident that $Y = \bigcup_{n \in D} X_n$ satisfies the conclusion of the lemma. $\square$

Theorem 3.4. *Let $p, q \in \omega^*$ and $\nu < \omega_1$. Then $S_\nu(p)$ is a* FU(q)*-space, if and only if $p^\nu \leq_{\text{RK}} q$.*

Proof: $\Rightarrow$). Assume that $S_\nu(p)$ is a FU(q)-space. Since $\omega \subseteq S_\nu(p)$, then there is a function $f \in {}^\omega\omega$ such that $\bar{f}(q) = p^\nu$, that is $p^\nu \leq_{\text{RK}} q$.

$\Leftarrow$). We proceed by transfinite induction. Suppose that the conclusion holds for all $\mu < \nu < \omega_1$. According to Theorem 3.2, we may assume that $\omega \leq \nu$. Let $X \subseteq S_\nu(p)$. We will verify that each point x in X^- is a q-limit of some sequence in X. Indeed, if $x \neq p^\nu$, then $x \in \text{Cl}_{S_\nu(p)}[S_{\nu(m)} \cap X] \subseteq \hat{A}_m$ for some $m < \omega$. Then, we apply the induction hypothesis, since $p^{\nu(m)} \leq_{\text{RK}} q$. If $x = p^\nu$, by Lemma 3.3, there is $Y \subseteq X$ discrete such that $p^\nu \in Y^-$. Choose a bijection $f \in {}^\omega Y$ and $r \in \omega^*$, for which $\bar{f}(r) = p^\nu$. Since f is an embedding, then $r \leq_{\text{RF}} p^\nu \leq_{\text{RK}} q$. Let $g \in {}^\omega\omega$ with $\bar{g}(q) = r$ and define $h = f \circ g$. Then $\bar{h}(q) = p^\nu$ and $h(\omega) \subseteq X$. $\square$

In [BM], the authors showed that every sequential (Hausdorff) space is a $\mathcal{F}$-Fréchet–Urysohn space (hence it is a FU(p)-space for some $p \in \omega^*$): The proof of their result involves infinite powers of filters. We slightly improve this result by using the basic idea from [BM].

Theorem 3.5. *Let $p, q \in \omega^*$. Then every p-sequential space is a* FU(q)*-space, if and only if $\forall\ \nu < \omega_1$ $(p^\nu \leq_{\text{RK}} q)$.*

Proof: $\Rightarrow$). This follows directly from Theorem 3.4.

$\Leftarrow$). Let $q \in \omega^*$ satisfy $\forall\ \nu < \omega_1$ $(p^\nu \leq_{\text{RK}} q)$ and let X be a p-sequential/ space. We show that X is a FU(q)-space. Indeed, fix $A \subseteq X$ and define $A_0 = A$ and $A_\mu = \{\bar{f}(p) : f \in {}^\omega[\bigcup_{\nu<\mu} A_\nu]$ and $\bar{f}(p) \in X\}$ for $\mu < \omega_1$. It is evident that $A^- = \bigcup_{\mu<\omega_1} A_\mu$, since X is p-sequential/. We claim that $A^- = B$, where $B = \{\bar{f}(p^\nu) : f \in {}^\omega A, \bar{f}(p^\nu) \in X$ and $\nu < \omega_1\}$. Assume that $\bigcup_{\nu<\mu} A_\nu \subseteq B$ and let $x \in A_\mu \setminus \bigcup_{\nu<\mu} A_\nu$ for $\mu < \omega_1$. Choose $f \in {}^\omega[\bigcup_{\nu<\mu} A_\nu]$ such that $\bar{f}(p) = x$. By induction hypothesis, for each $n < \omega$ there is $\nu_n < \omega_1$ and $g_n \in {}^\omega A$, for which $\bar{g}_n(p^{\nu_n}) = f(n)$. Let $\theta = \lim \nu_n + 1$. Then, by Lemma 1.2 (2), we have that

$\forall\, n < \omega(p^{\nu_n} \leq_{\mathrm{RF}} p^{\theta})$. Let $\{A_n : n < \omega\}$ and $\{B_n : n < \omega\}$ be a partition of ω in infinite sets. Without loss of generality, we may suppose that $p^{\nu_n} \in \hat{B}_n$ for $n < \omega$. For each $n < \omega$, pick $r_n \in T(p^{\theta}) \cap \hat{A}_n$ and $h_n : A_n \to B_n$ such that $\bar{h}_n(r_n) = p^{\nu_n}$ for $n < \omega$. Set $h = \bigcup_{n<\omega} g_n \circ h_n$ and define $e \in {}^{\omega}\omega^*$ by $e(n) = r_n$ for $n < \omega$. Then $\bar{h}(e(n)) = \bar{g}_n(p^{\nu_n}) = f(n)$ for all $n < \omega$. Hence $\bar{h}(\bar{e}(p)) = \bar{f}(p) = x$. Let $\mu < \omega_1$ be a limit ordinal with $\theta < \mu$. By Lemma 1.1 (3), we have that $p \otimes p^{\theta} \leq_{\mathrm{RF}} p^{\mu}$. Since $p \otimes p^{\theta} \approx \bar{e}(p)$ (by Lemma 1.1 (1)), then there is $d \in {}^{\omega}\omega$ such that $\bar{d}(p^{\mu}) = \bar{e}(p)$. Hence, we have that $\bar{\phi}(p^{\mu}) = x$, where $\phi = h \circ d$ and so $x \in B$. Thus $A^- \subseteq B$. By a similar argument, we may show that $B \subseteq A^-$. This proves our claim. It follows that X is a FU(q)-space. $\square$

Boldjiev and Malykhin [BM] asked whether it is consistent with ZFC that every sequential (compact) space is a FU(p)-space for all $p \in \omega^*$. Notice, from Theorem 3.8 below, that s_2 is a FU(p)-space, iff p is not a P-point. We will show that it is independent with ZFC that every sequential space X with $\sigma(X) < \omega$ is a FU(p)-space for all $p \in \omega^*$. First, we give some notation.

For $p, q \in \omega^*$, we say $p \leq_l q$, if there is $f \in {}^{\omega}\omega$ such that $f|_A$ is not finite-to-one for all $A \in p$ and $\bar{f}(q) = p$.

Theorem 3.6. *For $p \in \omega^*$ and $1 < n < \omega$, the following conditions are equivalent.*

(1) *S_n is a FU(p)-space;*
(2) *there is $\{p_1, \dots, p_{n-2}\} \subseteq \omega^*$ (here, $p_0 = p$) such that p_{n-2} is not a P-point and $p_{n-2} <_{\mathrm{RK}} \cdots <_{\mathrm{RK}} p_1 <_l p$;*
(3) *every sequential space X with $\sigma(X) \leq n$ is a FU(p)-space.*

Proof: (1) $\Rightarrow$ (2). Let $S_n = \{s\} \cup \{x_{k_1,\dots,k_i} : k_j < \omega \text{ for } 1 \leq j \leq i \leq n\}$ and assume that S_n is a FU(q)-space. Set $Y = \{x_{k_1,\dots,k_n} : k_1, \dots, k_n < \omega\}$. By assumption, there is a function $f \in {}^{\omega}Y$ such that $\bar{f}(p) = s$. Without loss of generality, we may assume that f is onto. For each sequence $(k_1, \dots, k_{n-1})$, let $E(k_1, \dots, k_{n-1}) = f^{-1}(\{x_{k_1,\dots,k_{n-1},k} : k < \omega\})$. By induction, for each $1 \leq j < n-1$ and each sequence $(k_1, \dots, k_j)$ we define $E(k_1, \dots, k_j) = \bigcup_{k<\omega} E(k_1, \dots, k_j, k)$. Notice that $\bar{f}(E(k_1, \dots, k_j)^*) = \{x_{k_1,\dots,k_j}\}$ for $1 \leq j < n$ and for $(k_1, \dots, k_j)$. For $1 \leq j < n$ and $A \in p$, we have that

$$(*) \qquad \begin{aligned}\{k_1 < \omega : |\{k_2 < \omega : \dots |\{k_{j-1} : |\{k_j < \omega : |E(k_1, \dots, k_j) \cap A| = \omega\}| \\ = \omega\}| = \omega \dots\}| = \omega\} \in p.\end{aligned}$$

This fact $(*)$ follows from the definition of the topology of S_n and $\bar{f}(p) = s$. Let $f_1 \in {}^{\omega}\omega$ with the fibers $\{E(k_1, \dots, k_{n-1}) : k_1, \dots, k_{n-1} < \omega\}$ and let $p_1 = \bar{f}_1(p)$. If $\exists\, A \in p(f_1|_A$ is finite-to-one), then we can find a neighborhood V of s in $\beta(S_n)$ such that $V \cap f(A) = \emptyset$, which is a contradiction, Hence $p_1 <_l p$ via f_1. For $j = n-1$, by $(*)$, we obtain that if $A \in p$, then A hits infinitely many sets of the partition $\{E(k_1, \dots, k_{n-1}) : k_1, \dots, k_{n-1} < \omega\}$ in an infinite set; hence p is not a P-point. In particular, for $n = 2$ the conclusion holds. Thus, we may suppose that $n > 2$. Assume that we have defined $f_j \in {}^{\omega}\omega$ and $p_j \in \omega^*$ for $1 \leq j < i \leq n-2$ such that

(1) $\bar{f}_j(p_{j-1}) = p_j$ for $1 < j < i$;
(2) $p_{i-1} <_{\mathrm{RK}} \cdots <_{\mathrm{RK}} p_1 <_l p$; and
(3) p_j is not a P-point for $j < i$.

Choose $f_i \in {}^\omega\omega$ with the fibers $\{f_{i-1} \circ \cdots \circ f_1(E(k_1, \ldots, k_{n-i})) : k_1, \ldots, k_{n-i} < \omega\}$, and let $p_i = \bar{f}_i(p_{i-1})$. We claim that p_i satisfies $p_i <_{\mathrm{RK}} p_{i-1}$ and p_i is not a P-point. Indeed, if $A \in p_i$, then $B = f_1^{-1} \circ \cdots \circ f_{i-1}^{-1}(A) \in p$ and, by $(*)$, B meets infinitely many elements of the partition $\{f_i \circ \cdots \circ f_1(E(k_1, \ldots, k_{n-i-1})) : k_1, \ldots, k_{n-i-1} < \omega\}$ in an infinite set. Thus, p_i is not a P-point. Moreover, the fact $(*)$ also implies that $f_i \circ \cdots \circ f_1 \mid_A$ cannot be one-to-one for $A \in p$. Then, by Theorem 9.2 of [CN2] (see [C1]), we have that p_i is not equivalent to $p_{i-1} = \bar{f}_{i-1} \circ \cdots \circ \bar{f}_1(p)$. This proves our claim. Therefore, $p_1, \ldots, p_{n-1}$ satisfy the conclusion.

(2) $\Rightarrow$ (3). Clearly, the conclusion holds for $n = 1$. Assume that (3) is true for all $0 < j < n$. Let X be a sequential space with $\sigma(X) = n$ and let $Y \subseteq X$. Define $Z = \{x \in X : \exists f \in {}^\omega Y(\bar{f}(p) = x)\}$. We verify that $Y^- = Z$. In fact, by assumption we have that $Y^- = \bigcup_{j<n} Y_j$ and $Y_1 \subseteq Z$. If $y \in Y_k \setminus Y_{k-1}$ for $1 < k < n$, the space $S_k = \{s\} \cup \{x_{n_1,\ldots,n_j} : n_1, \ldots, n_j < \omega \text{ and } 1 \le j \le k\}$ can be embedded in Y^- so that $s = y$ and $\{x_{n_1,\ldots,n_k} : n_1, \ldots, n_k < \omega\} \subseteq Y$. By induction hypothesis, $\exists g \in {}^\omega Y(\bar{g}(p) = y)$, that is, $y \in Z$. We only need to show that $Y_n \subseteq Z$. Fix $x \in Y_n \setminus Y_{n-1}$and let $S_n = \{x\} \cup \{x_{k_1,\ldots,k_j} : k_1, \ldots, k_j < \omega$ and $1 \le j \le n\} \subseteq Y^-$ so that $\{x_{k_1,\ldots,k_n} : k_1, \ldots, k_n < \omega\} \subseteq Y$. For $1 \le j < n-1$, let $f_j \in {}^\omega\omega$ such that $\bar{f}_j(p_{j-1}) = p_j$ and $\forall A \in p$ ($f_1 \mid_A$ is not finite-to-one). We may assume that f_j is onto for $1 \le j < n-1$. Let $\{A_{k_1} : k_1 < \omega\}$ be a partition of ω witnessing that p_{n-2} is not a P-point. Define

$$E(k_1) = f_1^{-1} \circ \cdots \circ f_{n-2}^{-1}(A_{k_1}) \text{ for } k_1 < \omega \text{ and } E(k_1, \ldots, k_j) =$$
$$= \{f_1^{-1} \circ \cdots \circ f_{n-j}^{-1}(\{k_j\}) : k_j \in f_{j-1}^{-1}(\{k_{j-1}\}),\ k_{j-1} \in f_{j-2}^{-1}(\{k_{j-2}\}), \ldots,$$
$$k_3 \in f_2^{-1}(\{k_2\}) \text{ and } k_2 \in A_{k_1}\} \text{ for } 1 < j \le n-1 \text{ and for } (k_1, \ldots, k_j).$$

Clearly, $\mathcal{A} = \{E(k_1, \ldots, k_j) : k_1, \ldots, k_j < \omega \text{ and } 1 \le j \le n-1\}$ satisfies $(*)$ of the proof of (1) $\Rightarrow$ (2) above. Choose $f \in {}^\omega Y$ so that f is a bijection between $E(k_1, \ldots, k_{n-1})$ and $\{x_{k_1,\ldots,k_{n-1},k} : k < \omega\}$ for each sequence $(k_1, \ldots, k_{n-1})$. Since $\mathcal{A}$ satisfies $(*)$, then $\bar{f}(E(k_1, \ldots, k_j)^*) = x_{k_1,\ldots,k_j}$ for each $1 \le j < n$ and each sequence $(k_1, \ldots, k_j)$. Thus $\bar{f}(p) = x$; that is, $x \in Z$.

(3) $\Rightarrow$ (1). This is evident. $\square$

Let M_S be the Shelah's model of ZFC in which $M_S \models \omega^*$ does not have P-points (see [M] and [W]). It follows from 3.6 that $M_S \models$ every sequential space X with $\sigma(X) < \omega$ is a FU(p)-space for all $p \in \omega^*$. On the other hand, it is a direct consequence from Theorem 3.6 and the next lemma that $MA \models \exists\, p \in \omega^*$ (S_3 is not a FU(p)-space), since every RK-minimal ultrafilter is a P-point (see [C] or [CN2]) and RK-minimal points exist assuming MA ([Bo]).

Lemma 3.7. *If $p \in \omega^*$ is* RK-*minimal, then $P_{\mathrm{RK}}(p^n) = \bigcup_{1 \le k \le n} T(p^k)$ for each $1 \le n < \omega$.*

Proof: Assume that the conclusion holds for each $1 \le j \le n$. According to Lemma 1.1 (1), $p^{n+1} = \bar{e}(p)$, where $e \in {}^\omega T(p^n)$ is an embedding. Let $q \in \omega^*$

with $q \leq_{\mathrm{RK}} p^{n+1}$ and let $f \in {}^{\omega}\omega$ such that $\bar{f}(p^{n+1}) = q$. By Lemma 9.4 of [C], we may suppose that $\bar{f} \circ e$ is an embedding and $\bar{f}(e(m)) \in \omega^*$ for $m < \omega$. By induction hypothesis, we have that $\bar{f}(e(\omega)) \subseteq \bigcup_{1 \leq k \leq n} T(p^k)$. This implies that $\{m \subset \omega : \bar{f}(e(m)) \in T(p^k)\} \in p$ for some $1 \leq k \leq n$. Then, by Lemma 1.1 (1), we obtain that $q = \bar{f}(\bar{e}(p)) \approx p^{k+1}$. □

For infinite ordinals $\omega \leq \nu < \omega_1$, we have the following corollary.

Lemma 3.8. *Let $q \in \omega^*$ such that there is $\{p_\nu : 1 \leq \nu \leq \theta\} \subseteq \omega^*$, for $\theta < \omega_1$, such that $\forall\, 1 \leq \mu < \nu \leq \theta$ $(p_\mu <_{\mathrm{RF}} p_\nu \leq_{\mathrm{RK}} q)$. If X is a sequential space and $x \in Y_\theta$ for $Y \subseteq X$, then $\exists\, f \in {}^{\omega}Y(\bar{f}(q) = x)$.*

PROOF: Clearly, the conclusion holds for $\theta = 1$. We proceed by transfinite induction. Assume that the lemma is true for all $1 \leq \nu < \theta < \omega_1$. We need the following fact (for a proof see [Bo, Lemma 2.20]):

$$(*)\qquad \text{If } f, g \in {}^{\omega}\omega^* \text{ are embeddings and } p \in \omega^*, \text{ then } \bar{f}(p) <_{\mathrm{RF}} \bar{g}(p), \text{ if and only if } \{n < \omega : f(n) <_{\mathrm{RF}} g(n)\} \in p.$$

For every $1 < \nu \leq \theta$, let $e_\nu \in {}^{\omega}\omega^*$ be an embedding such that $\bar{e}_\nu(p_1) = p_\nu$. In virtue of $(*)$, we may assume that $\forall\, 1 < \mu < \nu \leq \theta\ \forall\, n < \omega\ (e_\mu(n) <_{\mathrm{RF}} e_\nu(n))$. Let $Y \subseteq X$ and fix $x \in Y_\theta$. Without loss of generality, we may suppose that $\theta = \nu + 1$. Then there is a sequence $\{x_n\}_{n<\omega}$ in Y_ν, for which $x_n \to x$. Let $\{A_n : n < \omega\}$ be a partition of ω such that $\forall\, n < \omega(e_\theta(n) \in \hat{A}_n)$. Applying the induction hypothesis to ν and $e_\theta(n)$ for each $n < \omega$, we have that $\forall\, n < \omega\ \exists\, f_n : A_n \to Y(\bar{f}_n(e_\theta(n)) = x_n)$. Let $f = \bigcup_{n<\omega} f_n$ and $g \in {}^{\omega}\omega$ such that $\bar{g}(q) = p_\theta$. Define $h = f \circ g$. Then, $\bar{h}(q) = \bar{f}(\bar{g}(q)) = \bar{f}(p_\theta) = \bar{f}(\bar{e}_\theta(p_1)) = x$. □

As an immediate consequence of Lemma 3.8 we have:

Corollary 3.9. *If $q \in \omega^*$ satisfies the conditions of Lemma 3.8 for $\theta < \omega_1$, then every sequential space X with $\sigma(X) \leq \theta$ is a* FU(q)*-space.*

It was pointed out in [K1] and [Sa] that every weakly M-sequential space X satisfies $t(X) \leq \alpha$. However, the converse is not true. For instance, take $M \subseteq \alpha^*$ with $2^\alpha < |M|$. By Theorem 2.1, $t(\Xi(M)) = \alpha$ and $\Xi(M)$ cannot be p-sequential/, since $|P_{\mathrm{RK}}(p)| \leq 2^\alpha$. The next two theorems show that the opposite holds under certain additional assumptions. We need the following result due to Comfort and Negrepontis [CN1], [CN2].

Lemma 3.10 (Comfort–Negrepontis). *If $A \subseteq \alpha^*$ with $|A| \leq 2^\alpha$, then there is $p \in U(\alpha)$ such that $\forall\, q \in A(q \leq_{\mathrm{RK}} p)$.*

Theorem 3.11. *For $\omega < \alpha$, the following conditions are equivalent.*

(1) $|N(\alpha)| \leq 2^\alpha$;
(2) *if X is a space and $t(X) < \alpha$, then $\exists\, p \in U(\alpha)$ (X is p-sequential/);*
(3) *if X is a space with $t(X) < \alpha$, then $\exists\, p \in U(\alpha)$ (X is a* FU(p)*-space).*

PROOF: (1) $\Rightarrow$ (3). Let X be a space with $t(X) < \alpha$. For every $x \in X$ let $W_x = \{p \in N(\alpha) : \exists\, f \in {}^{\gamma}X(\gamma < \alpha \wedge p \in U(\gamma) \wedge \bar{f}(p) = x \notin f(\gamma))\}$. It is evident that $\bigcup_{x\in X} W_x \subseteq N(\alpha)_*$. By Lemma 3.10, $\exists\, p \in U(\alpha)\; \forall\, q \in \bigcup_{x\in X} W_x (q \leq_{\text{ RK}} p)$. We verify that X is a FU(p)-space. Indeed, let $A \subseteq X$ and $x \in A^- \setminus A$. Since $t(X) < \alpha$, then $\exists\, \omega \leq \gamma < \alpha\; \exists\, q \in U(\gamma)\; (\bar{f}(q) = x)$. By definition, we have that $q \in W_x$. Choose $g \in {}^{\alpha}\gamma$ such that $\bar{g}(p) = q$ and set $h = f \circ g$. Then $h(\alpha) \subseteq A$ and $\bar{h}(p) = x$. Thus X is a FU(p)-space.

(3) $\Rightarrow$ (2). This is evident.

(2) $\Rightarrow$ (1). We consider two cases:

(a) Assume that $\alpha = \gamma^+$. We then have that $t(\Xi(N(\alpha)_*)) = \gamma < \alpha$. By hypothesis, $\exists\, p \in U(\alpha)$ ($\Xi(N(\alpha)_*)$ is p-sequential/). Hence, $\forall\, q \in N(\alpha)_*$ ($\xi(q)$ is p-sequential/). By Theorem 2.1, $\forall\, q \in N(\alpha)_*$ ($q \leq_{\text{ RK}} p$) and so $|N(\alpha)| \leq 2^{\alpha}$.

(b) Assume that α is a limit cardinal. Suppose that $|N(\alpha)| > 2^{\alpha}$. Since $|N(\alpha)| = a^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^{\gamma}}$, we have that $a^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^{\gamma}} > 2^{\alpha}$; hence $\exists\, \gamma < \alpha(2^{\alpha} < 2^{2^{\gamma}})$. Set $\delta = \gamma^+ < \alpha$. By applying the same argument to $\Xi(N(\delta)_*)$ as in case (a), we have that $2^{2^{\gamma}} \leq |N(\delta)| = \delta^{<\delta} \cdot \sum_{\kappa<\delta} 2^{2^{\kappa}} \leq 2^{\delta} \leq 2^{\alpha}$, a contradiction. $\square$

We remark that if α is a strong limit cardinal, then any of the three conditions of Theorem 3.11 holds in ZFC, and assuming GCH, it is true for all cardinals. The clause (1) of Theorem 3.11 does not hold for ω_1 in any model M of ZFC, in which $M \models 2^{\omega_1} < 2^c$. Indeed, if M is such a model of ZFC, then $M \models |N(\omega_1)| = 2^c > 2^{\omega_1}$ and $t(\Xi(N(\omega_1)_*)) = \omega$, so in M the space $\Xi(N(\omega_1)_*)$ cannot be p-sequential/ for all $p \in U(\omega_1)$ (by Theorem 2.1). If we replace $t(X) < \alpha$ by $t(X) = \alpha$ in the clause (2) (or (3)) in Theorem 3.11, then this is not a theorem of ZFC. In fact, Fedorčuk [F] using diamond, defined a compact separable space x with $t(X) = \omega$ and $|X| = 2^c$; hence X is not p-sequential/ for all $p \in \omega^*$ (by Theorem 2.3). On the other hand, Balogh [Ba] proved that PFA implies that every compact space with countable tightness is sequential.

Theorem 3.12. *If X is a space satisfying $t(X) \leq \alpha$ and $|X| \leq 2^{\alpha}$, then there is $p \in U(\alpha)$ such that X is a* FU(p)*-space.*

PROOF: Enumerate X by $\{x_\xi : \xi < 2^{\alpha}\}$. For each $\xi < 2^{\alpha}$, define $F_\xi = \{f \in {}^{\alpha}X : \exists\, q \in \alpha^*\, (\bar{f}(q) = x_\xi)\}$. Clearly, $|F_\xi| \leq 2^{\alpha}$ for $\xi < 2^{\alpha}$. Hence, we may enumerate F_ξ by $\{f_{\xi,\zeta} : \zeta < 2^{\alpha}\}$ for $\xi < 2^{\alpha}$. For each $(\xi,\zeta) \in 2^{\alpha} \times 2^{\alpha}$, choose $p(\xi,\zeta) \in \alpha^*$ so that $\bar{f}_{\xi,\zeta}(p(\xi,\zeta)) = x_\xi$. Set $M = \{p(\xi,\zeta) : (\xi,\zeta) \in 2^{\alpha} \times 2^{\alpha}\}$. By Lemma 3.10, $\exists\, p \in U(\alpha)\; \forall\, q \in M(q \leq_{\text{ RK}} p)$. We claim that X is a FU(p)-space. Indeed, let $A \subseteq X$ and fix $x_\xi \in \mathrm{Cl}_X(A) \setminus A$ for some $\xi < 2^{\alpha}$. Since $t(X) \leq \alpha$, then $\exists\, g \in {}^{\alpha}A\; \exists\, q \in \alpha^*\; (\bar{g}(q) = x_\xi)$. Hence, there is $\zeta < 2^{\alpha}$ with $g = f_{\xi,\zeta}$ and $\bar{f}_{\xi,\zeta}(p(\xi,\zeta)) = x_\xi = \bar{g}(q)$. Let $f \in {}^{\alpha}\alpha$ be a function such that $\bar{f}(p) = p(\xi,\zeta)$ and define $h = f_{\xi,\zeta} \circ f$. Then $h(\alpha) \subseteq A$ and $\bar{h}(p) = \bar{f}_{\xi,\zeta}(\bar{f}(p)) = \bar{f}_{\xi,\zeta}(p(\xi,\zeta)) = x_\xi \in X \setminus A$. Therefore, X is a FU(p)-space. $\square$

Arhangel'skii [A] conjectured that the cardinality of every compact homogeneous space of countable tightness does not exceed 2^{ω}. It follows from Theorem 3.12 that a positive answer to Arhangel'skii's conjecture would respond the following question in the affirmative: Assuming PFA, the answer is positive, as we pointed out above (see [Ba]).

Question 3.13 (Comfort–Garcia)**.** Is every compact homogeneous space of countable tightness p-sequential/ for some $p \in \omega^*$?

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