Homology theory in the alternative set theory I. Algebraic preliminaries

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Abstract. The notion of free group is defined, a relatively wide collection of groups which enable infinite set summation (called **commutative π-group**), is introduced. Commutative π-groups are studied from the set-theoretical point of view and from the point of view of free groups. Commutativity of the operator which is a special kind of inverse limit and factorization, is proved. Tensor product is defined, commutativity of direct product (also a free group construction and tensor product) with the special kind of inverse limit is proved. Some important examples of tensor product are computed.

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0. Introduction.

The main goal of this paper is to give the algebraic foundations for creating at least first parts of homology theory in the Alternative set theory (AST), in spite of the fact that this theory has been developed from the opposite side. First we tried to create some algebraic topology in the AST. At the same time we believed that our results would be good at least for Sd-groups of coefficients starting from indiscernibility relation (i.e. π-equivalence with some additional properties). Each step of the construction of a homology theory which was of interest from the algebraic point of view was checked just from this point of view. At the same time we were looking for the special properties of groups, homomorphisms, operators of direct products, free group, tensor product etc. which have appeared in this process. A certain meaningful part of homology with Sd-groups of coefficients has been created. Then we tried to extend our results to the more general groups of coefficients. In this paper, just this step has been made.

The homology theory in the AST based on these algebraic foundations will be discussed in the next papers.

Throughout the paper we use usual principles and notations of the AST (see [V]).

1. Free groups, commutative π-groups.

By $\mathbf{Z}$ we denote the set-definable class of all integers, $+$ is the usual addition on $\mathbf{Z}$ (which is also an Sd operation). If it does not lead to any misunderstanding, we can use the sign $+$ also for operations in other groups.

By k, l, m, n (if necessary, with subscripts) we shall denote finite natural numbers, by $\alpha, \beta, \gamma, \delta, \mu, \nu$ (also if necessary with subscripts) we shall denote natural numbers (or integers), possible infinite.

By $[x_1, x_2, \ldots, x_n]$ we shall denote an ordered n-tuple of elements $x_1, x_2, \ldots, x_n$.

Definition 1.1. Let $\mathbf{X}$ be a class. Then the class

$$\mathcal{F}(\mathbf{X}) = \{\mathbf{f}; \mathrm{Fnc}\,(\mathbf{f})\,\&\,\mathrm{dom}\,(\mathbf{f}) \subseteq \mathbf{X}\,\&\,\mathrm{rng}\,(\mathbf{f}) \subseteq \mathbf{Z} - \{0\}\}$$

with the following operation $\oplus$:

$$\mathrm{dom}\,(\mathbf{f} \oplus \mathbf{g}) = \{x \in \mathrm{dom}\,(\mathbf{f}) \cup \mathrm{dom}\,(\mathbf{g});\ x \in \mathrm{dom}\,(\mathbf{f}) \cap \mathrm{dom}\,(\mathbf{g}) \Rightarrow \mathbf{f}(x) \neq -\mathbf{g}(x)\}$$

and

$$(\mathbf{f} \oplus \mathbf{g})(x) = \begin{cases} \mathbf{f}(x) + \mathbf{g}(x) & \text{if } x \in \mathrm{dom}\,(\mathbf{f}) \cap \mathrm{dom}\,(\mathbf{g})\,\&\,\mathbf{f}(x) \neq -\mathbf{g}(x) \\ \mathbf{f}(x) & \text{if } x \in \mathrm{dom}\,(\mathbf{f}) \setminus \mathrm{dom}\,(\mathbf{g}) \\ \mathbf{g}(x) & \text{if } x \in \mathrm{dom}\,(\mathbf{g}) \setminus \mathrm{dom}\,(\mathbf{f}), \end{cases}$$

is a commutative group. We shall call it the *commutative free group freely generated by the* (*class*) $\mathbf{X}$. (Its zero element is the empty set $\emptyset$.)

It is easy to see that if $\mathbf{X}$ is an Sd-class, then $\mathcal{F}(\mathbf{X})$ and also $\oplus$ are Sd-classes. Because in all our considerations we shall use only commutative groups, we shall omit this attribute.

According to the fact that for every $x \in \mathbf{X}$ the function $\mathbf{g}_x = \{[1, x]\} \in \mathcal{F}(\mathbf{X})$ there is the natural copy of $\mathbf{X}$ in $\mathcal{F}(\mathbf{X})$ and therefore we shall frequently consider $\mathbf{X}$ to be a subclass of $\mathcal{F}(\mathbf{X})$.

In the common circumstances it is usual to do only finite sums of elements of a given group. The serious problems appear if one wants to sum infinite sets (classes) of elements of a given group. These problems could be solved by means of topology. The first step is to show that we can speak about some kinds of infinite sums in the AST. That is why we introduce a special kind of the inverse system and its limit.

Definition 1.2. An *inverse system of Sd-classes and Sd-maps* (Sd-IS in short) is a codable system $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ which consists of a codable class of classes $\{\mathbf{G}_n; n \in \mathbf{FN}\}$ in which each of $\mathbf{G}_n$ is a set-definable class and a codable class of set-definable maps $\mathbf{H}_n^m : \mathbf{G}_m \longrightarrow \mathbf{G}_n$ for every pair $m \geq n$ such that

(a) each of $\mathbf{H}_n^n$ is the identity on $\mathbf{G}_n$,
(b) for $m \geq n \geq k$ $\mathbf{H}_k^m = \mathbf{H}_k^n \circ \mathbf{H}_n^m : \mathbf{G}_m \longrightarrow \mathbf{G}_k$.

An *inverse limit* of an Sd-IS $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ is a class $\mathbf{G}$ such that there are maps $\mathbf{H}_n : \mathbf{G} \longrightarrow \mathbf{G}_n$ such that

(1) for $m \geq n$ $\mathbf{H}_n = \mathbf{H}_n^m \circ \mathbf{H}_m$,
(2) for each sequence $x_0, x_1, \ldots$ such that $x_n \in \mathbf{G}_n$ and for every $m \geq n$ $x_n = \mathbf{H}_n^m(x_m)$ there is just one element $x \in \mathbf{G}$ such that for each n $x_n = \mathbf{H}_n(x)$,
(3) there is Sd-map $\mathbf{H}_n^{'}$ such that $\mathbf{H}_n = \mathbf{H}_n^{'} \restriction \mathbf{G}$ for each n.

Notation:A group $(\mathbf{G}, +)$ is said to be an Sd-group iff the class $\mathbf{G}$ and also the operation $+$ are set-definable. If all classes $\mathbf{G}_n$ are enriched by group operations $+_n$ such that $(\mathbf{G}_n, +_n)$ are Sd-groups and moreover $\mathbf{H}_n^m$ are homomorphisms, then $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ is said to be an Sd-IS *of groups*. A group structure induced from this Sd-IS of groups can be in a common way introduced on its inverse limit. Therefore this inverse limit can be considered to be a group.

Definition 1.3. A group $(\mathbf{G}, +)$ is said to be a *commutative π-group*, if it is an inverse limit of some Sd-IS of commutative groups.

Example 1.4. Let $+$ be an Sd-operation, $\mathbf{G}_n$ be such set-definable classes that $(\mathbf{G}_n, + \cap \mathbf{G}_n^3)$ is a commutative Sd-group and $\mathbf{G}_{n+1}$ is a subgroup of $\mathbf{G}_n$ for each n. Let us put $\mathbf{G} = \bigcap\{\mathbf{G}_n;\, n \in \mathbf{FN}\}$. Then $(\mathbf{G}, + \cap \mathbf{G}^3)$ is a commutative π-group.

Indeed, all homomorphisms required by Definition 1.2 can be chosen as appropriate inclusions.

Example 1.5. Let $\mathbf{X}$ be a π-class. Then $\mathcal{F}(\mathbf{X})$ is a commutative π-group.

Indeed, because $\mathbf{X}$ is an intersection of a decreasing sequence $\mathbf{X}_n$ of set-definable classes, then the sequence $(\mathcal{F}(\mathbf{X}_n), \oplus)$ and $(\mathcal{F}(\mathbf{X}), \oplus)$ can be considered to be the case of Example 1.4.

Theorem 1.6. *Let $(\mathbf{G}, +)$ be a commutative π-group, let $\mathbf{u}$ be a set. Let $\mathbf{f} : \mathbf{u} \longrightarrow \mathbf{Z}$ and $\mathbf{g} : \mathbf{u} \longrightarrow \mathbf{G}$ be set maps. Then it is possible to define correctly an expression $\mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}(x);\, x \in \mathbf{u}\}$ in such a way that it assigns the uniquely determined element of $\mathbf{G}$. Moreover, because $\mathbf{v} = \operatorname{rng}(\mathbf{g}) \subseteq \mathbf{G}$, we can consider the inclusion $\iota : \mathbf{v} \longrightarrow \mathbf{G}$. If we consider the function $\mathbf{h} : \mathbf{v} \longrightarrow \mathbf{Z}$ given by*

$$\mathbf{h} = \{[\alpha, x];\, x \in \mathbf{v} \,\&\, \alpha = \mathbf{\Sigma}\{\mathbf{f}(y);\, y \in \mathbf{g}^{-1}{}''\{x\}\}\}, \text{ then}$$
$$\mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}(x);\, x \in \mathbf{u}\} = \mathbf{\Sigma}\{\mathbf{h}(y) \cdot \iota(y);\, y \in \mathbf{v}\}.$$

PROOF: Let $(\mathbf{G}_n, +_n), \mathbf{H}_n^m, \mathbf{H}_n$ and $\mathbf{H}_n'$ be such as it is required by 1.2 and the above notation. First of all we prove that an expression $\alpha \cdot \mathbf{g}(x)$ can be correctly defined for given $\alpha \in \mathbf{Z}$ and $x \in \mathbf{u}$. Let $\alpha \geq 0$. A recursive prescription

$$\mathbf{M}_x^n(0) = e_n \text{ (the zero element of } (\mathbf{G}_n, +_n)), \mathbf{M}_x^n(\beta + 1) = \mathbf{M}_x^n(\beta) +_n x$$

can be written by a set formula. By means of the axiom of induction we can easily prove that there is the Sd-function $\mathbf{M}^n(-,-) : \mathbf{N} \times \mathbf{G}_n \longrightarrow \mathbf{G}_n$ which fulfils this prescription.

To be more precise, we shall write at least the relevant set formula (we shall omit this step in the next similar proofs):

$$\begin{aligned}\mathbf{M}^n = \{[z, \alpha, x]; z \in \mathbf{G}_n \,\&\, \alpha \in \mathbf{N} \,\&\, x \in \mathbf{G}_n \,\&\, (\exists\, \mathbf{h})(\operatorname{Fnc}(\mathbf{h}) \,\&\, \mathbf{h} : \alpha+1 \longrightarrow \mathbf{G}_n \,\&\\ \&\, \mathbf{h}(0) = e_n \,\&\, (\forall\, \beta \in \alpha)(\mathbf{h}(\beta + 1) = \mathbf{h}(\beta) +_n x) \,\&\, z = \mathbf{h}(\alpha))\}.\end{aligned}$$

By means of induction, it is also easy to prove that for every $m \geq n$ it holds that

$$\mathbf{M}^n(\alpha, \mathbf{H}_n'(\mathbf{g}(x))) = \mathbf{H}_n^m(\mathbf{M}^m(\alpha, \mathbf{H}_m'(\mathbf{g}(x))))$$

and therefore there is just one element $y \in \mathbf{G}$ such that

$$\mathbf{H}_n(y) = \mathbf{M}^n(\alpha, \mathbf{H}_n'(\mathbf{g}(x))).$$

We put $\alpha \cdot \mathbf{g}(x) = y$. Let $\alpha < 0$. Then we put $\alpha \cdot \mathbf{g}(x) = -((-\alpha) \cdot \mathbf{g}(x))$.

Now let $\mathbf{u} = \{x_1, x_2, \ldots, x_\alpha\}$ be a set ordering of the set $\mathbf{u}$.

Denote $\mathbb{M}^n = \{[z, \alpha, x]; [z, \alpha, x] \in \mathbf{M}^n \vee [-z, -\alpha, x] \in \mathbf{M}^n\}$. $\mathbb{M}^n$ is the extension of $\mathbf{M}^n$ to $\mathbf{Z} \times \mathbf{G}_n$. A recursive prescription

$$S^n(0) = e_n, S^n(\beta + 1) = S^n(\beta) +_n \mathbb{M}^n(\mathbf{f}(x_{\beta+1}), \mathbf{H}_n^{'}(\mathbf{g}(x_{\beta+1})))$$

can be written by a set formula and therefore there is the set function $s^n : \alpha+1 \longrightarrow \mathbf{G}_n$ which fulfils this prescription. Again it is easy to prove by means of induction that for every $m \geq n, \beta \leq \alpha$ it holds that

$$\mathbf{H}_n^m(s^m(\beta)) = s^n(\beta)$$

and therefore there is just one element $y \in \mathbf{G}$ such that for each n

$$\mathbf{H}_n(y) = s^n(\alpha).$$

We put $\mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}(x);\, x \in \mathbf{u}\} = y$.

The last thing we need to prove is that the above construction is independent on the set ordering of the set $\mathbf{u}$. Let us fix some ordering of $\mathbf{u} : \mathbf{u} = \{x_1, x_2, \ldots x_\beta, \ldots x_\gamma, \ldots, x_\alpha\}$. We prove that if we change some two elements, i.e. if we make some transposition, the sum will not change. So let $\mathbf{u} = \{x_1, x_2, \ldots x_\gamma, \ldots x_\beta, \ldots, x_\alpha\}$ be another ordering of $\mathbf{u}$. Let s^n be the function which we get from the first ordering, let $\underline{s}^n$ be the function which we get from the second one. By means of induction it can be easily proved that

$$\begin{aligned}
&s^n(\delta) = \underline{s}^n(\delta) \qquad \text{if } \ \delta < \beta,\\
&s^n(\delta) = \underline{s}^n(\delta) +_n \mathbb{M}^n(\mathbf{f}(x_\beta), \mathbf{H}_n^{'}(\mathbf{g}(x_\beta))) -_n \mathbb{M}^n(\mathbf{f}(x_\gamma), \mathbf{H}_n^{'}(\mathbf{g}(x_\gamma))) \text{ if } \beta \leq \delta < \gamma\\
&s^n(\delta) = \underline{s}^n(\delta) \qquad \text{if } \ \gamma \leq \delta \leq \alpha.
\end{aligned}$$

So $s^n(\alpha) = \underline{s}^n(\alpha)$ is the special case of these formulas.

Finally, for every two set orderings of $\mathbf{u}$ there is a set sequence of transpositions such that the second ordering is the composition of the original one with these transpositions in a given order. Thus, the desired independence is proved.

The second assertion can be now proved by induction. □

Let us make an agreement for summation through the empty set. If $\mathbf{u} = \emptyset$, then we put $\mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}(x);\, x \in \mathbf{u}\} = e$ (the zero element of $(\mathbf{G}, +)$).

And another agreement: let $\mathbf{u} \subseteq \mathbf{G}$, i.e. we can use the inclusion $\iota : \mathbf{u} \longrightarrow \mathbf{G}$. Let $\mathbf{f} : \mathbf{u} \longrightarrow \mathbf{Z}$ be a function. We shall write

$$\mathbf{\Sigma}\{\mathbf{f}(x) \cdot x;\, x \in \mathbf{u}\} \ \text{ instead of } \ \mathbf{\Sigma}\{\mathbf{f}(x) \cdot \iota(x);\, x \in \mathbf{u}\}.$$

Definition 1.7. Let $(\mathbf{A}, +)$ and $(\mathbf{B}, \oplus)$ be commutative π-groups. Let $\mathbf{H} : \mathbf{A} \longrightarrow \mathbf{B}$ be a map. $\mathbf{H}$ is said to be a *total homomorphism* iff for each element $\mathbf{\Sigma}\{\mathbf{f}(x) \cdot x;\, x \in \mathrm{dom}\,(\mathbf{f})\}$ in $(\mathbf{A}, +)$ (here $\mathbf{f}$ is a function $\mathbf{f} : \mathrm{dom}\,(\mathbf{f}) \longrightarrow \mathbf{Z}$ with $\mathrm{dom}\,(\mathbf{f}) \subseteq \mathbf{A}$) it holds that

$$\mathbf{H}(\mathbf{\Sigma}\{\mathbf{f}(x) \cdot x;\, x \in \mathrm{dom}\,(\mathbf{f})\}) = \mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{H}(x);\, x \in \mathrm{dom}\,(\mathbf{f})\}.$$

Clearly, if $\mathbf{H}$ is a homomorphism which is a restriction of some Sd-map $\mathbf{H}^{'}$, then it is a total homomorphism.

Theorem 1.8. *Let* $(\mathbf{G}, +)$ *be a commutative* π*-group. Let* $\mathbf{X}$ *be a* π*-class and* $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{G}$ *be such map which is a restriction of some Sd-map* $\mathbb{F}$ *to* $\mathbf{X}$*. Then there is a unique total homomorphism* $\mathbf{F}' : \mathcal{F}(\mathbf{X}) \longrightarrow \mathbf{G}$ *such that for each* $x \in \mathbf{X}$ $\mathbf{F}'(\mathbf{g}_x) = \mathbf{F}(x)$ *(remember the note after* 1.1*).*

PROOF: Let $\mathbf{f} \in \mathcal{F}(\mathbf{X})$, let $\mathbf{u} = \mathrm{dom}\,(\mathbf{f})$, i.e. $\mathbf{u} \subseteq \mathbf{X}$. Then $\mathbf{F} \restriction \mathbf{u}$ is a set map, $\mathbf{F} \restriction \mathbf{u} : \mathbf{u} \longrightarrow \mathbf{G}$, $\mathbf{f} : \mathbf{u} \longrightarrow \mathbf{Z}$. It follows from 1.6 that the expression $\mathbf{\Sigma}\{\mathbf{f}(x) \cdot (\mathbf{F} \restriction \mathbf{u})(x);\ x \in \mathbf{u}\}$ has a sense and it assigns a uniquely determined element $y \in \mathbf{G}$. We put $\mathbf{F}'(\mathbf{f}) = y$. After a simple analysis of the proof of 1.6, we can see that this map is a total homomorphism. Moreover, for each $x \in \mathbf{X}$ it holds that

$$\mathbf{F}'(\mathbf{g}_x) = \mathbf{F}'(\{[1, x]\}) = \mathbf{\Sigma}\{1 \cdot \mathbf{F}(x);\ x \in \{x\}\} = \mathbf{F}(x)$$

□

Remark: The homomorphism $\mathbf{F}'$ is said to be a linear extension of the map $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{G}$ to the free group freely generated by $\mathbf{X}$.

We can prove a certain analogy of this theorem for an arbitrary class $\mathbf{X}$, but in this case we can not assert that $\mathbf{F}'$ is a total homomorphism. This is because if $\mathbf{X}$ is not a π-class, then $(\mathcal{F}(\mathbf{X}), \oplus)$ need not be a commutative π-group.

Let $\mathbf{X}$ be a π-class. Then $(\mathcal{F}(\mathbf{X}), \oplus)$ is a commutative π-group. Let $\mathbf{f} \in \mathcal{F}(\mathbf{X})$. Then $x \in \mathrm{dom}\,(\mathbf{f}) \Rightarrow \mathbf{g}_x = \{[1, x]\} \in \mathcal{F}(\mathbf{X})$ and it is clear that for $\alpha \in \mathbf{Z}$ $\alpha \cdot \mathbf{g}_x = \{[\alpha, x]\}$. Then $\mathbf{f} = \mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}_x;\ x \in \mathrm{dom}\,(\mathbf{f})\}$.

There are some theorems which can be important from the set-theoretical point of view.

Theorem 1.9. *Let* $(\mathbf{G}, +)$ *be a commutative* π*-group which is an inverse limit of an Sd-IS* $\{(\mathbf{G}_n, +_n), \mathbf{H}_n^m, m \geq n\}$ *of groups such that at least one of the Sd-extensions* $\mathbf{H}_n'$ *of one projection* $\mathbf{H}_n$ *can be chosen as an injection. Then* $\mathbf{G}$ *and* $+$ *are revealed classes.*

PROOF: First of all let us note that if $\mathbf{H}_n$ is an injection, then according to $\mathbf{H}_n = \mathbf{H}_n^m \circ \mathbf{H}_m$, all $\mathbf{H}_m$ for $m \geq n$ are also injections. For our proof, it is enough to prove the revealness of the operation $+$, then the revealness of $\mathbf{G}$ follows from the equation $x + e = x$.

Let $x_1, x_2, \dots$ be a sequence of pairwise distinct elements of the operation $+$, i.e. $x_i = [a_i, b_i, a_i + b_i]$. Then for each n the sequence

$$\{x_i^n = [\mathbf{H}_n(a_i), \mathbf{H}_n(b_i), \mathbf{H}_n(a_i) + \mathbf{H}_n(b_i)];\ i \in \mathbf{FN}\}$$

is a sequence in $+_n$. Because $+_n$ is set-definable, there is its prolongation to $\{x_\beta^n;\ \beta \in \alpha_n\}$ such that for each $\beta \in \alpha_n$ $x_\beta^n = [y_\beta^n, z_\beta^n, u_\beta^n] \in +_n$, i.e. $u_\beta^n = y_\beta^n +_n z_\beta^n$. And according to the fact that for $i \in \mathbf{FN}, m \geq n$, it holds:

$$(*) \qquad [\mathbf{H}_n^m(y_i^m), \mathbf{H}_n^m(z_i^m), \mathbf{H}_n^m(y_i^m +_m z_i^m)] = [y_i^n, z_i^n, u_i^n]$$

these equalities hold up to some infinite natural number. It means that there is $\alpha \in \mathbf{N}\text{-}\mathbf{FN}$ such that:

(1) for each n $\{x_\beta^n;\ \beta \in \alpha\}$ is a set of elements of $+_n$,
(2) equations of type $(*)$ hold for $m \geq n$ and each $\iota \in \alpha$.

Now let n be the number for which $\mathbf{H}_n^{'}$ is an injection. Then $\{x_\beta^n;\ \beta \in \alpha\}$ is an infinite set and moreover, according to (1) and (2), $\mathbf{H}_n^{'-1}{}''\{x_\beta^n;\ \beta \in \alpha\} \subseteq +$. But because $\mathbf{H}_n^{'}$ is an Sd-map, $\mathbf{H}_n^{'-1}{}''\{x_\beta^n;\ \beta \in \alpha\}$ is a set which contains all $x_1, x_2, \dots$. □

Let us note that we do not know whether such $(\mathbf{G}, +)$ has to be a π-class or not. (Because many of $\mathbf{H}_n^{'}$ need not be surjections.)

We can also notice that if we omit the assumption about injectivity of at least one $\mathbf{H}_n^{'}$, the proof could be repeated up to the assertion that some $\{x_\beta^n;\ \beta \in \alpha\}$ is an **infinite** set. In this case we can only state that there is some element of $\mathbf{G}$ which seems to be a natural continuator (limit) of the sequence $x_1, x_2, \dots$ .This theorem also suggests that the assumption that there is an injective Sd-extension of at least one projection $\mathbf{H}_n^{'}$ yields that for infinitely many $\mathbf{H}_n$ their Sd-extension can be chosen to be injective. This fact can be really proved.

Theorem 1.10. *Let $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ be such Sd-IS of groups that there is n such that for $m \geq n$ $\mathbf{H}_n^m$ are isomorphisms. Then an inverse limit of this Sd-IS of groups can be chosen to be an Sd-group. Moreover, if $(\mathbf{G}, +)$ is an inverse limit of this Sd-IS of groups and for infinitely many n $\mathbf{H}_n^{'}$ is injective, then $(\mathbf{G}, +)$ is an Sd-group.*

PROOF: Let $x_0, x_1, \dots, x_i \in \mathbf{G}_i$ be such a sequence that for every $k \geq l$ $\mathbf{H}_l^k(x_k) = x_l$. Then for $n \geq i$ it holds that $x_i = \mathbf{H}_i^n(x_n)$ and for $i \geq n$ it holds that $x_i = (\mathbf{H}_n^i)^{-1}(x_n)$. It means that we can choose $(\mathbf{G}_n, +_n)$ to be an inverse limit of this Sd-IS of groups. To prove this, we put

$$\mathbf{H}_k = (\mathbf{H}_n^k)^{-1}, \quad \text{if } \ k \geq n,$$

and

$$\mathbf{H}_k = \mathbf{H}_k^n, \qquad \text{if } \ k < n.$$

Now let $(\mathbf{G}, +)$ be an inverse limit of this Sd-IS of groups and infinitely many of $\mathbf{H}_n^{'}$ are injections. It follows from the above consideration that $\mathbf{H}_n$ secures bijectivity between $(\mathbf{G}, +)$ and $(\mathbf{G}_n, +_n)$. By $\mathbf{H}_n = \mathbf{H}_n^{'} \restriction \mathbf{G}$ and the set-definability of $\mathbf{H}_n^{'}$ (without loss of generality, we can assume that $\mathbf{H}_n^{'}$ is injective) we have $\mathbf{G} = \{x;\ \mathbf{H}_n^{'}(x) \in \mathbf{G}_n\}$ and hence $\mathbf{G}$ is an Sd-class. It is clear that also $+$ is an Sd-class. □

Theorem 1.11. *Let $\{(\mathbf{G}_n, +_n), \mathbf{H}_n^m, m \geq n\}$ be an Sd-IS of groups such that for every $m \geq n, \mathbf{H}_n^m$ are injective. Suppose that there is an Sd-group $(\mathbf{G}, +)$ which is its inverse limit. Then there is n such that for $m \geq n$ $\mathbf{H}_n^m$ are isomorphisms.*

PROOF: Homomorphisms $\mathbf{H}_n$, required by 1.2, are Sd-classes (a restriction of an Sd-map to an Sd-class be an Sd-map) in this case.

First we prove that if one $\mathbf{H}_n$ is an isomorphism, then all $\mathbf{H}_m$ for $m \geq n$ are isomorphisms, too. Indeed, if $\mathbf{H}_m$ is not injective, then neither $\mathbf{H}_n = \mathbf{H}_n^m \circ \mathbf{H}_m$

is, which is a contradiction. For next, suppose that $\mathbf{H}_m$ is not surjective, i.e. there is $x \in \mathbf{G}_m$ such that for no $y \in \mathbf{G}$ it holds that $\mathbf{H}_m(y) = x$. But because $\mathbf{H}_n$ is surjective, there is $z \in \mathbf{G}$ such that $\mathbf{H}_n(z) = \mathbf{H}_n^m(x)$. Then according to $\mathbf{H}_n = \mathbf{H}_n^m \circ \mathbf{H}_m$ it holds that $\mathbf{H}_n^m(\mathbf{H}_m(z)) = \mathbf{H}_n^m(x)$. But $\mathbf{H}_m(z) \neq x$ and it is a contradiction with the injectivity of $\mathbf{H}_n^m$.

Now we shall prove that at least one of $\mathbf{H}_n$ is an isomorphism. We shall do it by contradiction. There are three possibilities there:

(a) almost all of $\mathbf{H}_n$ are not injective
(b) almost all of $\mathbf{H}_n$ are not surjective
(c) infinitely many of $\mathbf{H}_n$ are not injective, infinitely many of $\mathbf{H}_n$ are not surjective

("almost all" means that "there is m such that for every $n \geq m$").
The case (c) follows from each of (a) and (b), because if $(\mathbf{G}, +)$ is an inverse limit of given Sd-IS. then it is an inverse limit of any of its cofinal "subsystem".
(a) We can assume that all $\mathbf{H}_n$ are not injective. It means that for every n there are $x_n \neq y_n \in \mathbf{G}$ such that $\mathbf{H}_n(x_n) = \mathbf{H}_n(y_n)$. If we prolong this statement, we can see that there are elements $x, y \in \mathbf{G}$, $x \neq y$ ($\mathbf{G}$ is an Sd-class!) such that for every n, $\mathbf{H}_n(x) = \mathbf{H}_n(y)$. But this is a contradiction, because in this case for the sequence $x_0 = \mathbf{H}_0(x), x_1 = \mathbf{H}_1(x), \ldots$ there are <u>two</u> elements $x, y \in \mathbf{G}$ such that $\mathbf{H}_n(x) = x_n = \mathbf{H}_n(y)$ for each n.
(b) Let all $\mathbf{H}_n$ be not surjective. It means that for each n there exists an element $x_n \in \mathbf{G}_n$ which is not in the image of $\mathbf{H}_n$. By means of this, we get the following sequences:

$$\begin{array}{lllll}
\mathbf{H}_0^0(x_0) & & & & \\
\mathbf{H}_0^1(x_1), & \mathbf{H}_1^1(x_1) & & & \\
\mathbf{H}_0^2(x_2), & \mathbf{H}_1^2(x_2), & \mathbf{H}_2^2(x_2) & & \\
\quad\vdots & \quad\vdots & \quad\vdots & & \\
\mathbf{H}_0^n(x_n), & \mathbf{H}_1^n(x_n), & \mathbf{H}_2^n(x_n), & \ldots, & \mathbf{H}_n^n(x_n) \\
\quad\vdots & \quad\vdots & \quad\vdots & & \quad\vdots
\end{array}$$

Of course, as x_n is not in the image of $\mathbf{H}_n$, none of the elements $\mathbf{H}_i^n(x_n)$ is in the image of $\mathbf{H}_i$ (this follows from injectivity of $\mathbf{H}_i^n$ and the equality $\mathbf{H}_i = \mathbf{H}_i^n \circ \mathbf{H}_n$). After prolongation of a given triangle up to some row (with superscripts) $\alpha \in \mathbf{N}\text{-}\mathbf{FN}$, we obtain a row $\beta \in \alpha\text{-}\mathbf{FN}$, in which (for $\mu \geq \kappa \geq \nu$) it holds that

$$\mathbf{H}_\nu^\mu(\mathbf{H}_\mu^\beta(x_\beta)) = \mathbf{H}_\nu^\beta(x_\beta)$$

and none of $\mathbf{H}_\mu^\beta(x_\beta)$ is in the image of $\mathbf{H}_\mu$ and each of $\mathbf{H}_\mu^\beta(x_\beta)$ is an element of $\mathbf{G}_\mu$. But this means that for the sequence $\mathbf{H}_0^\beta(x_\beta), \mathbf{H}_1^\beta(x_\beta), \ldots, \mathbf{H}_n^\beta(x_\beta), \ldots$ there is no element $x \in \mathbf{G}$ such that for each n it holds that $\mathbf{H}_n(x) = x_n$. And this is a contradiction with 1.2. $\square$

Theorem 1.12 (P. Zlatoš). *Let $\{(\mathbf{G}_n, +_n), \mathbf{H}_n^m, m \geq n\}$ be an Sd-IS of groups such that for every $m \geq n$ $\mathbf{H}_n^m$ are injective. Then its inverse limit $(\mathbf{G}, +)$ can be chosen in such a way that $\mathbf{G}$ and $+$ are π-classes.*

Proof: Let us denote $\mathbf{G} = \bigcap\{\mathbf{H}_0^{n\prime\prime}\mathbf{G}_n;\ n \in \mathbf{FN}\}$. It is clear that $\mathbf{G}$ is a π-class and moreover, it is a subgroup of $\mathbf{G}_0$. Therefore $+ = (+_0) \cap \mathbf{G}^3$ is also a π-class and $(\mathbf{G}, +)$ is a group. We state that this is an inverse limit of the given Sd-IS. To prove this, it is enough to take $\mathbf{H}_n^{\prime} = (\mathbf{H}_0^n)^{-1}$ (this is correct because each $\mathbf{H}_0^n$ is an injection). It is clear from the definition of $\mathbf{G}$ and $+$ that each $\mathbf{H}_n^{\prime}$ is a homomorphism which maps $\mathbf{G}$ into $\mathbf{G}_n$ and also that all formulas required by 1.2 hold. □

Remark: As it was mentioned in the proof of the Theorem 1.11, sentences "for all" and "for infinitely many" (naturally used in the right situation) have the same consequences (see e.g the formulation of 1.10–1.12).

Also it is clear that 1.9–1.12 can be reformulated for all Sd-IS's (without requirements on its algebraic structure). Perhaps they could throw a new insight to the well-known facts about π-classes (each inclusion is an injection).

2. Homomorphisms of Sd-IS.

In this section we are going to develop the technique being later useful at least for the comparison of our homology theory with some classical ones (Čech's and Vietoris's).

Theorem 2.1. *Let $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ be an Sd-IS of groups. Then there is its inverse limit.*

Proof: We use usual construction of the inverse limit. According to the axiom of prolongation, for each sequence $x_0, x_1, \dots$ such that if $m \geq n$, then $x_n \in \mathbf{G}_n$ and $x_n = \mathbf{H}_n^m(x_m)$, there is a function $\mathbf{g} : \alpha \longrightarrow \mathbf{V}$ ($\alpha \in \mathbf{N}$) such that $\mathbf{g}(n) = x_n$. According to the axiom of choice, we can consider a class $\mathbf{G}$ such that for every sequence described above there is just one function of the kind described, and conversely.

Now let $\mathbf{f}, \mathbf{g} \in \mathbf{G}$. Then the sequence $\mathbf{f}(0) +_0 \mathbf{g}(0), \mathbf{f}(1) +_1 \mathbf{g}(1), \dots, \mathbf{f}(n) +_n \mathbf{g}(n), \dots$ has all required properties and therefore there is just one $\mathbf{h} \in \mathbf{G}$ such that $\mathbf{h}(n) \in \mathbf{G}$ such that $\mathbf{h}(n) = \mathbf{f}(n) +_n \mathbf{g}(n)$. We put $\mathbf{f} + \mathbf{g} = \mathbf{h}$. It is clear that $(\mathbf{G}, +)$ is the group. For each n let $\mathbf{H}_n^{\prime} = \{[x, y]; \mathrm{Fnc}\,(y)\,\&\, x = y(n)\}$. It is clear that $\mathbf{H}_n^{\prime}$ is set-definable and that $\mathbf{H}_n^{\prime} \restriction \mathbf{G}$ is the homomorphism $\mathbf{H}_n^{\prime} \restriction \mathbf{G} : (\mathbf{G}, +) \longrightarrow (\mathbf{G}_n, +_n)$. Let $\mathbf{g} \in \mathbf{G}$. As for $\mathbf{H}_n = \mathbf{H}_n^{\prime} \restriction \mathbf{G}$ it holds that $\mathbf{H}_n(\mathbf{g}) = \mathbf{g}(n)$ and $\mathbf{H}_m(\mathbf{g}) = \mathbf{g}(m)$ and for $m \geq n$ $\mathbf{H}_n^m(\mathbf{g}(m)) = \mathbf{g}(n)$, we have $\mathbf{H}_n = \mathbf{H}_n^m \circ \mathbf{H}_m$. □

Theorem 2.2. *Let $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ be an Sd-IS. Let for each $n, \mathbf{G}_n$ be the nonempty class. Then its inverse limit is nonempty.*

Proof: We can prolong the Sd-IS $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$ to $\{\mathbf{G}_\nu, \mathbf{H}_\nu^\mu, \mu \geq \nu\}$ where (a) and (b) of 1.2 are fulfilled for all $\mu, \nu, \kappa \in \alpha \in \mathbf{N}\text{-}\mathbf{FN}$ and also for $\nu \in \alpha$ $\mathbf{G}_\nu \neq \emptyset$ (for appropriate α). Let $x \in \mathbf{G}_{\alpha-1}$. The sequence

$$\mathbf{H}_0^{\alpha-1}(x), \mathbf{H}_1^{\alpha-1}(x), \mathbf{H}_2^{\alpha-1}(x), \dots, \mathbf{H}_n^{\alpha-1}(x), \dots$$

fulfils the condition (2) of 1.2 and therefore the inverse limit is nonempty. □

Agreement: Instead of the Sd-IS $\{\mathbf{G}_n, \mathbf{H}_n^m, m \geq n\}$, we shall write the Sd-IS $(\mathbf{G}, \mathbf{H})$.

Definition 2.3. Let $(\mathbf{X},\mathbf{P}),(\mathbf{Y},\mathbf{Q})$ be two Sd-IS. A *map* $\Phi : (\mathbf{X},\mathbf{P}) \longrightarrow (\mathbf{Y},\mathbf{Q})$ consists of an order preserving function $\varphi : \mathbf{FN} \longrightarrow \mathbf{FN}$, and for each n, an Sd-map $\varphi_n : \mathbf{X}_{\varphi(n)} \longrightarrow \mathbf{Y}_n$ such that for $m \geq n$ the following diagram

$$\begin{tikzcd} \mathbf{X}_{\varphi(n)} \arrow[d, "\varphi_n"] & \mathbf{X}_{\varphi(m)} \arrow[l, "\mathbf{P}^{\varphi(m)}_{\varphi(n)}"'] \arrow[d, "\varphi_m"] \\ \mathbf{Y}_n & \mathbf{Y}_m \arrow[l, "\mathbf{Q}^m_n"'] \end{tikzcd}$$

is commutative. If both $(\mathbf{X},\mathbf{P})$ and $(\mathbf{Y},\mathbf{Q})$ are Sd-IS of groups, we require in addition φ_n to be homomorphisms.

Let $\Phi : (\mathbf{X},\mathbf{P}) \longrightarrow (\mathbf{Y},\mathbf{Q})$ be a map between two Sd-IS. Let $\mathbf{X}_\infty$ with the maps $\{\mathbf{P}_n;\, n \in \mathbf{FN}\}$ and $\mathbf{Y}_\infty$ with the maps $\{\mathbf{Q}_n;\, n \in \mathbf{FN}\}$ be the inverse limits of these Sd-IS. Then we can define a *limit map of* Φ, in sign φ_∞ (or, if necessary, Φ_∞) as follows:
For $x \in \mathbf{X}_\infty$, put $y_n = \varphi_n(\mathbf{P}_{\varphi(n)}(x))$. For $m \geq n$, it follows from the above commutative diagram that

$$y_n = \varphi_n(\mathbf{P}^{\varphi(m)}_{\varphi(n)} \circ \mathbf{P}_{\varphi(m)}(x)) = \mathbf{Q}^m_n(\varphi_m(\mathbf{P}_{\varphi(m)}(x))) = \mathbf{Q}^m_n(y_m).$$

Therefore for the sequence $y_0, y_1, \dots$ there is just one $y \in \mathbf{Y}_\infty$ such that $y_n = \mathbf{Q}_n(y)$. We put $\varphi_\infty(x) = y$.

Again it is clear that if $(\mathbf{X},\mathbf{P})$ and $(\mathbf{Y},\mathbf{Q})$ are two Sd-IS of groups, then the limit φ_∞ of the map $\Phi : (\mathbf{X},\mathbf{P}) \longrightarrow (\mathbf{Y},\mathbf{Q})$ is a homomorphism.

Let $\Phi : (\mathbf{X},\mathbf{P}) \longrightarrow (\mathbf{Y},\mathbf{Q})$ and $\Psi : (\mathbf{Y},\mathbf{Q}) \longrightarrow (\mathbf{Z},\mathbf{T})$ be two maps of the Sd-IS, their *composition* $\Psi \circ \Phi$ consists of the $\psi \circ \varphi : \mathbf{FN} \longrightarrow \mathbf{FN}$ and the maps $\psi_n \circ \varphi_{\psi(n)}$.

Lemma 2.4. *Let* $\Phi : (\mathbf{X},\mathbf{P}) \longrightarrow (\mathbf{Y},\mathbf{Q})$ *be a map of two Sd-IS,* $n \in \mathbf{FN}$. *Then commutativity holds in the diagram*

$$\begin{tikzcd} \mathbf{X}_{\varphi(n)} \arrow[d, "\varphi_n"] & \mathbf{X}_\infty \arrow[l, "\mathbf{P}_{\varphi(n)}"'] \arrow[d, "\varphi_\infty"] \\ \mathbf{Y}_n & \mathbf{Y}_\infty \arrow[l, "\mathbf{Q}_n"'] \end{tikzcd}.$$

Proof: This follows immediately from the definition. □

Lemma 2.5. *Let* $\Phi : (\mathbf{X},\mathbf{P}) \longrightarrow (\mathbf{Y},\mathbf{Q})$ *and* $\Psi : (\mathbf{Y},\mathbf{Q}) \longrightarrow (\mathbf{Z},\mathbf{T})$ *be two maps of the Sd-IS. Then* $(\Psi \circ \Phi)_\infty = \psi_\infty \circ \varphi_\infty$.

Proof: This follows readily from the fact that $\varphi_\infty(x)$ is defined by mapping the "coordinates" (i.e. $\mathbf{P}_{\varphi(n)}(x)$) of x by means of the coordinate functions φ_n and the fact that the composition $\Psi \circ \Phi$ was defined by composing "coordinate" functions. □

Definition 2.6. An *Sd-chain of groups* is a codable class $\mathbf{C} = \{\mathbf{G}_\nu, \mathbf{H}_\nu\}$ where $(\mathbf{G}_\nu, +_\nu)$ are Sd-groups and $\mathbf{H}_\nu : (\mathbf{G}_\nu, +_\nu) \longrightarrow (\mathbf{G}_{\nu-1}, +_{\nu-1})$ are homomorphisms which are Sd-maps. Here ν runs over $\mathbf{N}$.

If we omit the requirements of set-definability in the previous definition then we talk about a *chain of groups.*

Definition 2.7. An *inverse system of Sd-chains of groups* $(\mathbb{C}, \Pi)$ is a function which attaches to each $n \in \mathbf{FN}$ an Sd-chain of groups $\mathbf{C}_k = \{{}^k\mathbf{G}_\nu, {}^k\mathbf{H}_\nu\}$ and for every $k \geq l$ and Sd-homomorphism $\Pi^k_l : \mathbf{C}_k \longrightarrow \mathbf{C}_l$ of these Sd-chains of groups (it means that $\Pi^k_l = \{{}^\nu\Pi^k_l ; {}^\nu\Pi^k_l : {}^k\mathbf{G}_\nu \longrightarrow {}^l\mathbf{G}_\nu \,\&\, \nu \in \mathbf{N}\}$ and all ${}^\nu\Pi^k_l$ are Sd-maps and homomorphisms and ${}^l\mathbf{H}_\nu \circ {}^\nu\Pi^k_l = {}^{\nu-1}\Pi^k_l \circ {}^k\mathbf{H}_\nu$) such that

(a) for each k, Π^k_k is the appropriate identity (i.e. each ${}^\nu\Pi^k_k$ is identity),
(b) for $k \geq l \geq m$, $\Pi^k_m = \Pi^l_m \circ \Pi^k_l$ (again coordinatewise as in (a)).

Then, for any fixed ν $\{{}^n\mathbf{G}_\nu, {}^\nu\Pi^m_n, m \geq n\}$ form an Sd-IS of groups. Its limit group is denoted by ${}^\infty\mathbf{G}_\nu$. Again for fixed ν, the homomorphism $\{{}^n\mathbf{H}_\nu, n \in \mathbf{FN}\}$ together with the identity map of $\mathbf{FN}$ form a map $\mathbf{H}^\nu : \{{}^n\mathbf{G}_\nu, {}^\nu\Pi^m_n, m \geq n\} \longrightarrow \{{}^n\mathbf{G}_{\nu-1}, {}^{\nu-1}\Pi^m_n, m \geq n\}$. The limit of $\mathbf{H}^\nu$ is denoted by ${}^\infty\mathbf{H}_\nu : {}^\infty\mathbf{G}_\nu \longrightarrow {}^\infty\mathbf{G}_{\nu-1}$. The chain of groups (which need not be Sd-groups) $\mathbf{C}_\infty = \{{}^\infty\mathbf{G}_\nu, {}^\infty\mathbf{H}_\nu, \nu\}$ so obtained is called the *inverse limit* of the system $(\mathbb{C}, \Pi)$.

Definition 2.8. A chain of groups $\{\mathbf{G}_\nu, \mathbf{H}_\nu\}$ is said to be *exact* iff for each ν $\mathrm{Im}\ \mathbf{H}_{\nu+1} = \mathrm{Ker}\ \mathbf{H}_\nu$.

Theorem 2.9. *Let* $(\mathbb{C}, \Pi)$ *be an inverse system of Sd-chains of groups such that each* $\mathbf{C}_k$ *is an exact chain of groups. Then* $\mathbf{C}_\infty$ *is an exact chain of groups.*

Proof: The composition $\mathbf{H}^{\nu-1} \circ \mathbf{H}^\nu$ (mapping $\{{}^n\mathbf{G}_\nu, {}^\nu\Pi^m_n, m \geq n\}$ into $\{{}^n\mathbf{G}_{\nu-2}, {}^{\nu-2}\Pi^m_n, m \geq n\}$) consists of the identity map of $\mathbf{FN}$ and the maps ${}^n\mathbf{H}_{\nu-1} \circ {}^n\mathbf{H}_\nu = 0$. Hence the inverse limit of $\mathbf{H}^{\nu-1} \circ \mathbf{H}^\nu$ is zero. By 2.5, this is the composition ${}^\infty\mathbf{H}_{\nu-1} \circ {}^\infty\mathbf{H}_\nu$. So $\mathrm{Im}\ {}^\infty\mathbf{H}_{\nu+1} \subseteq \mathrm{Ker}\ {}^\infty\mathbf{H}_\nu$.

Conversely, let $\mathbf{g} \in {}^\infty\mathbf{G}_\nu$ and ${}^\infty\mathbf{H}_\nu(\mathbf{g}) = 0$. Let $\{{}^\nu\Pi_n, n \in \mathbf{FN}\}$ be the projections ${}^\nu\Pi_n : {}^\infty\mathbf{G}_\nu \longrightarrow {}^n\mathbf{G}_\nu$ and $\{{}^\nu\Pi'_n, n \in \mathbf{FN}\}$ its Sd-extensions, as it is required in 1.2. Let $\mathbf{g}_n = {}^\nu\Pi'_n(\mathbf{g})$ be the "coordinate" of $\mathbf{g}$ in ${}^n\mathbf{G}_\nu$. Since ${}^\infty\mathbf{H}_\nu(\mathbf{g}) = 0$, it follows ${}^n\mathbf{H}_\nu(\mathbf{g}_n) = 0$ for each n. Since $\mathbf{C}_n$ is exact, $\mathbf{X}_n = {}^n\mathbf{H}^{-1}_{\nu+1}(\mathbf{g}_n)$ is a nonempty Sd-class which is the subclass of ${}^n\mathbf{G}_{\nu+1}$. From the relation ${}^l\mathbf{H}_{\nu+1} \circ {}^{\nu+1}\Pi^k_l = {}^\nu\Pi^k_l \circ {}^k\mathbf{H}_{\nu+1}$ it follows that ${}^{\nu+1}\Pi^k_l$ maps $\mathbf{X}_k$ into $\mathbf{X}_l$ $(k \geq l)$. Indeed, let $x \in \mathbf{X}_k$. It means that ${}^k\mathbf{H}_{\nu+1}(x) = g_k$. Hence ${}^l\mathbf{H}_{\nu+1} \circ {}^{\nu+1}\Pi^k_l(x) = {}^\nu\Pi^k_l \circ {}^k\mathbf{H}_{\nu+1}(x) = {}^\nu\Pi^k_l(g_k) = g_l$, so that ${}^{\nu+1}\Pi^k_l(x) \in \mathbf{X}_l$. According to this fact it follows that $(\mathbf{X}, {}^{\nu+1}\Pi \restriction \mathbf{X})$ is an Sd-IS which has by 2.2 a nonempty inverse limit, say $\mathbf{X}_\infty$. It is easily seen that $\mathbf{X}_\infty \subseteq {}^\infty\mathbf{G}_{\nu+1}$ and ${}^\infty\mathbf{H}_{\nu+1}$ maps $\mathbf{X}_\infty$ into $\{\mathbf{g}\}$. □

Definition 2.10. Let $(\mathbf{G}, \Pi)$ be Sd-IS of groups. Suppose that for each n, $\mathbf{H}_n$ is a subgroup of $\mathbf{G}_n$ which is an Sd-group, and suppose that for each $k \geq l$, Π^k_l maps $\mathbf{H}_k$ into $\mathbf{H}_l$. Let $P^k_l : \mathbf{H}_k \longrightarrow \mathbf{H}_l$ be a map defined by Π^k_l. Clearly $(\mathbf{H}, P)$ is an Sd-IS of groups. It is called a *system of Sd-subgroups* of $(\mathbf{G}, \Pi)$. For each n define $\mathbf{K}_n = \mathbf{G}_n/\mathbf{H}_n$, and for each $k \geq l$, define $\Sigma^k_l : \mathbf{K}_k \longrightarrow \mathbf{K}_l$ to be the map induced by Π^k_l. Then $(\mathbf{K},\Sigma)$ is Sd-IS of groups called the *system of factor groups* of $(\mathbf{G},\Pi)$ by $(\mathbf{H},P)$. The *inclusion* map $\Phi : (\mathbf{H},P) \longrightarrow (\mathbf{G},\Pi)$ and the *natural* map $\Psi : (\mathbf{G},\Pi) \longrightarrow (\mathbf{K},\Sigma)$ are defined in the obvious way. According to the definition, each element of the limit group $\mathbf{H}_\infty$ of $(\mathbf{H},P)$ is an element of $\mathbf{G}_\infty$, and $\varphi_\infty : \mathbf{H}_\infty \longrightarrow \mathbf{G}_\infty$ is the inclusion.

Theorem 2.11. *Let $(\mathbf{G},\Pi),(\mathbf{H},P)$ and $(\mathbf{K},\Sigma)$ be Sd-IS of groups, subgroups and factor groups, respectively. Then $\psi_\infty : \mathbf{G}_\infty \longrightarrow \mathbf{K}_\infty$ induces an isomorphism $\mathbf{G}_\infty \,/\, \mathbf{H}_\infty \cong \mathbf{K}_\infty$.*

Proof: For each n, let adjoin an infinite class (indexed by $\mathbf{N}$) of trivial groups and maps to

$$\mathbf{H}_n \xrightarrow{\varphi_n} \mathbf{G}_n \xrightarrow{\psi_n} \mathbf{K}_n$$

so as to obtain an Sd-chain of groups $\mathbf{C}_n$

$$\dots \to 0 \to \mathbf{H}_n \xrightarrow{\varphi_n} \mathbf{G}_n \xrightarrow{\psi_n} \mathbf{K}_n \to 0 \to \dots$$

It is clear that $\mathbf{C}_n$ is an exact Sd-chain of groups. For each $k \geq l$, adjoin to P^k_l, Π^k_l and Σ^k_l, an infinite class of trivial maps so as to obtain a map $T^k_l : \mathbf{C}_k \longrightarrow \mathbf{C}_l$. Then $(\mathbb{C},T)$ is an inverse system of Sd-chains of groups which are all exact. It is also clear that the limit chain of groups $\mathbf{C}_\infty$ consists of

$$\dots \to 0 \to \mathbf{H}_\infty \xrightarrow{\varphi_\infty} \mathbf{G}_\infty \xrightarrow{\psi_\infty} \mathbf{K}_\infty \to 0 \to \dots$$

By 2.9 $\mathbf{C}_\infty$ is an exact chain of groups. It follows that ψ_∞ must be onto and its kernel is $\mathbf{H}_\infty$. □

Later, the following corollary will be essential for us.

Corollary 2.12. *Let for each n, $\mathbf{G}_{n+1}$ be a subgroup of $\mathbf{G}_n$, let $\Pi^k_l : \mathbf{G}_k \longrightarrow \mathbf{G}_l$ be the inclusion restricted to $\mathbf{G}_k$ (in this case it is a homomorphism) and let $(\mathbf{G},\Pi)$ be Sd-IS of groups. Let $(\mathbf{H},P)$ be a system of Sd-subgroups of $(\mathbf{G},\Pi)$ and let $(\mathbf{K},\Sigma)$ be a system of factor groups of $(\mathbf{G},\Pi)$ by $(\mathbf{H},P)$. Then*

$$\Big(\bigcap\{\mathbf{G}_n;\, n \in \mathbf{FN}\}\Big) \,/\, \Big(\bigcap\{\mathbf{H}_n;\, n \in \mathbf{FN}\}\Big) \cong \mathbf{K}_\infty\,.$$

We state that $\bigcap\{\mathbf{G}_n;\, n \in \mathbf{FN}\} \cong \mathbf{G}_\infty$ and also $\bigcap\{\mathbf{H}_n;\, n \in \mathbf{FN}\} \cong \mathbf{H}_\infty$. For $x \in \mathbf{G}_\infty$, let $\{\Pi_n;\, n \in \mathbf{FN}\}$ be the projections of $\mathbf{G}_\infty$ into $\mathbf{G}_n$. As Π^k_l is an inclusion and according to the definition we have $\Pi_l(x) = \Pi^k_l \circ \Pi_k(x)$, hence $\Pi_k(x) = \Pi_l(x)$ and, moreover, $\Pi_n(x)$ is the element of $\bigcap\{\mathbf{G}_n;\, n \in \mathbf{FN}\}$. Therefore the map $\varphi : \mathbf{G}_\infty \longrightarrow \bigcap\{\mathbf{G}_n;\, n \in \mathbf{FN}\}$ defined by the equality $\varphi(x) = \Pi_0(x)$ is a bijection and also a homomorphism. It is also the prof of $\bigcap\{\mathbf{H}_n;\, n \in \mathbf{FN}\} \cong \mathbf{H}_\infty$. Now according to 2.11, we have

$$\Big(\bigcap\{\mathbf{G}_n;\, n \in \mathbf{FN}\}\Big) / \Big(\bigcap\{\mathbf{H}_n;\, n \in \mathbf{FN}\}\Big) \cong \mathbf{G}_\infty \;\mathbf{H}_\infty \cong \mathbf{K}_\infty\,.$$

The part of this paragraph beginning by 2.3 up to 2.11 is a relevant reformulation of some definitions and results of [E-S, Chapter VIII].

3. Tensor product.

This part is very important for our construction of a homology theory, because it will allow us to concentrate our effort to $\mathbf{Z}$ as a group of coefficients and then to transfer many results to some other groups of coefficients.

Theorem 3.1. *Let* $(\mathbf{A}, +)$ *and* $(\mathbf{B}, \oplus)$ *be commutative* π*-groups. Then* $(\mathbf{A} \times \mathbf{B}, + \times \oplus)$ (*the operation* $+ \times \oplus$ *on* $\mathbf{A} \times \mathbf{B}$ *is defined coordinatewise*) *is a commutative* π*-group.*

Proof: Let$\{(\mathbf{A}_n, +_n), \mathbf{H}^m_n, m \geq n\}$ and $\{(\mathbf{B}_n, \oplus_n), \underline{H}^m_n, m \geq n\}$ be two corresponding Sd-IS of groups, let $\mathbf{H}_n, \mathbf{H}'_n, \underline{H}_n, \underline{H}'_n$ be the corresponding projections. Take a codable class $\{(\mathbf{A}_n \times \mathbf{B}_n, +_n \times \oplus_n), \mathbf{H}^m_n \times \underline{H}^m_n, m \geq n\}$ where $\mathbf{H}^m_n \times \underline{H}^m_n : (\mathbf{A}_m \times \mathbf{B}_m, +_m \times \oplus_m) \longrightarrow (\mathbf{A}_n \times \mathbf{B}_n, +_n \times \oplus_n)$ are defined coordinatewise, i.e.

$$\mathbf{H}^m_n \times \underline{H}^m_n([a, b]) = [\mathbf{H}^m_n(a), \underline{H}^m_n(b)].$$

All the classes $\mathbf{A}_n \times \mathbf{B}_n, +_n \times \oplus_n, \mathbf{H}^m_n \times \underline{H}^m_n$ are set-definable, hence our codable class is an Sd-IS of groups.

The maps $\mathbf{H}_n \times \underline{H}_n : \mathbf{A} \times \mathbf{B} \longrightarrow \mathbf{A}_n \times \mathbf{B}_n$ are homomorphisms and they are the restrictions of the Sd-maps $\mathbf{H}'_n \times \underline{H}'_n$ and, moreover, they satisfy all required equations.

Let $[a_0, b_0], [a_1, b_1], \ldots$ be such a sequence that $[a_n, b_n] \in \mathbf{A}_n \times \mathbf{B}_n$ and for $m \geq n$, $[a_n, b_n] = \mathbf{H}^m_n \times \underline{H}^m_n([a_m, b_m])$. Hence $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are such sequences that for $m \geq n$, $a_n = \mathbf{H}^m_n(a_m)$ and $b_n = \underline{H}^m_n(b_m)$. Therefore there is just one $\mathbf{a} \in \mathbf{A}$ and just one $\mathbf{b} \in \mathbf{B}$ such that $\mathbf{H}_n(\mathbf{a}) = a_n$ and $\underline{H}_n(\mathbf{b}) = b_n$. So that $\mathbf{H}^m_n \times \underline{H}^m_n([\mathbf{a}, \mathbf{b}]) = [a_n, b_n]$ and there is no other pair $[\mathbf{a}', \mathbf{b}']$ with this property.

So $\mathbf{A} \times \mathbf{B}$ is an inverse limit of $\{(\mathbf{A}_n \times \mathbf{B}_n, +_n \times \oplus_n), \mathbf{H}^m_n \times \underline{H}^m_n, m \geq n\}$. □

Remark: If some of the pairs $(\mathbf{H}^m_n, \underline{H}^m_n)$ and $(\mathbf{H}'_n, \underline{H}'_n)$ are the pairs of injective maps, then $\mathbf{H}^m_n \times \underline{H}^m_n$ and $\mathbf{H}'_n \times \underline{H}'_n$ would be so, too.

Definition 3.2. Let $(\mathbf{A}, +)$ be a commutative π-group such that there is an Sd-IS of groups $\{(\mathbf{A}_n, +), \mathbf{H}^m_n, m \geq n\}$ such that $(\mathbf{A}, +)$ is its inverse limit and all $\mathbf{H}'_n$ are injective. Then we shall call $(\mathbf{A}, +)$ a commutative π-group *with injective projections* (a commutative π-group with i.p. in short) and $\{(\mathbf{A}_n, +), \mathbf{H}^m_n, m \geq n\}$ its injective representation, $\mathbf{H}'_n$ injective Sd-projections.

Theorem 3.3. *Let* $(\mathbf{A}, +)$ *be a commutative* π*-group with i.p. Then* $\mathcal{F}(\mathbf{A})$ *is a commutative* π*-group with i.p.*

Proof: For $(\mathbf{A}, +)$ take an Sd-IS as in 3.1. Take a codable class $\{(\mathcal{F}(\mathbf{A}_n), \oplus_n), \mathbf{P}^m_n, m \geq n\}$ where $\mathbf{P}^m_n$ is defined for $m \geq n$ as follows:
Let $\mathbf{f} \in \mathcal{F}(\mathbf{A}_m)$, i.e. $\mathbf{f} = \mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}_x; x \in \operatorname{dom}(\mathbf{f})\}$. Then we put

$$\mathbf{P}^m_n(\mathbf{f}) = \mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}_{\mathbf{H}^m_n(x)}; x \in \operatorname{dom}(\mathbf{f})\}.$$

Because $\mathbf{H}^m_n \restriction \operatorname{dom}(\mathbf{f}) : \operatorname{dom}(\mathbf{f}) \longrightarrow \mathbf{A}_n$ is a set function, $\mathbf{P}^m_n(\mathbf{f})$ is a well defined element of $\mathcal{F}(\mathbf{A}_n)$, $\mathbf{P}^m_n$ is a homomorphism and because $\mathcal{F}(\mathbf{A}_m)$ and $\mathcal{F}(\mathbf{A}_n)$ are Sd-groups, $\mathbf{P}^m_n$ is an Sd-map.

An easy computation yields to the fact that for $m \geq k \geq n$ it holds $\mathbf{P}_n^m = \mathbf{P}_n^k \circ \mathbf{P}_k^m$. Hence our system is an Sd-IS of groups.

The maps $\mathbf{P}_n^{'}$ are defined in a similar way as $\mathbf{P}_n^m$:
Let $\mathbf{X} = \mathrm{dom}\,(\mathbf{H}_n^{'}), u \subseteq \mathbf{X}$ and $\mathbf{f} : \mathbf{u} \longrightarrow \mathbf{Z} - \{0\}$. Then

$$\mathbf{P}_n^{'}(\mathbf{f}) = \mathbf{\Sigma}\{\mathbf{f}(x) \cdot \mathbf{g}_{\mathbf{H}_n^{\prime}(x)};\, x \in \mathrm{dom}\,(\mathbf{f})\}.$$

It is clear that $\mathbf{P}_n^{'}$ is an injective Sd-map and $\mathbf{P}_n^{'} \restriction \mathcal{F}(\mathbf{A}) : \mathcal{F}(\mathbf{A}) \longrightarrow \mathcal{F}(\mathbf{A}_n)$ is a homomorphism.

Now let $\mathbf{f}_0, \mathbf{f}_1, \dots$ be such a sequence that $\mathbf{f}_n \in \mathcal{F}(\mathbf{A}_n)$ and, for $m \geq n$, $\mathbf{P}_n^m(\mathbf{f}_m) = \mathbf{f}_n$. Let $x \in \mathrm{dom}\,(\mathbf{f}_n)$. According to the definition of $\mathbf{P}_n^m$, for $m \geq n$ there must be at least one element $y \in \mathrm{dom}\,(\mathbf{f}_m)$ such that $\mathbf{H}_n^m(y) = x$. We state that there is just one such element. On the contrary, let y' be a second element with these properties. Now for $k \geq m$ there are elements $x_k, and x_k'$ such that $\mathbf{H}_m^k(x_k) = y, \mathbf{H}_m^k(x_k') = y'$ in $\mathrm{dom}\,(\mathbf{f}_k)$ and therefore the sequences $\mathbf{H}_0^m(y), \mathbf{H}_1^m(y), \dots, x, \dots, y, x_{m+1}, x_{m+2}, \dots$ and $\mathbf{H}_0^m(y'), \mathbf{H}_1^m(y'), \dots, x, \dots, y', x_{m+1}', x_{m+2}', \dots$ are distinct and therefore there are the distinct elements $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ which correspond to these two sequences by the projections $\mathbf{H}_n$.

These considerations yield the function

$$\mathbf{f} = \mathbf{\Sigma}\{\mathbf{f}_0(x) \cdot \mathbf{g}_{\mathbf{H}_0^{\prime\,-1}(x)};\, x \in \mathrm{dom}\,(\mathbf{f}_0)\} \in \mathcal{F}(\mathbf{A}).$$

We have $\mathbf{P}_n^{'}(\mathbf{f}) = \mathbf{f}_n$. So $\mathcal{F}(\mathbf{A})$ is an inverse limit of Sd-IS of groups $\{(\mathcal{F}(\mathbf{A}_n), \oplus_n), \mathbf{P}_n^m, m \geq n\}$. □

If $(\mathbf{G}, \oplus)$ is a commutative π-group with i.p., then according to 1.8, $\mathbf{G}$ and $\oplus$ are revealed classes. Combining 3.1 and 3.3 yields

Corollary 3.4. *Let* $(\mathbf{A}, +)$ *and* $(\mathbf{B}, \oplus)$ *be commutative* π*-groups with i.p. Then* $\mathcal{F}(\mathbf{A} \times \mathbf{B})$ *is a commutative* π*-group with i.p. which is an inverse limit of* $\{\mathcal{F}(\mathbf{A}_n \times \mathbf{B}_n), \mathbf{P}_n^m \times \underline{P}_n^m, m \geq n\}$.

($\underline{P}_n^m$ is defined similarly as $\mathbf{P}_n^m$ in the proof of 3.3, but in terms of $\underline{H}_n^m$ instead of $\mathbf{H}_n^m$.)

Definition 3.5. Let $(\mathbf{A}, +)$ and $(\mathbf{B}, \boxplus)$ be commutative groups. Let $\mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}, \mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2, \mathbf{b} = \mathbf{b}_1 \boxplus \mathbf{b}_2$. Let $\mathcal{R}(\mathbf{A}, \mathbf{B})$ be the minimal symmetric relation on $\mathcal{F}(\mathbf{A} \times \mathbf{B})$ such that it holds that

$$[\mathbf{a}, \mathbf{b}]\ \mathcal{R}(\mathbf{A}, \mathbf{B})\,([\mathbf{a}_1, \mathbf{b}] \oplus [\mathbf{a}_2, \mathbf{b}]) \quad \text{and} \quad [\mathbf{a}, \mathbf{b}]\ \mathcal{R}(\mathbf{A}, \mathbf{B})\,([\mathbf{a}, \mathbf{b}_1] \oplus [\mathbf{a}, \mathbf{b}_2]).$$

Now we put

$$\mathbf{x}\mathbf{R}(\mathbf{A}, \mathbf{B})\mathbf{y} \equiv (\exists\, \alpha \in \mathbf{N})\,(\exists\, \mathbf{f})\,(\mathrm{Fnc}\,(\mathbf{f})\,\&\,\mathrm{dom}\,(\mathbf{f}) = \alpha{+}1\,\&\,\mathrm{rng}\,(\mathbf{f}) \subseteq \mathcal{F}(\mathbf{A} \times \mathbf{B})\,\&$$
$$\&\,\mathbf{f}(0) = \mathbf{x}\,\&\,\mathbf{f}(\alpha) = \mathbf{y}\,\&\,(\forall\, \beta \in \alpha)\,(\mathbf{f}(\beta)\mathcal{R}(\mathbf{A}, \mathbf{B})\mathbf{f}(\beta + 1))).$$

The relation $\mathbf{R}(\mathbf{A}, \mathbf{B})$ is an equivalence relation on $\mathcal{F}(\mathbf{A}, \mathbf{B})$. If $\mathbf{A}, \mathbf{B}$ are set-definable groups, then also $\mathbf{R}(\mathbf{A}, \mathbf{B})$ is a set-definable relation. Now we put

$$\mathbf{A} \otimes \mathbf{B} = \mathcal{F}(\mathbf{A}, \mathbf{B}) / \mathbf{R}(\mathbf{A}, \mathbf{B})$$

and we call $\mathbf{A} \otimes \mathbf{B}$ the *tensor product of* $\mathbf{A}$ *and* $\mathbf{B}$.

Theorem 3.6. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be commutative* π*-groups with i.p. Let* $\mathbf{a}, \mathbf{b} \in \mathcal{F}(\mathbf{A} \times \mathbf{B}), \mathbf{a} = \Sigma\{\mathbf{f}(a_j, \underline{a}_j)[a_j, \underline{a}_j];\ j \in J\}, \mathbf{b} = \Sigma\{\mathbf{g}(b_k, \underline{b}_k)[b_k, \underline{b}_k];\ k \in K\}$. *We put*

$$\begin{aligned}\mathbf{aD}(\mathbf{A}, \mathbf{B})\mathbf{b} \equiv &(\forall\, j \in J)(\forall\, k \in K)(\exists\, a_j^f, a_j^s, \underline{a}_j^f, \underline{a}_j^s, b_k^f, b_k^s, \underline{b}_k^f, \underline{b}_k^s)\\ &\Big(a_j^s \subseteq \mathbf{A}\ \&\ \mathrm{Fnc}\,(a_j^f)\ \&\ a_j^f : a_j^s \longrightarrow \mathbf{Z} - \{0\}\ \&\ \dots\ \&\ \underline{b}_k^s \subseteq \mathbf{B}\ \&\\ &\&\ \mathrm{Fnc}\,(\underline{b}_k^f)\ \&\ \underline{b}_k^f : \underline{b}_k^s \longrightarrow \mathbf{Z} - \{0\}\ \&\ a_j = \Sigma\{a_j^f(x) \cdot x;\ x \in a_j^s\}\ \&\\ &\&\ \dots\ \&\ \underline{b}_k = \Sigma\{\underline{b}_k^f(x) \cdot x;\ x \in \underline{b}_k^s\}\ \&\\ &\&\ \mathbf{\Sigma}\{\Sigma\{\mathbf{f}(a_j, \underline{a}_j) \cdot a_j^f(y)[x, y];\ [x, y] \in a_j^s \times \underline{a}_j^s\};\ j \in J\} =\\ &= \mathbf{\Sigma}\{\Sigma\{\mathbf{g}(b_k, \underline{b}_k) \cdot \mathbf{b}_k^f(x) \cdot \underline{b}_k^f(y)[x, y];\ [x, y] \in b_k^s \times \underline{b}_k^s\};\ k \in K\}\Big).\end{aligned}$$

(*The meaning of this formula is similar to that one in the classical definition of the tensor product. The unique difference is that we use an operation of the infinite set summation in* $\mathcal{F}(\mathbf{A} \times \mathbf{B})$ *instead of a binary operation in this formula.*)

If $\mathbf{aD}(\mathbf{A}, \mathbf{B})\mathbf{b}$, *then* $\mathbf{aR}(\mathbf{A}, \mathbf{B})\mathbf{b}$.

Proof: According to 3.4 $\mathcal{F}(\mathbf{A} \times \mathbf{B})$ is a commutative π-group with i.p. Let $\mathbf{F}_0'$ be its 0-th injective Sd-projection. Let $\mathbf{x} \subseteq \mathbf{A} \times \mathbf{B}$ and let $\mathbf{f} : \mathbf{x} \longrightarrow \mathbf{Z} - \{0\}$. Then according to the proof of 1.6 we can see that

$$\mathbf{\Sigma}\{\mathbf{f}(y) \cdot y;\ y \in \mathbf{x}\} = \mathbf{F}_0'^{-1}(\mathbf{\Sigma}\{\mathbf{f}(y) \cdot \mathbf{F}_0'(y);\ y \in \mathbf{x}\})$$

because $\mathbf{F}_0'$ is an injection. Hence the infinite summation in $\mathcal{F}(\mathbf{A} \times \mathbf{B})$ described in 1.6 is the restriction of an Sd-operation. The rest of the proof can be made by means of induction. □

Example 3.7. $\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{Z}$.

Proof: Let $\mathbf{f} \in \mathcal{F}(\mathbf{Z} \times \mathbf{Z})$. Then $\mathbf{f} = \Sigma\{\mathbf{f}(\gamma)[\alpha_\gamma, \beta_\gamma];\ \gamma \in \alpha\}$ for an appropriate $\alpha \in \mathbf{N}$. Then $\mathbf{f}\,\mathbf{R}(\mathbf{Z}, \mathbf{Z})\,\Sigma\{[1, \mathbf{f}(\gamma) \cdot \alpha_\gamma \cdot \beta_\gamma];\ \gamma \in \alpha\}$.

Now we shall define $\mathbf{H} : \mathbf{Z} \otimes \mathbf{Z} \longrightarrow \mathbf{Z}$ as follows. Let $\mathbf{f}$ be as above. Then put $\mathbf{H}([\mathbf{f}]) = \Sigma\{\mathbf{f}(\gamma) \cdot \alpha_\gamma \cdot \beta_\gamma;\ \gamma \in \alpha\}$. (Here $[\mathbf{f}]$ assigns the equivalence class in $\mathbf{R}(\mathbf{Z}, \mathbf{Z})$ relevant to $\mathbf{f}$.)

Obviously this map is surjective and it preserves the operation. Also it is clear that it is correctly defined, injective and it is a homomorphism. □

Next, let $(\mathbf{A}, +)$ and $(\mathbf{B}, \oplus)$ be commutative π-groups with i.p. Let $\{(\mathbf{A}_n, +_n), \mathbf{H}_n^m, m \geq n\}$ and $\{(\mathbf{B}_n, \oplus_n), \underline{H}_n^m, m \geq n\}$ respectively be their injective representation and let $\mathbf{H}_n'$ and $\underline{H}_n'$ respectively be injective Sd-projections.

Take a codable class $\{\mathbf{A}_n \otimes \mathbf{B}_n, \mathbf{Q}_n^m, m \geq n\}$ where $\mathbf{Q}_n^m$ are defined for $m \geq n$ as follows:
Let $\mathbf{f} \in \mathcal{F}(\mathbf{A}_m \times \mathbf{B}_m)$. Then

$$\mathbf{Q}_n^m([\mathbf{f}]\mathbf{R}(\mathbf{A}_m, \mathbf{B}_m)) = [\mathbf{P}_n^m \times \underline{P}_n^m(\mathbf{f})]\,\mathbf{R}(\mathbf{A}_n, \mathbf{B}_n).$$

As $\mathcal{F}(\mathbf{A}_\times \mathbf{B}_n), \mathbf{P}^m_n, \underline{P}^m_n$, $\mathbf{R}(\mathbf{A}_n, \mathbf{B}_n)$ are set-definable classes, then $\mathbf{Q}^m_n$ are Sd-classes, too.

It is clear that $\mathbf{Q}^m_n$ are correctly defined maps and homomorphisms. Also it is clear that for $m \geq k \geq n$ $\mathbf{Q}^m_n = \mathbf{Q}^k_n \circ \mathbf{Q}^m_k$. Hence $\{\mathbf{A}_n \otimes \mathbf{B}_n, \mathbf{Q}^m_n, m \geq n\}$ is an Sd-IS of groups.

Let us consider the following map $\mathbf{Q}'_n$:

$$\mathbf{Q}'_n(\mathbf{f}) = [\mathbf{P}'_n \times \underline{P}'_n(\mathbf{f})]\mathbf{R}(\mathbf{A}_n, \mathbf{B}_n).$$

Each of $\mathbf{Q}'_n$ is an Sd-map and $\mathbf{Q}_n = \mathbf{Q}'_n \restriction \mathcal{F}(\mathbf{A} \times \mathbf{B})$ are homorphisms satisfying $\mathbf{Q}_n = \mathbf{Q}^m_n \circ \mathbf{Q}_m$ and stable under $\mathbf{R}(\mathbf{A}, \mathbf{B})$ (i.e. if $\mathbf{f}\, \mathbf{R}(\mathbf{A}, \mathbf{B})\mathbf{g}$, then $\mathbf{Q}_n(\mathbf{f}) = \mathbf{Q}_n(\mathbf{g})$). Instead of $\mathbf{R}(\mathbf{A}_n, \mathbf{B}_n)$ and $\mathbf{R}(\mathbf{A}, \mathbf{B})$ we shall write $\mathbf{R}_n$ and $\mathbf{R}$, respectively.

Let $[\mathbf{f}_0]\mathbf{R}_0, , [\mathbf{f}_1]\mathbf{R}_1, \dots$ be such a sequence that $\mathbf{Q}^m_n([\mathbf{f}_m]\mathbf{R}_m) = [\mathbf{f}_n]\mathbf{R}_n$. First of all we prove that there is $\mathbf{f} \in \mathcal{F}(\mathbf{A} \times \mathbf{B})$ such that $\mathbf{Q}_n(\mathbf{f}) = [\mathbf{f}_n]\mathbf{R}_n$.

For we can prolong the sequence $\mathbf{f}_0, \mathbf{f}_1, \dots$ to $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_\alpha$, $\alpha \in$ **N-FN** which is a set. For each n it holds that

$$\begin{aligned}[\mathbf{f}_0]\mathbf{R}_0 = \mathbf{Q}^n_0([\mathbf{f}_n]\mathbf{R}_n), [\mathbf{f}_1]\mathbf{R}_1 = \mathbf{Q}^n_1([\mathbf{f}_n]\mathbf{R}_n), \dots, [\mathbf{f}_{n-1}]\mathbf{R}_{n-1} &= \\ = \mathbf{Q}_{n-1}([\mathbf{f}_n]\mathbf{R}_n), [\mathbf{f}_n]\mathbf{R}_n &= \mathbf{Q}^n_n([\mathbf{f}_n]\mathbf{R}_n).\end{aligned}$$

We can also prolong all the countable codable systems which could be prolonged—($\{\mathbf{A}_n; n \in \mathbf{FN}\}$, $\{\mathbf{B}_n; n \in \mathbf{FN}\}$, $\{\mathbf{H}^m_n; m \geq n, m, n \in \mathbf{FN}\}$, $\{\underline{H}^m_n; m \geq n, m, n \in \mathbf{FN}\}$, $\{\mathbf{H}'_n; n \in \mathbf{FN}\}$, $\{\underline{H}'_n; n \in \mathbf{FN}\}$—and at the same time also $\{\mathbf{P}^m_n; m \geq n\}$, $\{\underline{P}^m_n; m \geq n\}$, $\{\mathbf{Q}^m_n; m \geq n\}$, $\{\mathbf{P}'_n; n \in \mathbf{FN}\}$, $\{\underline{P}'_n; n \in \mathbf{FN}\}, \{\mathbf{Q}'_n; n \in \mathbf{FN}\}$) to the appropriate Sd* systems which fulfil all required properties—appropriate maps are identities, appropriate are injections, they fulfil equations of the kind (b) of 1.2, etc.—up to some infinite α. By this procedure, we can obtain an infinite $\beta \in \alpha$ such that

$$[\mathbf{f}_0]\mathbf{R}_0 = \mathbf{Q}^\beta_0([\mathbf{f}_\beta]\mathbf{R}_\beta), \dots, [\mathbf{f}_n]\mathbf{R}_n = \mathbf{Q}^\beta_n([\mathbf{f}_\beta]\mathbf{R}_\beta), \dots, [\mathbf{f}_\beta]\mathbf{R}_\beta = \mathbf{Q}^\beta_\beta([\mathbf{f}_\beta]\mathbf{R}_\beta).$$

We have $\mathbf{f}_\beta \in \mathcal{F}(\mathbf{A}_\beta \times \mathbf{B}_\beta)$. The sequence $\mathbf{P}^\beta_0 \times \underline{P}^\beta_0(\mathbf{f}_\beta), \mathbf{P}^\beta_1 \times \underline{P}^\beta_1(\mathbf{f}_\beta), \dots$ is such that $\mathbf{P}^m_n \times \underline{P}^m_n(\mathbf{P}^\beta_m \times \underline{P}^\beta_m(\mathbf{f}_\beta)) = \mathbf{P}^\beta_n \times \underline{P}^\beta_n(\mathbf{f}_\beta)$ and hence there is $\mathbf{f} \in \mathcal{F}(\mathbf{A} \times \mathbf{B})$ which corresponds to this sequence by projections. So we have

$$\begin{aligned}&\mathbf{P}'_m \times \underline{P}'_m(\mathbf{f}) = \mathbf{P}^\beta_m \times \underline{P}^\beta_m(\mathbf{f}_\beta) \ \text{ and hence} \\ &\mathbf{Q}'_m(\mathbf{f}) = \mathbf{Q}^\beta_m([\mathbf{f}_\beta]\mathbf{R}_\beta) = [\mathbf{f}_m]\mathbf{R}_m \quad \text{(for every } m).\end{aligned}$$

In this moment, we can represent this result, as $\{\mathbf{Q}_n, n \in \mathbf{FN}\}$ is a "map" onto the Sd-IS $\{\mathbf{A}_n \oplus \mathbf{B}_n, \mathbf{Q}^m_n, m \geq n\}$ and that it is a "homomorphism". Later, we shall give its more precise and correct representation.

Moreover, this "map" is injective in the following sense:

$$\text{not}\,(\mathbf{f}\ \mathbf{R}\ \mathbf{g}) \Rightarrow (\exists\, n \in \mathbf{FN})\,(\mathbf{Q}_n(\mathbf{f}) \neq \mathbf{Q}_n(\mathbf{g}).$$

For let $\mathbf{f},\mathbf{g} \in \mathcal{F}(\mathbf{A}\times\mathbf{B})$ be such that $(\forall\ n)(\mathbf{Q}_n'(\mathbf{f}) = \mathbf{Q}_n'(\mathbf{g}))$. Let $\mathbf{h}_n : \alpha_n + 1 \longrightarrow \mathcal{F}(\mathbf{A}_n\times\mathbf{B}_n)$ be a function which satisfies $\mathbf{h}_n(0) = \mathbf{P}_n'\times\underline{P}_n'(\mathbf{f}), \mathbf{h}_n(\alpha_n) = \mathbf{P}_n'\times\underline{P}_n'(\mathbf{g})$ and $(\forall\ \beta\in\alpha_n)(\mathbf{h}_n(\beta)\mathcal{R}_n\,\mathbf{h}_n(\beta+1))$. Let us prolong this sequence and all what is necessary. We get $\gamma\in\mathbb{N}$-$\mathbf{FN}$ such that $\mathbf{h}_\gamma : \alpha_\gamma+1\longrightarrow\mathcal{F}(\mathbf{A}_\gamma\times\mathbf{B}_\gamma)$ is a function which fulfils $\mathbf{h}_\gamma(0) = \mathbf{P}_\gamma'\times\underline{P}_\gamma'(\mathbf{f}), \mathbf{h}_\gamma(\alpha_\gamma) = \mathbf{P}_\gamma'\times\underline{P}_\gamma'(\mathbf{g})$ and $(\forall\ \beta\in\alpha_\gamma)(\mathbf{h}_\gamma(\beta)\,\mathcal{R}_\gamma\,\mathbf{h}_\gamma(\beta+1))$.

Now we can proceed in a similar way as in previous considerations. Put $\mathbf{a} = \{x; (\exists\ \beta\in\alpha_\gamma+1)(x\in\operatorname{dom}(\mathbf{h}_\gamma(\beta))$, or x is necessary to secure some of the relations $\mathbf{h}_\gamma(\beta)\mathcal{R}_\gamma\mathbf{h}_\gamma(\beta+1)\}$. This $\mathbf{a}$ is a set, because $\mathcal{R}_\gamma$ is an Sd* class. Let $x\in\mathbf{a}$. Then $\underline{h} = (\mathbf{H}_0'\times\mathbf{H}_0')^{-1}(\mathbf{H}_0^\gamma\times\underline{H}_0^\gamma)(x)\in\mathbf{A}\times\mathbf{B}$ is uniquely determined, because of injectivity of $\mathbf{H}_n'\times\underline{H}_n'$ for each n. Next, putting

$$\underline{a} = (\mathbf{H}_0'\times\underline{H}_0')^{-1}{}''\,\mathbf{H}_0^\gamma\times\underline{H}_0^{\gamma}{}''\,\mathbf{a},\ \text{ we obtain that}$$
$$\underline{a} = (\mathbf{H}_n'\times\underline{H}_n')^{-1}{}''\,\mathbf{H}_n^\gamma\times\underline{H}_n^{\gamma}{}''\,\mathbf{a}\ \text{ for each } n.$$

Therefore there is $\nu\in\gamma$-$\mathbf{FN}$ such that

$$\underline{a} = (\mathbf{H}_\nu'\times\underline{H}_\nu')^{-1}{}''\,\mathbf{H}_\nu^\gamma\times\underline{H}_\nu^{\gamma}{}''\,\mathbf{a}$$

and hence the function $\underline{h}_\nu : \alpha_\gamma+1\longrightarrow\mathcal{F}(\mathbf{A}_\nu\times\mathbf{B}_\nu)$ defined by

$$\underline{h}_\nu(\beta) = \mathbf{P}_\nu^\gamma\times\underline{P}_\nu^\gamma(\mathbf{h}_\gamma(\beta))$$

fulfils $\underline{h}_\nu(0) = \mathbf{P}_\nu'\times\underline{P}_\nu'(\mathbf{f}), \underline{h}_\nu(\alpha_\gamma) = \mathbf{P}_\nu'\times\underline{P}_\nu'(\mathbf{g})$ and $(\forall\ \beta\in\alpha_\gamma)(\underline{h}_\nu(\beta)\mathcal{R}_\nu\,\underline{h}_\nu(\beta+1))$. But as we can invert the set $\mathbf{H}_\nu^\gamma\times\underline{H}_\nu^{\gamma}{}''\,\mathbf{a}$ to $\underline{a}$, we can invert the all function $\underline{h}_\nu$ and also all what is necessary to secure the relations $\underline{h}_\nu(\beta)\mathcal{R}_\nu\underline{h}_\nu(\beta+1)$. By means of this, we obtain $\mathbf{h} : \alpha_\gamma+1\longrightarrow\mathcal{F}(\mathbf{A}\times\mathbf{B})$ such that $\mathbf{h}(0) = \mathbf{f}, \mathbf{h}(\alpha_\gamma) = \mathbf{g}$ and $(\forall\ \beta\in\alpha_\gamma)(\mathbf{h}(\beta)\mathcal{R}\mathbf{h}(\beta+1))$. This function $\mathbf{h}$ secures that $\mathbf{f}\,\mathbf{R}\,\mathbf{g}$.

According to the axiom of choice, we can choose a selector from the relation $\mathbf{R}$ on the class $\mathcal{F}(\mathbf{A}\times\mathbf{B})$ (i.e. we choose one function from each class of the equivalence $\mathbf{R}$). Denote this selector by $\mathbf{G}$. The class $\mathbf{G}$ can be enriched by the group structure induced from $\mathcal{F}(\mathbf{A}\times\mathbf{B})$ and $\mathbf{R}$. (this means that in fact $\mathbf{G}$ is (isomorphic to) $\mathcal{F}(\mathbf{A}\times\mathbf{B})/\mathbf{R}$ or that $\mathbf{G} = \mathbf{A}\otimes\mathbf{B}$.)

The above considerations yield

Theorem 3.8. *Let $(\mathbf{A},+)$ and $(\mathbf{B},\oplus)$ be commutative π-groups with i.p. Then $\mathbf{G}$ is an inverse limit of the Sd-IS of groups $\{\mathbf{A}_n\otimes\mathbf{B}_n, \mathbf{Q}_n^m, m\geq n\}$ and hence it is a commutative π-group.*

Proof: The required projections and their extensions are induced by $\mathbf{Q}_n$ and $\mathbf{Q}_n'$ from the above considerations. □

Example 3.9. Let $\mathbf{G}$ be a commutative π-group with i.p. Then $\mathbf{Z}\otimes\mathbf{G}\cong\mathbf{G}$.

Proof: This fact is a special case of the next example. □

Example 3.10. Let $\mathbf{G}$ be a commutative π-group with i.p., let $\mathbf{X}$ be a π-class. Put

$$\mathbf{G}^{\mathbf{X}} = \{\mathrm{Fnc}\,(\mathbf{f})\,\&\ \mathrm{dom}\,(\mathbf{f}) = \mathbf{u} \subseteq \mathbf{X}\,\&\,\mathbf{f} : \mathbf{u} \longrightarrow \mathbf{G}\text{-}\{0\}\}$$

together with the operation induced from the group $\mathbf{G}$ in the same way as it is done in 1.1. Then $\mathcal{F}(\mathbf{X}) \otimes \mathbf{G} \cong \mathbf{G}^{\mathbf{X}}$.

Proof: According to the assumptions, the group $\mathcal{F}(\mathcal{F}(\mathbf{X}) \times \mathbf{G})$ is a commutative π-group. Therefore, if $\mathbf{f} \in \mathcal{F}(\mathcal{F}(\mathbf{X}) \times \mathbf{G})$, then there are $l_\gamma \in \mathcal{F}(\mathbf{X}), \mathbf{g}_\gamma \in \mathbf{G}$ such that $\mathbf{f} = \Sigma\{\mathbf{f}(\gamma) \cdot [l_\gamma, g_\gamma];\ \gamma \in \varepsilon\}$. The universal class $\mathbf{V}$ can be enumerated by an Sd-bijection by $\mathbf{N}, \mathbf{V} = \{x_0, x_1, \dots\}$. Because $l_\gamma \in \mathcal{F}(\mathbf{X})$, we can write $l_\gamma = \Sigma\{\beta_\delta^\gamma \cdot x_\delta;\ \delta \in \mathbf{v}_\gamma\}$. Here $\mathbf{v}_\gamma$ is an appropriate subset of $\mathbf{N}$.

Instead of $\mathbf{R}(\mathcal{F}(\mathbf{X}), \mathbf{G})$ we shall write $\equiv$. Then we have

$$\begin{aligned}\mathbf{f} \equiv &\Sigma\{\mathbf{f}(\gamma) \cdot [\Sigma\{\beta_\delta^\gamma \cdot x_\delta;\ \delta \in \mathbf{v}_\gamma\}, g_\gamma];\ \gamma \in \varepsilon\} \equiv \\ \equiv &\Sigma\{\mathbf{f}(\gamma) \cdot (\Sigma\{\beta_\delta^\gamma \cdot [x_\delta, g_\gamma];\ \delta \in \mathbf{v}_\gamma\});\ \gamma \in \varepsilon\} \equiv \\ \equiv &\Sigma\{[x_\delta, \Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot \mathbf{g}_\gamma;\ \gamma \in \varepsilon\}];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\}.\end{aligned}$$

The last relation is fulfilled, if the element

$$\mathbf{F} = \Sigma\{[x_\delta, \Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\}];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\}$$

is correct, i.e. if it is an element of $\mathcal{F}(\mathcal{F}(\mathbf{X}) \times \mathbf{G})$. For this, it is enough to show that $\{[x_\delta, \Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\}];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\}$ is a set. (Indeed, we have $\mathbf{h} : \mathbf{u} \longrightarrow \mathbf{Z}$, the constant function $\mathbf{h}(x) = 1$, so that if the domain of $\mathbf{h}$ is a set, then $\mathbf{h}$ is also a set, hence the assumptions of 1.6 are fulfilled.)

So we must prove that $\{[x_\delta, \Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\}];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\}$ is a set.

Let $\mathbf{H}_n'$ be the injective Sd-extension of the projection of $\mathbf{H}_n : \mathbf{G} \longrightarrow \mathbf{G}_n$. Then $\Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\} = \mathbf{H}_n'^{-1}(\Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot \mathbf{H}_n'(g_\gamma);\ \gamma \in \varepsilon\})$ and therefore

$$\begin{aligned}&\{[x_\delta, \Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\}];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\} = \\ &\qquad = \{[x_\delta, \mathbf{H}_n'^{-1}(\Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot \mathbf{H}_n'(g_\gamma);\ \gamma \in \varepsilon\})];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\}\end{aligned}$$

is an Sd-class, its domain is a set. Hence this class is a set as well.

Now we can define a map $\mathbf{H} : \mathcal{F}(\mathbf{X}) \otimes \mathbf{G} \longrightarrow \mathbf{G}^{\mathbf{X}}$ as follows: (Sign $[\mathbf{f}]$ has a similar meaning as in 3.7)

$$\begin{aligned}\mathbf{H}([\mathbf{f}]) = \{[\Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\}, x_\delta];\ \delta \in \cup\{\mathbf{v}_\gamma;\ \gamma \in \varepsilon\}\,\& \\ \&\,\Sigma\{\mathbf{f}(\gamma) \cdot \beta_\delta^\gamma \cdot g_\gamma;\ \gamma \in \varepsilon\} \neq 0\}.\end{aligned}$$

According to the assumption that $\mathbf{G}$ is a commutative π-group with i.p., all projections $\mathbf{H}_n$ can be extended by some injective Sd-map $\mathbf{H}_n'$. As we have seen, the (infinite set) summation in $\mathbf{G}$ can be made by means of this Sd-map. Therefore $\mathbf{H}([\mathbf{f}])$ is really an element of $\mathbf{G}^{\mathbf{X}}$. The surjectivity of $\mathbf{H}$ is trivial and we can also see that the relation $\equiv$ is chosen just in such a way that this map is a homomorphism, correctly defined (i.e. this definition is independent on the choice of an element of $[\mathbf{f}]$), and injective. $\square$

Now let us consider $\mathbf{H}^{-1}$ (the inverse of $\mathbf{H}$ from the above proof). This is an effective selector from the relation $\equiv$ on the class $\mathcal{F}(\mathcal{F}(\mathbf{X})\times\mathbf{G})$, which is a restriction of some Sd-class. This means that $\mathbf{H}^{-1}$ (and also $\mathbf{H}$) is a total homomorphism.

From this moment, we shall mean by $\mathcal{F}(\mathbf{X}) \otimes \mathbf{G}$ the class $\mathbf{H}^{-1}{}'' \mathbf{G}^{\mathbf{X}}$ (for a π-class $\mathbf{X}$ and a commutative π-group with i.p. $\mathbf{G}$, of course).

Theorem 3.11. *Let $(\mathbf{A}, \mathcal{O}^a), (\mathbf{B}, \mathcal{O}^b), (\mathbf{C}, \mathcal{O}^c)$ and $(\mathbf{D}, \mathcal{O}^d)$ be commutative π-groups with i.p. with the injective representations $\{(\mathbf{X}_n, \mathcal{O}^x_n), {}^x\mathbf{H}^m_n, m \geq n\}$ and the injective Sd-projections ${}^x\mathbf{H}'_n$ for $x \in \{a, b, c, d\}$. Let $\mathbf{F} : \mathbf{A} \longrightarrow \mathbf{C}$ and $\mathbf{G} : \mathbf{B} \longrightarrow \mathbf{D}$ be homomorphisms which are restrictions of some Sd-maps $\mathbf{F}'$ and $\mathbf{G}'$ to the class $\mathbf{A}$ and $\mathbf{B}$ respectively. Then there are total homomorphisms*

$$\begin{aligned}
&\mathbf{F} \times \mathbf{G} : \mathbf{A} \times \mathbf{B} \longrightarrow \mathbf{C} \times \mathbf{D}\\
&\mathbf{P}_{\mathbf{F}\times\mathbf{G}} : \mathcal{F}(\mathbf{A} \times \mathbf{B}) \longrightarrow \mathcal{F}(\mathbf{C} \times \mathbf{D}) \quad \textit{and finally}\\
&\mathbf{F} \otimes \mathbf{G} : \mathbf{A} \otimes \mathbf{B} \longrightarrow \mathbf{C} \otimes \mathbf{D}
\end{aligned}$$

which are induced by $\mathbf{F}$ and $\mathbf{G}$.

Proof: The first homomorphism is described in the proof of 3.1. As $\mathbf{F}$ and $\mathbf{G}$ are total homomorphisms then $\mathbf{F} \times \mathbf{G}$ is so, too. We can define the second one as follows: let $\mathbf{f} \in \mathcal{F}(\mathbf{A} \times \mathbf{B})$, i.e. $\mathbf{f} = \Sigma\{\mathbf{f}([a, b]) \cdot \mathbf{g}_{[a,b]}; [a, b] \in \operatorname{dom}(\mathbf{f})\}$. Then we put $\mathbf{P}_{\mathbf{F}\times\mathbf{G}}(\mathbf{f}) = \Sigma\{\mathbf{f}([a, b]) \cdot \mathbf{g}_{[\mathbf{F}'(a),\mathbf{G}'(b)]}; [a, b] \in \operatorname{dom}(\mathbf{f})\}$. Again it is clear that it is a total homomorphism. For the last one let $\mathbf{f} \in \mathcal{F}(\mathbf{A} \times \mathbf{B})$. Then $\mathbf{P}_{\mathbf{F}\times\mathbf{G}}(\mathbf{f})$ is an element of $\mathcal{F}(\mathbf{C} \times \mathbf{D})$ and it is clear that $\mathbf{h}\,\mathbf{R}(\mathbf{A}, \mathbf{B})\mathbf{h}'$ implies $\mathbf{P}_{\mathbf{F}\times\mathbf{G}}(\mathbf{h})\,\mathbf{R}(\mathbf{C}, \mathbf{D})\mathbf{P}_{\mathbf{F}\times\mathbf{G}}(\mathbf{h}')$. Hence we can put $\mathbf{F} \otimes \mathbf{G}([\mathbf{f}]\mathbf{R}(\mathbf{A}, \mathbf{B})) = [\mathbf{P}_{\mathbf{F}\times\mathbf{G}}(\mathbf{f})]\, \mathbf{R}(\mathbf{C}, \mathbf{D})$ and it is a total homomorphism. (In the second and the third statement, we use the assumption of injectivity of the Sd-projections under which $\mathcal{F}(\mathbf{A} \times \mathbf{B})$ is a commutative π-group.) $\square$

Theorem 3.12. *Under the assumptions similar to those in* 3.9*, we have*

(a) $(\mathbf{F} \otimes \mathbf{G}) \circ (\mathbf{K} \otimes \mathbf{L}) = (\mathbf{F} \circ \mathbf{K}) \otimes (\mathbf{G} \circ \mathbf{L})$,
(b) $(\mathbf{F} + \underline{F}) \otimes \mathbf{G} = \mathbf{F} \otimes \mathbf{G} + \underline{F} \otimes \mathbf{G}$ *and* $\mathbf{F} \otimes (\mathbf{G} + \underline{G}) = \mathbf{F} \otimes \mathbf{G} + \mathbf{F} \otimes \underline{G}$.

Proof: The meaning and the proof of these statements are clear. $\square$

In the construction of homology theory we shall use groups of the kind $\mathcal{F}(\mathbf{X})$ for certain π-classes $\mathbf{X}$ which are the commutative π-groups with i.p. (see 1.5). By results of this paragraph, we can use the commutative π-groups with i.p. as the groups of coefficients.

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