# Plenary stable quasigroups

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*Abstract.* Some results are obtained for quasigroups in which the operation of iterated squaring always leads to stability irrespective of the initial element.

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An algebra $A$, not necessarily associative, that admits a homomorphism $w : A \to F$ into its ground field $F$, is called baric; and a baric algebra that satisfies the identity $(a^2)^2 = w^2(a)a^2$ is said to be Bernstein. Since their introduction in [4], Bernstein algebras have been the subject of much research. In a nonassociative system the right principal powers $a^j$ of an element $a$ are defined by $a^1 = a, a^j = a^{j-1}a$, and the plenary powers $a^{[j]}$ by $a^{[1]} = a, a^{[j]} = (a^{[j-1]})^2$. If instead of the above identity, $A$ satisfies $a^{[k+1]} = w^{2^k}(a)a^{[k]}$, it is called a generalised Bernstein algebra (see [5]). Since quasigroups stand in the same relation to groups that nonassociative algebras do to associative algebras, it is natural to investigate the classes $V_k$ of quasigroups defined by the identities $a^{[k]} = a^{[k+1]}, k \geq 1$. A quasigroup $Q$ satisfying such an identity will be called plenary stable. The difference class $U_k = V_k/V_{k-1}$ will be said to have plenary stability index $k$. This note presents some properties of these quasigroups.

Bruck [2] defined and obtained some results for unipotent and idempotent quasigroups, which are respectively those in which (i) $a^2 = b$ for some fixed $b$, all $a \in Q$, and (ii) $a^2 = a$, all $a \in Q$. The idempotent quasigroups make up $V_1$, while the unipotent quasigroups belong to $V_2$, and correspond to the 'quasiconstant' Bernstein algebras. He [2, p. 34] showed that every finite commutative quasigroup of odd order, and every finite group of odd order, is isotopic to an idempotent quasigroup. He showed further that, given an idempotent quasigroup of order $n$, it is possible to construct from it a unipotent quasigroup of order $n + 1$, and that this construction can moreover be carried out within the class of totally symmetric quasigroups. On the other hand, every quasigroup is isotopic to a unipotent quasigroup [4, p. 31].

For each quasigroup $Q$ of order $n$, we construct a directed graph $D(Q)$. The $n$ nodes of $D(Q)$ are labelled with the elements of $Q$, and if $a^2 = b$, an edge is directed from $a$ to $b$. Thus $D(Q)$ has no multiple edges, but may have single loops attached to some nodes. Then $Q$ is plenary stable if and only if $D(Q)$ contains no proper cycles. In this case $D(Q)$ is a union (often called a forest) of disjoint trees, each of which contains one node representing an idempotent. These nodes have loops attached, and otherwise only in-edges, and can be taken as the roots of their respective trees. Every edge of $D(Q)$ is directed towards the root of the tree to which it belongs. Plenary stable quasigroups corresponding to the same directed rooted tree will be said to be of the same type, and $D(Q)$ will be called the type graph. However, not every rooted forest with loops attached to each root corresponds to a non-empty class of plenary stable quasigroups.

At one extreme the type graph with $n$ unconnected nodes arises from the idempotent quasigroups, which exist for every order $n \geq 3$. The tree with a single root from which spring $n-1$ stems corresponds to the unipotent quasigroups, which exist for every order $n \geq 2$. At the other extreme is the single directed chain of length $n$. A quasigroup of this type has maximal index of plenary stability. For $n \leq 3$, the corresponding quasigroup set is empty.

For $n = 4$, we have the example

$$\begin{array}{c|cccc} & \mathrm{a} & b & c & d \\ \hline a & b & d & a & c \\ b & d & c & b & a. \\ c & c & a & d & b \\ d & a & b & c & d \end{array}$$

The isotope of this obtained by applying the transposition (c, d) to the head and sidelines also has maximal index of plenary stability, but that obtained by applying (b, d) does not. In general, stability properties are not preserved under isotopy.

**Proposition 1.** *Every quasigroup $Q$ has a symmetric principal isotope that is plenary stable. Every quasigroup of order $n \geq 3$ has a principal isotope that is not plenary stable. If $Q$ is not unipotent or diagonal, the isotope can be taken symmetric principal.*

Proof: (i) By making corresponding changes in the order of rows and columns while keeping head and sidelines fixed, we can arrange the diagonal elements of the multiplication table in ascending order. (ii) If there are at least two elements on the diagonal of the multiplication table, we can make corresponding interchanges of rows and columns to produce a cycle. If this condition is not satisfied, we can produce it by interchanges of either rows or columns. □

Plenary stability is well behaved with respect to elementary quasigroup structure.

**Proposition 2.** *Every subquasigroup and every homomorphic image of a plenary stable quasigroup is plenary stable. The direct product of a finite collection of plenary stable quasigroups of indices $k_1, k_2, \ldots, k_m$, is plenary stable, with index* $\max(k_1, k_2, \ldots, k_m)$.

Proof: The last result follows from $(a_1 \otimes a_2 \otimes \cdots \otimes a_k)^{[s]} = a_1^{[s]} \otimes a_2^{[s]} \otimes \cdots \otimes a_k^{[s]}$. □

The product of two plenary stable subquasigroups $Q_1, Q_2$, of a quasigroup, in the sense of the subquasigroup generated by $Q_1 \cap Q_2$, is not necessarily plenary stable. This was pointed out by the referee who supplied the following example:

| | a | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $c$ | $e$ | $g$ | $f$ | $b$ | $d$ |
| $b$ | $c$ | $b$ | $g$ | $f$ | $e$ | $d$ | $a$ |
| $c$ | $e$ | $g$ | $d$ | $b$ | $a$ | $f$ | $c$. |
| $d$ | $g$ | $f$ | $b$ | $c$ | $d$ | $a$ | $e$ |
| $e$ | $f$ | $e$ | $a$ | $d$ | $g$ | $c$ | $b$ |
| $f$ | $b$ | $d$ | $f$ | $a$ | $c$ | $e$ | $g$ |
| $g$ | $d$ | $a$ | $c$ | $e$ | $b$ | $g$ | $f$ |

The commutative quasigroup $Q$ is generated by its subquasigroups $Q_1 = (a)$, $Q_2 = (b)$, which are trivially plenary stable, but since $c^2 = d, d^2 = c$, it is not itself plenary stable. Certain properties prevent plenary stability. A diagonal quasigroup is one that contains each of its $n$ elements on the diagonal.

**Proposition 3.** *A diagonal quasigroup $Q$ which is not idempotent cannot be plenary stable.*

PROOF: Since all elements occur as squares, the sequence of plenary powers must contain cycles, proper or degenerate. If they are all of length one $Q$ is idempotent, and otherwise there is a proper cycle. □

Every commutative quasigroup of odd order is diagonal, as is every Bol-loop, i.e. loop satisfying $(a((bc)b)) = (((ab)c)b)$, [3, pp. 31, 32]. A square with one $a$, and $n-1$ $b$'s on its diagonal cannot be completed to a Latin square. In this case there are two possible type graphs. If $a^2 = a$, the node corresponding to $a$ is isolated, and the node corresponding to $b$ has arrows to each of the remaining $n-2$ nodes. This partial multiplication table can be completed to the multiplication table of a groupoid satisfying $a^{[3]} = a^{[2]}$, but it cannot be completed to that of a quasigroup. If $a^2 = b$, we obtain the type graph with $n-2$ nodes joined to the root, with one of the $n-2$ leading to a further node.

**Lemma.** *The type graph of a plenary stable loop $Q$ is a single rooted tree.*

PROOF: In a loop the identity is the unique idempotent. Hence it must be the only root. □

**Proposition 4.** *No group of odd order can be plenary stable.*

PROOF: The node adjacent to the root is of order 2, hence the result. □

The tables of quasigroups and loops provided in [7] and [3, pp. 129–137] enable us to examine the prevalence of plenary stability among quasigroups and loops of small order. For $n = 4$, the two possible non-isomorphic loops are both groups, of which Klein's 4-group is plenary stable, while the cyclic group of order 4 is not. There are 15 non-isomorphic sets of quasigroups of order 4. The following contingency table shows the numbers of these sets, and in brackets the corresponding numbers of distinct quasigroups that they account for, classified by whether they are isotopic to the cyclic group $Z_4$ or to Klein's 4-group $K$, and whether or not they are plenary stable.

| | Stable | Not Stable | Total |
|---|---|---|---|
| Isotopic to $Z_4$ | 3 (42) | 3 (66) | 6 (108) |
| Isotopic to $K$ | 5 (288) | 4 (180) | 9 (468) |
| All cases | 8 (330) | 7 (246) | 15 (576) |

It appears that there is a statistical propensity to greater plenary stability among the isotopes of $K$ compared with those of $Z_4$. For $n = 5$, two of the six isomorphism classes of loops are plenary stable. One of these is unipotent and the other satisfies $a^2 = e, b^2 = c^2 = d^2 = a$. There are 109 isomorphism classes of loops for $n = 6$, of which 40 are plenary stable. These are distributed among 15 different type graphs. Two of the plenary stable isomorphism classes have maximal index, and there are 4 distinct isomorphism classes of unipotent loops. Of the 18 possible unlabelled rooted trees on six nodes, three do not correspond to non-empty classes of plenary stable loops. One of these, the root joined to four nodes, one of which is joined to the sixth, does not even have a corresponding plenary stable quasigroup.

The above statistics suggest that a high proportion of quasigroups are plenary stable. For larger order I conjecture that this is not so. Consider the category of groupoids. Let the head and sidelines of the multiplication table of a random groupoid $G^*$ be fixed, and each cell filled in independently and randomly, each of the $n$ possible entries being equally likely. We investigate the probability that the random graph $D(G^*)$ contains a cycle. Each of the ordered sets of $k$ nodes will give rise to a cycle with probability $(n^{-1})^k$, each of the $n^{(k)}$ possible cycles being counted $k$ times. Hence the expected number of cycles of length $k$ is $k^{-1}n^{(k)}n^{-k}$. The expected number of cycles of any length is $\mu_n = \sum_2^n k^{-1}n^{(k)}n^{-k}$. Only a small proportion of pairs of possible cycles are dependent, and the distribution of the number of cycles is asymptotically Poisson. The probability of zero, and hence of $G^*$ being plenary stable, is approximately $\exp(-\mu_n) = O(n^{-1})$. Thus of the $n^3$ groupoids on $n$ distinguishable elements, about $cn^2$ are plenary stable. Quasigroups constitute an infinitesimal proportion of groupoids, and it would be interesting to make carry out a similar calculation for them.
We have seen that, given a rooted forest, it is not always possible to construct a plenary stable quasigroup for which it is the type. We may ask which forests are admissible in the sense that there is a corresponding type. It is of particular interest to investigate the existence of plenary stable quasigroups with maximal possible index.

**Proposition 5.** *For every even $n \geq 4$ there exists a quasigroup of order $n$ which is plenary stable of index $n-1$.*

We present two proofs. The first, provided by the referee, leads to the explicit construction of a plenary stable quasigroup with maximal index for each order $n \geq 4$.

First Proof: Consider the group $Z_n = \{0, 1, \ldots, n-1\}$ of integers with addition modulo $n$. Let $t(i)$ be the transformation $t(i) = i+1$ if $i \neq \frac{1}{2}n, n-1, \frac{1}{2}n-1$; $t(\frac{1}{2}n) =$

$t(n-1) = \frac{1}{2}n$; $t(\frac{1}{2}n-1) = \frac{1}{2}n+1$. Then $\sum t(i) = 0$, and hence by a result of M. Hall Jr. quoted in [6], there are permutations $f$ and $g$ of $Z_n$ such that $t(i) = f(i) + g(i)$ for every $i \in Z_n$. Now if we define $i^*j = f(i) + g(j)$, we obtain a quasigroup $Z_n^*$ isotopic to $Z_n$. Now $0^*0 = 0, 1^*1 = 2, \ldots, (\frac{1}{2}n-2)^*(\frac{1}{2}n-2) = \frac{1}{2}n-1, (\frac{1}{2}n-1)^*(\frac{1}{2}n-1) = \frac{1}{2}n+1, (\frac{1}{2}n+1)^*(\frac{1}{2}n+1) = \frac{1}{2}n+2, \ldots, (n-2)^*(n-2) = n-1, (n-1)^*(n-1) = \frac{1}{2}n, \frac{1}{2}n^*\frac{1}{2}n = \frac{1}{2}n$. Thus $Z_n^*$ is plenary stable of index $n-1$. □

The second proof does not involve computation but relies on properties of Latin squares. If $Q$ is a quasigroup on whose diagonal one element occurs $k_1$ times, another $k_2$ times, etc., we say that it has diagonal partition $(k_1, \ldots, k_p)$. The diagonal partition of a plenary stable quasigroup is determined by, but does not determine, its type graph.

**Lemma.** *There exists a quasigroup with diagonal partition $(k_1, \ldots, k_p)$ provided that $\sum x_j k_j \equiv 0 \pmod n$ has a solution in distinct integers $x_1, \ldots, x_p$.*

This is equivalent to a result of Marica & Schönheim [6], which in turn follows from the theorem of M. Hall Jr. already mentioned.

Second Proof: We begin with the set $1, 2, \ldots, n$, and replace one of its elements $y$ by the integer $x \neq y, x \leq n$. By the Lemma, the new set can form the diagonal of a Latin square if $\frac{1}{2}n(n+1) + x - y \equiv 0 \pmod n$, which has the solution $x - y = \frac{1}{2}n$. The set $2, 3, \ldots, \frac{1}{2}n+1$ (twice) $, \ldots, n$ satisfies this condition. We now interchange the numbers $\frac{1}{2}n$ and $n$ in the table, and make equivalent changes in the order of rows and columns so as to bring the diagonal into the order $2, 3, \ldots, n, n$. □

**Corollary 1.** *The quasigroup of Proposition 5 can be taken to be a symmetric isotope of an idempotent quasigroup.*

**Corollary 2.** *The single chain with $n \geq 4$ nodes, with a loop at the initial node, is admissible.*

We have seen that there is no quasigroup of this type for $n = 3$. A plenary stable quasigroup of odd order and maximal index, proposed by the referee, is the following, which has canonical diagonal:

| | a | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $e$ | $d$ | $c$ |
| $b$ | $e$ | $c$ | $b$ | $a$ | $d$ |
| $c$ | $c$ | $e$ | $d$ | $b$ | $a$ |
| $d$ | $a$ | $d$ | $c$ | $e$ | $b$ |
| $e$ | $d$ | $b$ | $a$ | $c$ | $e$. |

The parastrophes of a quasigroup $Q$, namely itself, its conjoint, reciprocal, adjoint, transpose, and reverse transpose, are the quasigroups with composition rules $ab; ba; a/b; a\backslash b; b\backslash a; b/a$.

**Proposition 6.** *The conjoint of $Q$ has the same type of plenary stability as $Q$ itself. If $Q$ is a loop, all the other parastrophes are unipotent.*

Despite the similar position of plenary stable quasigroups and Bernstein nonassociative algebras, plenary stability cannot be transferred.

**Proposition 7.** *A quasigroup algebra cannot be Bernstein.*

PROOF: A quasigroup algebra is baric with $w(\sum x_i a_i) = \sum x_i$. A Bernstein algebra has a direct decomposition $B = Fe \oplus U \oplus V$, where $e$ is an idempotent, $F$ is the ground field, $U = \{u; eu = \frac{1}{2}u\}, V = \{v; ev = 0\}$ (see e.g. [5]). Now if $B$ were a quasigroup algebra, the left multiplication by any element $a_i$, hence by any element $a$ with $w(a) \neq 0$, and in particular by $e$, is nonsingular on $A$. Hence it is nonsingular on Ker $w$. Thus the component $V$ must be absent in this case. This implies that $a_i - a_j \in U$. The Bernstein property now implies that $e(a_i - a_j) = \frac{1}{2}(a_i - a_j)$, while the quasigroup property implies $e(a_i - a_j) = a_k - a_l, k \neq l$. This is a contradiction. □

The noncommutative duplicate of an algebra $A$ with basis $a_1, a_2, \ldots, a_n$ is the algebra $A^D$ with basis the ordered pairs $(a_i, a_j)$ and multiplication table $(a_i, a_j)\,(a_u, a_v) = (a_i a_j, a_u a_v)$. This construction is important in genetic applications. Duplication cannot be carried out within the category of quasigroups. However, let $G^*$ be a groupoid. The set $G^{*D}$ of ordered pairs of elements of $G^*$, with multiplication rule $(a, b)\,(c, d) = (ab, cd)$ is also a groupoid.

**Proposition 8.** *The noncommutative duplicate of a plenary stable groupoid (in particular of a plenary stable quasigroup) of index $d$ is plenary stable with index $d + 1$.*

PROOF: We have $(a, b)^{[d+1]} = (ab, ab)^{[d]} = ((ab)^{[d]}, (ab)^{[d]})$. □

If for an element $a \in Q$, we have $a^{[k]} = a^{[k+1]}$, we say that $a$ is a plenary stable element. The order of a subquasigroup need not divide the order of the quasigroup, but it cannot exceed half the order. Plenary stability imposes a further condition.

**Proposition 9.** *Let $Q$ have order $2n$. The set $S$ of plenary stable elements cannot form a subquasigroup of order $n$.*

PROOF: Suppose the contrary and label the elements of $S$ $a_1, a_2, \ldots, a_n$. None of these elements can appear in the diagonal positions in rows $n + 1$ to $2n$, since if e.g. $a_i^2 = a_j, n + 1 \leq i \leq 2n, 1 \leq j \leq n$, $a_i$ would also be plenary stable. But the first $n$ columns of row $i$ must contain the elements $a_{n+1}, a_{n+2}, \ldots, a_{2n}$ by the Latin square property. □

A quasigroup can be realised as a geometric 3-net [3]. To obtain results on plenary stability other than changes in terminology, we may consider special cases defined by geometric conditions. Let $N$ be a 5-net whose parallel classes are determined by points $U, V, W, X, Y$ and let $l, m$ be two lines not passing through $U, V$, or $W$. Let the lines through $U$ be labelled $a_1, a_2, \ldots, a_n$, and let $\lambda, \mu$ be the perspectivities determined by $l, m$, mapping the line bundle through $U$ onto those through $V, W$, respectively. Let $Q$ be a quasigroup given by the multiplication rule $a_i a_j = \mu^{-1}\{a_i \cap \lambda(a_j) W\}$.

**Lemma.** (i) *If* $l = m$, *$Q$ is an idempotent quasigroup.* (ii) *If* $l \neq m$ *and* $l \cap m$ *lies on a line through* $U$, *then* $Q$ *has exactly one idempotent element.* (iii) *If* $l, m$ *intersect in* $X$ *or* $Y$, *then* $Q$ *has no idempotent elements.*

Proof: In case (i), it may be seen geometrically that the lines $a, \lambda(a)$ and $\mu(a)$ are coincident on $l$, for every $a \in Q$. More formally, we have $\lambda(a) = (a \cap l)V, \mu(a) = (a \cap l)W$. Then the square of $a$ is $\{(a \cap a)\}W = \{a \cap (a \cap l)V\}W = (a \cap l)W = \mu(a)$. In case (ii) this calculation can be carried through for the point $l \cap m$, but not for any other, while in case (iii) it does not hold anywhere. □

Consider any line $b$ through $U$ other than $a$. To find $b^2$ we determine $b \cap l$, and the join of this to $W, (b \cap l)W$ is $\mu(b^2)$. The join of its intersection with $m$, and $U$ is $b^2$.

**Proposition 10.** *In the* 5*-net* $N$, *let* $b$ *be a line through* $U$ *not corresponding to the idempotent, and let* $L_1 = b \cap l$. *Let the sequence* $L_1, M_1, L_2, M_2, \dots$ *be defined by alternating projections from* $l$ *to* $m$ *with vertex* $U$, *and from* $m$ *to* $l$ *with vertex* $W$. *If the sequence closes to a polygon, then* $Q$ *is not a plenary stable quasigroup, and conversely.*

Proof: A closed polygon corresponds to a cycle in the sequence of iterated squares. □

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