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Locally convex topologies in linear orthogonality spaces

Jerzy Kąkol, Pekka Sorjonen

Abstract. In this paper, we investigate the existence and characterizations of locally convex topologies in a linear orthogonality space.

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1. Introduction.

Let $(E,\perp)$ be a linear orthogonality space (for the terminology, see below). We show that if $(E,\perp)$ has a vector space topology, which gives the same closed subspaces as the linear orthogonality relation, then there exists always a locally convex topology with the same property. On the other hand, even in a Hilbert space there exist non locally convex topologies with this property.

Furthermore, we characterize those topologies which give the same closed subspaces as the linear orthogonality relation with the aid of Hahn-Banach property. In a special case, this gives us a necessary and sufficient condition for a topology to be locally convex. As a by-product, we achieve new characterizations of a Hilbert space.

2. Terminology.

In this paper, the symbol E denotes always a vector space over the field K, which is either the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. We suppose further that E is infinite dimensional. If nothing else is said, τ is a Hausdorff vector space topology of the vector space E. Especially, the pair (E,τ) means a Hausdorff topological vector space. The algebraic dual of E is denoted by E^* and the topological dual by E' or $(E,\tau)'$. The set of all τ-closed vector subspaces is denoted by $L_\tau(E)$.

A vector space $E=(E,\perp)$ is called a *linear orthogonality space*, if $\perp$ is a binary relation on E such that

1. $x \perp y$ if and only if $y \perp x$,
2. $\{x\}^\perp := \{y \in E \mid y \perp x\}$ is a subspace for all $x \in E$,
3. $x \perp y$ for all $y \in E$ implies $x = 0$.

For the general theory of linear orthogonality spaces, see [3]. Recall that every indefinite and definite inner product space is a linear orthogonality space in a natural way, but the converse does not hold.

Let $(E,\perp)$ be a linear orthogonality space. The *orthogonal* $X^\perp$ of a set X in E is the subspace $\{y \in E \mid y \perp x \ \forall x \in X\}$. A vector subspace M of E is called *orthoclosed*, if $M = M^{\perp\perp}$. Orthoclosed subspaces form a lattice $L_{\perp\perp}(E)$.

The *orthodual* $(E, \perp)'$ of a linear orthogonality space E consists of *orthocontinuous* linear functionals f, i.e. $f \in E^*$ and the kernel $\ker f$ is orthoclosed.

3. Existence of a locally convex topology.

Theorem 3.1. *Let $(E, \perp)$ be a linear orthogonality space endowed with a Hausdorff vector space topology τ. If $L_\tau(E) = L_{\perp\perp}(E)$, then there exists a locally convex vector space topology α on E such that $L_\alpha(E) = L_\tau(E)$.*

Proof: The assumptions imply that $(E, \tau)' = (E, \perp)'$ and that the relation $\perp$ has the Mackey property

$$M \in L_{\perp\perp}(E) \,\&\, x \in E \Longrightarrow M + \langle x \rangle \in L_{\perp\perp}(E);$$

here $\langle \cdot \rangle$ denotes the vector subspace spanned by $\{\cdot\}$. This in turn implies that the space $(E, \perp)$ is a quadratic space, i.e., there exists an automorphism ν of the field $\mathbf{K}$ and a non-degenerate orthosymmetric ν-sesquilinear form $[\cdot \mid \cdot]$ on E. Furthermore, $[x \mid y] = 0$ if and only if $x \perp y = 0$; see [3].

The Fréchet–Riesz representation theorem [4] implies that the orthodual $(E, \perp)'$ consists of the functions of the form $[\cdot \mid x]$ with $x \in E$. This together with the relation $(E, \perp)' = (E, \tau)' = E'$ means that (E, E') is a dual pair. The weak topology α induced by this duality is a locally convex vector space topology on E such that $(E, \alpha)' = (E, \tau)'$.

As the weak topology α is the coarsest topology on E for which the forms $f \in (E, \tau)'$ are continuous, we have the relation $L_\alpha(E) \subset L_\tau(E)$. Thus $L_\alpha(E) \subset L_{\perp\perp}(E)$.

To prove the opposite inclusion, note first that the subspaces of the form $\{x\}^\perp$ are orthoclosed hyperplanes which in turn are all of the form $\ker f_x$ with $f_x \in (E, \perp)'$. Using this, we get for all subspaces $M \in L_\tau(E) = L_{\perp\perp}(E)$

$$M = M^{\perp\perp} = \left(\bigcup_{x \in M^\perp} \{x\} \right)^\perp = \bigcap_{x \in M^\perp} \{x\}^\perp = \bigcap_{x \in M^\perp} \ker f_x \in L_\alpha(E),$$

because $f_x \in (E, \perp)' = (E, \alpha)'$. Thus $L_\alpha(E) \supset L_{\perp\perp}(E)$. □

Corollary 3.2. *Let (E, τ) be a topological vector space. Suppose that there is a mapping $'$ on $L_\tau(E)$ with the properties*

1. *$M \subset N$ implies $M' \supset N'$,*
2. *$M'' = M$ for all $M \in L_\tau(E)$.*

Then there exists a locally convex vector space topology α on E such that $L_\alpha(E) = L_\tau(E)$.

Proof: Define a relation $\perp$ by the rule $x \perp y$ if $\langle x \rangle \subset \langle y \rangle'$. It is easy to see that the space $(E, \perp)$ is a linear orthogonality space with the property $\{x\}^\perp = \langle x \rangle'$ for all $x \in E$. A little calculation shows then that $L_\tau(E) = L_{\perp\perp}(E)$. Thus we can apply the previous theorem. □

As an immediate corollary to this corollary, we get Theorem 3.3 of [7], where it is supposed that the mapping $'$ is an orthocomplementation.

Theorem 3.1 shows that if a linear orthogonality space has a topology which gives the same closed vector subspaces as the linear orthogonality relation, we can always find a *locally convex* topology with the same property. But we can also always find a *non locally convex* topology with this property even in the Hilbert space case:

Theorem 3.3. *Let E be a Hilbert space considered as a linear orthogonality space, and let τ be the topology induced by the inner product. There exists a non locally convex vector space topology γ on E such that $L_\gamma(E) = L_\tau(E)$ and $\sigma(E, E^{'}) < \gamma < \tau$; here $\sigma(E, E^{'})$ denotes the weak topology of E.*

PROOF: From [1, Theorem 2], it follows that there exists a non locally convex vector space topology γ on E such that $\sigma(E, E^{'}) < \gamma < \tau$. Furthermore, one can easily show that $L_\gamma(E) = L_\tau(E)$. □

4. Characterization of locally convex topologies.

Let $(E, \perp)$ be a linear orthogonality space and let τ be a vector space topology on E. In this section, we consider under what conditions τ is locally convex.

Theorem 4.1. *Let $(E, \perp)$ be a linear orthogonality space which has the Hahn–Banach separation property $(HBSP)$:*

$$(HBSP) \begin{cases} M \in L_{\perp\perp}(E) \,\&\, x \notin M \\ \Longrightarrow \\ \text{there exists } f \in (E,\perp)' s.t. \ker f \supset M \,\&\, x \notin \ker f. \end{cases}$$

Let τ be a Hausdorff vector space topology on E such that $(E, \tau)' = (E, \perp)'$.

Then $L_\tau(E) = L_{\perp\perp}(E)$, if and only if the topology τ has the Hahn–Banach separation property $(HBSP)_\tau$:

$$(HBSP)_\tau \begin{cases} M \in L_\tau(E) \,\&\, x \notin M \\ \Longrightarrow \\ \text{there exists } f \in (E,\tau)' s.t. \ker f \supset M \,\&\, x \notin \ker f. \end{cases}$$

PROOF: As the scalar field is either $\mathbf{R}$ or $\mathbf{C}$, the topology τ has the Mackey property

$$M \in L_\tau(E) \,\&\, x \in E \Longrightarrow M + \langle x \rangle \in L_\tau(E).$$

Theorems 5.4 and 5.5 of [6] imply now the result. □

Corollary 4.2. *In addition to the assumptions of the previous theorem, suppose that the topology τ is metrizable and complete. Then τ is locally convex, if and only if $L_\tau(E) = L_{\perp\perp}(E)$.*

PROOF: If τ is locally convex, then it has the Hahn–Banach separation property $(HBSP)_\tau$, which implies the result by the previous theorem.

Conversely, we get from the previous theorem that τ has the Hahn–Banach separation property $(HBSP)_\tau$, which implies that it has also the Hahn–Banach extension property $(HBEP)_\tau$. Then a result of Kalton, see e.g. [2, Theorem 4.8], implies that τ is locally convex. □

Note that Theorem 3.3 implies that this corollary fails to be true without the assumption of the metrizability of the topology τ.

Corollary 4.3. *Let a linear orthogonality space $(E, \perp)$ be also a p-Banach space with $0 < p \leq 1$, and let τ be the topology induced by the p-norm. If the linear orthogonality relation $\perp$ is definite, i.e., $x \perp x$ implies $x = 0$, and $L_\tau(E) = L_{\perp\perp}(E)$, then E is a Hilbert space with the natural norm equivalent to the p-norm of E.*

Proof: By Corollary 4.2, the topology τ is locally convex. Thus we must have $p = 1$ and hence the p-Banach space E is a usual Banach space. Now [5, Corollary 3.6] implies the result. $\square$

The following result is an extension of a theorem of S. Kakutani and G.W. Mackey characterizing lattice-theoretically Hilbert spaces among Banach spaces. Recall that an *orthocomplementation* on $L_\tau(E)$ is a mapping $'$ on $L_\tau(E)$ with the properties

1. $M \subset N$ implies $M' \supset N'$,
2. $M'' = M$ for all $M \in L_\tau(E)$,
3. $M \cap M' = \{0\}$ for all $M \in L_\tau(E)$.

Corollary 4.4. *Let E be a p-Banach space with $0 < p \leq 1$, and let τ be the topology induced by the p-norm. If $L_\tau(E)$ admits an orthocomplementation, then E is a Hilbert space with the natural norm equivalent to the p-norm of E, and the orthocomplementation is the usual one.*

Proof: Defining a linear orthogonality relation as in the proof of Corollary 3.2, one can apply the previous corollary. $\square$

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J. Kąkol – Institute of Mathematics, A. Mickiewicz University, Matejki 48/49 Poznań, Poland

J. Kąkol – Mathematisches Institut der Ludwig–Maximilians–Universität, D–8000 München 2, Theresienstrasse 39, Federal Republic of Germany

P. Sorjonen – Department of Mathematics, University of Jyväskylä, 40100 Jyväskylä, Finland

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