Automorphisms of concrete logics

Mirko Navara, Josef Tkadlec

Abstract. The main result of this paper is Theorem 3.3 : Every concrete logic (i.e., every set-representable orthomodular poset) can be enlarged to a concrete logic with a given automorphism group and with a given center. Since every sublogic of a concrete logic is concrete, too, and since not every state space of a (general) quantum logic is affinely homeomorphic to the state space of a concrete logic [8], our result seems in a sense the best possible. Further, we show that every group is an automorphism group of a concrete lattice logic and, on the other hand, we prove that this is not true for Boolean logics with a dense center. As a technical tool for pursuing the latter type of problems, we investigate the correspondence between homomorphisms of concrete logics and pointwise mappings of their domain. We prove that in a suitable topological representation of concrete logics, every automorphism is carried by a homeomorphism.

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1. Basic notions and historical remarks.

By a (*quantum*) *logic* we mean an orthomodular poset (see e.g. [5],[2], [12] for more information). The concrete logics are exactly those logics that are represented as collections of subsets of a set. They can be explicitly defined as follows.

Definition 1.1. A *concrete logic* is a pair (X, L), where X is a nonempty set and $L \subset \exp X$ such that

(1) $\emptyset \in L$,
(2) $\forall\, A \in L : A^c \in L$, where $A^c = X \setminus A$,
(3) $\forall\, A, B \in L : A \cap B = \emptyset \Rightarrow A \cup B \in L$.

For simplicity, we shall denote the concrete logic (X, L) by L, if the character of the problem we pursue does not involve the domain, X, of L.

Definition 1.2. Let L_1, L_2 be concrete logics. A mapping $h : L_1 \to L_2$ is called a *homomorphism* if

(1) $h(\emptyset) = \emptyset$,
(2) $\forall\, A \in L_1 : h(A^c) = h(A)^c$,
(3) $\forall\, A, B \in L_1 : A \cap B = \emptyset \Rightarrow h(A \cup B) = h(A) \cup h(B)$.

A homomorphism h is called an *isomorphism*, if $h(L_1) = L_2$ and

(4) $\forall\, A, B \in L_1 : h(A) \subset h(B) \Rightarrow A \subset B$.

An isomorphism $h : L_1 \to L_1$ is called an *automorphism*. Let us denote the group of all automorphisms of a concrete logic L by $\mathcal{A}(L)$.

Let (X, L) be a concrete logic. Two elements $A, B \in L$ are called *compatible* (abbr. $A \leftrightarrow B$) if $A \cap B \in L$. In this case both A and B are contained in Boolean subalgebra of L. A maximal Boolean subalgebra of a concrete logic is called a *block*. Obviously, every concrete logic is the union of all its blocks. The intersection of all blocks of L is called the *center* of L (denoted by $\mathcal{C}(L)$). In other words, $\mathcal{C}(L) = \{A \in L : (\forall B \in L : A \leftrightarrow B)\}$. The center $\mathcal{C}(L)$ of L is a Boolean algebra and it is called *trivial*, if $\mathcal{C}(L) = \{\emptyset, X\}$.

Suppose that (X, L) is a concrete logic and K is a subset of L closed under the formation of complements (in X) and disjoint unions of two elements. Then (X, K) is a concrete logic. In this case, (X, K) is called a *sublogic* of (X, L). Dually, every concrete logic containing a sublogic isomorphic to (X, K) is called an *enlargement* of (X, K). Let us call (X, K) a *strong sublogic* [12] of (X, L), if each $A, B \in K$ satisfy the following condition: A, B are compatible in (X, K), iff A, B are compatible in (X, L). A concrete logic containing a strong sublogic isomorphic to (X, K) is called a *strong enlargement* of (X, K).

Definition 1.3. A *state* on a concrete logic (X, L) is a mapping $s : L \to [0, 1]$ such that

$$s(X) = 1,$$
$$s(A \cup B) = s(A) + s(B) \text{ whenever } A \cap B = \emptyset.$$

A *two-valued state* is a state attaining only the values 0 and 1.

A nonempty set $A \in L$ is called an *atom* if L contains no nonempty proper subset of A. If all elements of L can be expressed as joins of atoms, L is said to be *atomistic*.

According to [6], every group is the automorphism group of a (not necessarily concrete) logic. This result was improved by Kallus and Trnková [4] – they proved that logics with given automorphism groups can be taken from the class of enlargements of a given atomistic logic (it should be observed that in [10] the authors proved that the assumption of atomisticity was superfluous). In the quantum axiomatics, the automorphism group of a general logic was investigated in the connection with the central properties of the logic (see [7], [11]). Here we study the interplay of the center and the automorphism group for concrete logics and, in certain lines, we obtain a considerable improvement of [7]. Obviously, the proof technique then becomes quite different.

2. Homomorphisms of concrete logics.

In this section we shall study the question of the fact when a homomorphism of concrete logics is carried by a point mapping. We shall need the following definitions.

Definition 2.1. Let $(X_1, L_1), (X_2, L_2)$ be concrete logics. We say that a homomorphism $h : L_1 \to L_2$ is *carried by a point mapping* $f : X_2 \to X_1$, if $h(A) = f^{-1}(A)$ for any $A \in L_1$.

Every two-valued state can be viewed as a homomorphism into the two-element concrete logic $\{\emptyset, X\}$. This leads us to the following definition.

Definition 2.2. We say that a two-valued state s on a concrete logic (X, L) is *carried by a point* $x \in X$, if $s(A) = 1$ iff $x \in A$ (for all $A \in L$). The state carried by a point x is denoted by s_x.

Let (X, L) be a concrete logic. Each point $x \in X$ carries a two-valued state, hence the set $S = \{s_x : x \in X\}$ of two-valued states is *full* (i.e., for each $A, B \in L$ such that $A \not\subset B$, there is a state $s \in S$ such that $s(A) \not\leq s(B)$). On the other hand, a logic with a full set of two-valued states has a concrete representation – e.g. such that its domain is the set of all two-valued states, see [16], [2], [12] – and it is usually called a concrete logic, too.

Definition 2.3. We say that a set $M \subset \exp X$ is *separating* on X if for each pair of points $x, y \in X$ there is an $A \in M$ such that $x \in A$ and $y \notin A$.

A concrete logic (X, L) is separating on X, iff there is no pair of points from X that carry the same two-valued state on L. If (X, L) is non-separating we may identify the points of X that carry the same two-valued state and, as a result, we obtain a separating representation $(\widetilde{X}, \widetilde{L})$ of (X, L) $(\widetilde{X} \subset X, \widetilde{L} = \{A \cap \widetilde{X} : A \in L\})$.

Let us call a *generalized Stone representation* (abbr. *GSR*) of a concrete logic such a separating representation that each two-valued state is carried by a point (it is a Stone-like representation by means of all two-valued states). It is (in a sense) the greatest separating representation.

Lemma 2.4. *Suppose that $(X_1, L_1), (X_2, L_2)$ are concrete logics and $h : L_1 \to L_2$ is a homomorphism carried by a point mapping $f : X_2 \to X_1$. Then $s_y \circ h = s_{f(y)}$ for all $y \in X_2$.*

PROOF: For all $y \in X_2$ and all $A \in L_1$ the following statements are equivalent: $(s_y \circ h)(A) = 1,\ y \in h(A),\ f(y) \in A,\ s_{f(y)}(A) = 1.$ □

Proposition 2.5. *Suppose that $(X_1, L_1), (X_2, L_2)$ are concrete logics and suppose that $h : L_1 \to L_2$ is a homomorphism. Let us denote by $\mathcal{D}(f)$ the set of all $y \in X_2$ for which the set $\{x \in X_1 : \ s_x = s_y \circ h\}$ is nonempty. For all $y \in \mathcal{D}(f)$, let us choose some $f(y)$ from this set. Then the mapping $f : \mathcal{D}(f) \to X_1$ satisfies the following conditions:*

1. *$f^{-1}(A) = h(A) \cap \mathcal{D}(f)$ for all $A \in L_1$; particularly, if $\mathcal{D}(f) = X_2$ then h is carried by the mapping f,*
2. *if $h(L_1)$ is separating on X_2 then f is one-to-one,*
3. *if $\mathcal{D}(f) = X_2$ and h is an isomorphism then $A = \bigwedge\{B \in L_1 : \ B \supset f(h(A))\}$ for all $A \in L_1$.*

PROOF: (1) We have $y \in f^{-1}(A)$ iff $y \in \mathcal{D}(f)$ and $f(y) \in A$. The latter condition is equivalent to each of the following conditions: $s_{f(y)}(A) = 1,\ (s_y \circ h)(A) = 1,\ y \in h(A)$.

(2) Suppose that $y, z \in X_2$ and that $f(y) = f(z)$. Then $s_y \circ h = s_z \circ h$, i.e., $s_y(h(A)) = s_z(h(A))$ for all $A \in L_1$. Since $h(L_1)$ is separating on X_2, we obtain $y = z$.

(3) According to part (1), $h(A) = f^{-1}(A)$, hence $f(h(A)) = A$. Suppose that $B \supset f(h(A))$. According to (1), we then obtain $h(B) = f^{-1}(B) \supset h(A)$. Hence, $B \supset A$. □

Corollary 2.6. *Let $(X_1, L_1), (X_2, L_2)$ be concrete logics. A homomorphism $h : L_1 \to L_2$ is carried by a point mapping iff for each two-valued state s_y on L_2 (carried by a point $y \in X_2$) the two-valued state $s_y \circ h$ on L_1 is carried by a point.*

Proof: It follows from Lemma 2.4 and Proposition 2.5 (1). (It should be noted that this result can be viewed as a generalization of [14, §11B].) □

Corollary 2.7. (1) *Every homomorphism of a GSR into a concrete logic is carried by a point mapping.*

(2) *If there is an isomorphism of a separating concrete logic (X, L) to its GSR $(\widetilde{X}, \widetilde{L})$ that is carried by a point mapping f, then f is a one-to-one mapping of $\widetilde{L}$ onto L.*

Proof: (1) It follows from Proposition 2.5 (1).

(2) According to Proposition 2.5 (2), the mapping f is one-to-one. Suppose that $x \in X$. Then $s_x \circ h^{-1}$ is a two-valued state on $\widetilde{L}$ and hence it is carried by a point $y \in \widetilde{X}$. According to Lemma 2.4, $s_{f(y)} = s_y \circ h = s_x \circ h^{-1} \circ h = s_x$. Since L is separating, we obtain $x = f(y)$. Thus, $f(\widetilde{X}) = X$. □

Corollary 2.8. *Suppose that (X, L) is a separating concrete logic, $(\widetilde{X}, \widetilde{L})$ is its GSR and that $h : \widetilde{L} \to L$ is an isomorphism. Then there is a one-to-one mapping $f : X \to \widetilde{X}$ such that*

$$h(A) = f^{-1}(A) \quad \text{for all} \quad A \in \widetilde{L},$$
$$h^{-1}(C) = \bigwedge\{B \in \widetilde{L} :\ B \supset f(C)\} \quad \text{for all} \quad C \in L.$$

Proof: It follows from Proposition 2.5. □

Let us recall that a *closure space* (see e.g. [3]) is a pair $(X, ^{-})$ such that $^{-} : \exp X \to \exp X$ is a *closure operation*, i.e., (1) $\overline{\emptyset} = \emptyset$, (2) $A \subset \overline{A}$, (3) $A \subset B \Longrightarrow \overline{A} \subset \overline{B}$, (4) $\overline{\overline{A}} = \overline{A}$. A set $A \subset X$ is called *closed*, if $\overline{A} = A$ and it is called *open* if $X \setminus A$ is closed.

The union of two closed sets (the intersection of two open sets, resp.) need not be closed (open, resp.). On the other hand, the intersection of any family of closed sets (the union of any family of open sets, resp.) has to be closed (open, resp.). If we replace the condition (3) by the stronger condition (3') $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we obtain the definition of a topological space.

Each family $\mathcal{B} \subset \exp X$ such that $\bigcup \mathcal{B} = X$ is a *base of open sets* for some closure space $(X, ^{-})$ (we put $\overline{A} = X \setminus \bigcup\{B \in \mathcal{B} :\ A \cap B = \emptyset\}$ for all $A \in X$) and a subbase of open sets for the *associated topological space* (X, τ) (we put $\overline{A} = X \setminus \bigcup\{\bigcap \mathcal{B}_1 :\ \mathcal{B}_1 \subset \mathcal{B}, \text{card}\colon \mathcal{B}_1 < \infty,\ A \cap \bigcap \mathcal{B}_1 = \emptyset\}$ for each $A \subset X$).

With any concrete logic (X, L), we associate the closure space $(X, ^{-})$ such that L is a base of open sets. This space is *0-dimensional* (i.e., clopen sets form a base of open sets). If L is separating then $(X, ^{-})$ is totally disconnected. Moreover, it can be proved that, if L is a GSR then $(X, ^{-})$ is compact (see e.g. [15]).

Also, it can be shown that the closure space $(X, ^{-})$ is topological, if and only if each point of X carries a two-valued Jauch–Piron state [15]. (Recall that a state s on L is called *Jauch–Piron*, if for each $a, b \in L$ with $s(a) = s(b) = 1$ there is a $c \in L$ such that $c \leq a, c \leq b$ and $s(c) = 1$.)

A mapping $f : (X_1, ^{-}) \to (X_2, ^{-})$ is called *continuous*, if for each open set $B \subset X_2$ the set $f^{-1}(B)$ is open. It is called a *homeomorphism*, if there exists f^{-1} and both f and f^{-1} are continuous.

It is well-known that a continuous mapping on a closure space is also continuous on the associated topological space. The converse does not have to be true. According to Alexander sub-base theorem, a closure space is compact iff the associated topological space is compact.

Proposition 2.9. *Suppose that $(X_1, L_1), (X_2, L_2)$ are concrete logics, $(X_1, ^{-})$, $(X_2, ^{-})$ are their associated closure spaces and that $h : L_1 \to L_2$ is a homomorphism carried by a point mapping $f : X_2 \to X_1$. Then the mapping $f : (X_2, ^{-}) \to (X_1, ^{-})$ is continuous.*

Proof: Suppose that $A \subset X_1$ is open. Then $A = \bigcup M$ for some $M \subset L_1$ and therefore the set $f^{-1}(A) = f^{-1}(\bigcup M) = \bigcup f^{-1}(M) = \bigcup h(M)$ is open. $\square$

Theorem 2.10. *Every automorphism of a GSR (X, L) is carried by a homeomorphism of $(X, ^{-})$, where $(X, ^{-})$ is the closure space with the base L of open sets.*

Proof: According to Corollary 2.7, there is a one-to-one mapping f of X onto X that carries the given automorphism h of L. It is easy to see that the isomorphism h^{-1} is carried by f^{-1} and, according to Proposition 2.9, both f and f^{-1} are continuous.

$\square$

Remark 2.11. (1) The separating property of the GSR is not essential in Corollary 2.7 (1).

(2) The condition of L being separating can be replaced with a weaker condition in Theorem 2.10, namely with a condition that all sets of all mutually non-separable points of X (i.e., $\{x \in X : s_x = s_y\}$, $y \in X$) have the same cardinality.

(3) The concrete logic (X, L) in Theorem 2.10 need not be a GSR. Theorem 2.10 remains valid also for "Stone-like" representations by means of a suitable set of two-valued states (see Corollary 2.6), e.g., for representations by means of all two-valued Jauch–Piron states.

3. Concrete logics with given centers and automorphism groups.

In this section, we show that every group can be represented as the automorphism group of a concrete logic with a given center. Moreover, we prove that this logic may contain a given sublogic. An important role in our proof will be played by the logics that are *rigid*, i.e., by the logics that have only one automorphism (the identity).

An element A of a logic L is called *stable*, if each automorphism of L maps A onto A.

We shall need the following two lemmas.

Lemma 3.1. *Let (X, L) be a concrete logic such that*

$$\text{(A)} \qquad \forall\, h \in \mathcal{A}(L)\ \forall\, A \in L : \ \textit{ if } \ A \neq h(A) \ \textit{ then } \ A \vee h(A) = X.$$

Then L satisfies also the following condition

$$\text{(B)}\qquad \begin{array}{l} \forall\, g,h\in\mathcal{A}(L)\ \forall\, A,B\in L:\ \ \textit{if}\ \ A\cap B=\emptyset\ \ \textit{and}\ \ g(A)\cap h(B)=\emptyset \\ \quad \textit{then there is a } k\in\{g,h\}\ \textit{ such that }\ g(A)=k(A)\ \ \textit{and} \\ h(B)=k(B). \end{array}$$

Proof: Suppose that (A) is fulfilled and that $g,h\in\mathcal{A}(L)$. Suppose also that $A,B\in L$ satisfy $A\cap B=\emptyset=g(A)\cap h(B)$. If $A=\emptyset$, then h can be taken for k. Suppose that $A\neq\emptyset$. Applying (A) to the element $g(B)$ and the automorphism $h\circ g^{-1}$, we obtain either $g(B)=h(B)$ or $g(B)\vee h(B)=X$. Both $g(B)$ and $h(B)$ are disjoint with a nonempty set $g(A)$. Thus, they have a common upper bound $g(A)^c\subsetneqq X$. So the only possible case is $g(B)=h(B)$ and we can take g for k. □

Lemma 3.2. *Suppose that*

K is a concrete logic,
G is a group, and
$\mathcal{P}$ is a poset.

Then there is a collection of concrete logics $\{L_i : i\in\mathcal{P}\}$ such that for each $i,j\in\mathcal{P}$ the following conditions hold:

(1) *L_i is a strong enlargement of K,*
(2) *$\mathcal{C}(L_i)$ is trivial,*
(3) *$\mathcal{A}(L_i)\cong G$,*
(4) *L_i satisfies the condition* (B) *from Lemma* 3.1, *and*
(5) *L_j is a (strong) enlargement of L_i iff $i\le j$.*

If K is a lattice, all L_i, $i\in\mathcal{P}$, are also lattices.

Proof: We only outline the basic steps of the proof because they have already appeared in the preceding papers, though not together. If K is atomistic and if we omit the condition (4), the proof is given by Kallus and Trnková in [4]. In order to describe our modifications, we need to review the steps of their proof.

Step 1: The basic tool is the construction of a proper class of undirected graphs with the following properties: the automorphism group of the graph is isomorphic to G, each vertex is of an order greater than 1, and there are no cycles of a length less than 5 (see e.g. [13]). Moreover, this graph may contain arbitrarily many stable vertices (vertices preserved by all automorphisms). To see this, one may add such a graph representing the trivial group.

Step 2: If we add one vertex to each edge of the graph, we obtain a hypergraph which is the Greechie diagram of a logic, M_i. The automorphism group of M_i is again isomorphic to G. Each block of M_i is isomorphic to 2^3 and it contains an *outer point* – an atom which belongs to exactly one block. It should be noted that an atomistic logic with an outer point in each block is necessarily concrete. Indeed, in order to prove the concreteness of an atomistic logic it suffices to show (see [16], [2], [12]) that for each two atoms a,b satisfying $a\not\leq b'$ there is a two-valued state s such that $s(a)=s(b)=1$. Such a state s can be defined in the following way: It attains the value 1 on a and on b and on exactly one outer point from each block which contains neither a nor b.

Step 3: In order to embed the logic K, each atom of K is connected by a 2^3-block with an appropriate stable atom of M_i (for this, M_i must have a sufficiently large "stable part", see [4] for details). This procedure gives the logic L_i. All automorphisms of L_i are uniquely determined by their values on M_i. Hence, $\mathcal{A}(L_i) \cong \mathcal{A}(M_i) \cong G$.

For our purpose, let us modify some parts of the proof of Kallus and Trnková.

Modification 1: In Step 3, we have to fix the automorphisms not only on all atoms but also on sufficiently many further elements. This is done (using a generalized pasting technique for logics) in [10, Theorem 7.1]. Going through the construction, one can easily see that starting with a concrete logic we obtain again a concrete logic.

Modification 2: In order to satisfy (4), we have to insert the following procedure between Step 1 and Step 2:

Step 1.5: Each edge (a, b) of the graph is replaced with a copy of the graph F in Fig. 1.

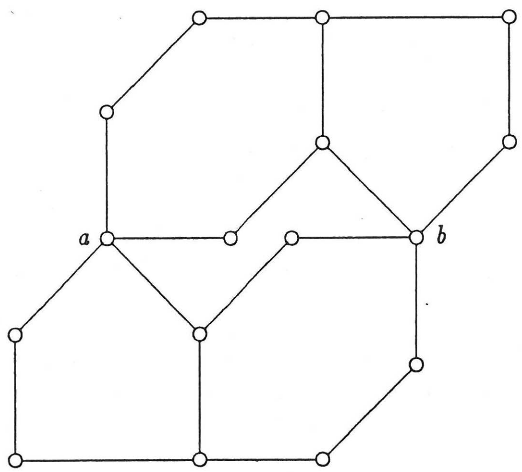

Fig. 1

One can easily see that all cycles of the lengths 5 and 6 in the resulted graph are the cycles of the copies of F. This enables us to check that the automorphism group did not change in this procedure. The distance of a vertex and its image in an automorphism is either 0 or it is greater than 2. This ensures that all L_i, $(i \in P)$, satisfy the condition (A) from Lemma 3.1.

With the two modifications described above, the proof of Kallus and Trnková gives the assertion of Lemma 3.2. $\square$

Theorem 3.3. *Suppose that*

K is a concrete logic,
B is a Boolean algebra, and

G is a group.

Then there is a concrete logic L such that

(1) *L is a strong enlargement of K,*
(2) *the center of L is isomorphic to $\mathcal{B}$,*
(3) *the automorphism group of L is isomorphic to G.*

Proof: Applying Lemma 3.2, we obtain a concrete strong enlargement H of K such that $\mathcal{A}(H) \cong G$, $\mathcal{C}(H)$ is trivial and H satisfies the condition (B) from Lemma 3.1.

We may assume that $\mathcal{B}$ is the Boolean algebra of clopen subsets of its Stone space, X. We fix a point $z \in X$ and we take the filter $\mathcal{F} = \{B \in \mathcal{B} : \; z \in B\}$.

We apply Lemma 3.2, where we take H for K, the trivial group for G and the poset $\mathcal{F}$ (partially ordered by inclusion) for $\mathcal{P}$. We obtain a collection of rigid concrete logics $\{L_B : \; B \in \mathcal{F}\}$ such that L_B is a strong enlargement of L_A iff $A \subset B$. As L_X is a common enlargement of all these logics, we may (and shall) suppose that $H \subset L_X$ and $L_A \subset L_B$ for each $A, B \in \mathcal{F}$ with $A \subset B$.

Let us denote by Y the domain of a representation of L_X. Further, let us denote by $\mathcal{T}$ the collection of all subsets of $X \times Y$ which are of one of the following forms:

(F1) $$B \times A, \quad \text{where} \quad B \in \mathcal{F} \quad \text{and} \quad A \in H,$$

(F2) $$B \times A, \quad \text{where} \quad B \in \mathcal{B} \setminus F \quad \text{and} \quad A \in L_{B'}.$$

The collection $\mathcal{T}$ generates a concrete logic $M \subset \exp(X \times Y)$; M is the collection of all unions of finitely many disjoint elements of $\mathcal{T}$.

For each $x \in X$, let us define the "restriction mapping" $P_x : \exp(X \times Y) \to \exp(\{x\} \times Y)$ by the formula $P_x(U) = U \cap (\{x\} \times Y)$. The set $P_x(M) = \{P_z(U) : U \in M\}$ forms a concrete logic (with the domain $\{x\} \times Y$). The logic $P_z(M)$ is isomorphic to H, hence it satisfies the condition (B) of Lemma 3.1. For each $x \in X$ and for each mapping h on $\exp(\{x\} \times Y)$, we define a mapping h_x on $\exp(X \times Y)$ by the formula $h_x(U) = (U \setminus P_x(U)) \cup h(P_x(U))$. Now, we put $L = \{h_z(U) : \; h \in \mathcal{A}(P_z(M)), \; U \in M\}$.

First, we have to prove that L is a logic. If $\{z\} \in \mathcal{B}$ then $L = M$. Suppose that $\{z\} \notin \mathcal{B}$. Trivially, L is closed under the formation of complements. Suppose that $V, W \in M, \; g, h \in \mathcal{A}(P_z(M))$ such that $g_z(V), h_z(W) \in L$ are disjoint. Then V, W are disjoint, hence $V \cup W \in M$. According to condition (B) of Lemma 3.1 (where we take $P_z(M)$ for L, $P_z(V)$ for A and $P_z(W)$ for B), there is a $k \in \{g, h\}$ such that $g_z(V) \cup h_z(W) = k_z(V \cup W) \in L$.

Now we are ready to check the assertions (1)–(3) of the Theorem.

(1) The collection $\{X \times A : A \in H\}$ forms a sublogic of L isomorphic to H.

(2) Suppose that $B \times A \in L$. where $B \in \mathcal{B}$ and $A \notin \{\emptyset, Y\}$. If $B \times A$ satisfies (F1), we can find an element $C \in H$ which is non-compatible (in H) to A. As L is a strong enlargement of H, the elements $B \times A, B \times C \in L$ (corresponding to $A, C \in H$) are non-compatible in L. If $B \times A$ satisfies (F2), the element C with the same properties can be chosen from $L_{B'}$. In both cases $B \times A$ does not belong to the center of L. The same arguments are applicable to all elements of L which contain

$B \times A$ and which are disjoint to $B \times (Y \setminus A)$. Thus, all elements of $\mathcal{C}(L)$ have to be of the form $B \times Y, B \in \mathcal{B}$. On the other hand, all such elements are central, so $\mathcal{C}(L) = \{B \times Y : B \in \mathcal{B}\} \cong \mathcal{B}$.

(3) First of all, notice that each automorphism of L maps the center of L onto itself. For all $x \in X$, take the ideal $\mathcal{I}_x = \{A \times Y : A \in \mathcal{B}, x \notin A\} \subset \mathcal{C}(L)$ (it is a usual maximal ideal in the Boolean algebra $\mathcal{C}(L)$). The factorization of L over the ideal $\mathcal{I}_x$ (see [5]) gives a logic $L/\mathcal{I}_x$ isomorphic to $P_x(L)$ $(= P_x(M))$. If $B \in \mathcal{B}, x \in B$, then the collection $\{B \times C : C \in L_{B'}\}$ forms a concrete logic (with the domain $B \times Y$) which is mapped by the factorization onto a sublogic of $P_x(L)$ isomorphic to $L_{B'}$. Thus, for all $B \in \mathcal{B}$ such that $B \times Y \notin \mathcal{I}_x$, $P_x(L)$ is an enlargement of $L_{B'}$. Our construction ensures that $P_x(L)$ is not isomorphic to $P_y(L)$ for $x \neq y$. Thus each automorphism h of L maps $\mathcal{I}_x$ onto $\mathcal{I}_x$ and hence coincides with the identity on $\mathcal{C}(L)$. It also induces an automorphism of $L/\mathcal{I}_x \cong P_x(L)$. We have obtained that h induces automorphisms of $P_x(L)(x \in X)$, and h is fully described by this collection of automorphisms. For $x \neq z$, the logic $P_x(L)$ is rigid, so $\mathcal{A}(L) = \{h_z : h \in \mathcal{A}(P_z(L))\} \cong \mathcal{A}(P_z(L)) \cong \mathcal{A}(H)$. □

Remark 3.4. Provided that K is a lattice, the concrete logic M constructed in the proof of Theorem 3.3 becomes a lattice. If $\mathcal{B}$ has an atom, we can choose $z \in X$ so that $\{z\} \in \mathcal{B}$ and hence $L = M$. Theorem 3.3 holds for concrete lattice logics with the additional assumption that $\mathcal{B}$ has at least one atom. We do not know for the time being whether this assumption can be omitted.

4. Automorphism groups of concrete logics.

In the previous section, we have shown that every group is representable as the automorphism group of a concrete (lattice) logic (Remark 3.4). On the other hand, it is well-known (see e.g. [1]) that there are groups that are not representable as automorphism groups of Boolean algebras. An interesting question arises of how small class of concrete logics allows to represent arbitrary groups as automorphism groups of logics of the class. Several classes of concrete logics which are "near" to Boolean algebras have been introduced in [9]. We present here a negative result for the largest class studied there - the class of so-called Boolean logics.

Definition 4.1. A concrete logic (X, L) is called a *Boolean logic*, if for any pair $A, B \in L$, the condition $A \wedge B = \emptyset$ implies that $A \cap B = \emptyset$.

It should be noted (see e.g. [9]) that every Boolean lattice logic is a Boolean algebra.

Lemma 4.2. *Suppose that (X, L) is a Boolean logic and that h is a nontrivial automorphism of L. Then there is an element $A \in L \setminus \{\emptyset\}$ such that $A \cap h(A) = \emptyset$.*

Proof: Since h is nontrivial, there is a $B \in L$ such that $h(B) \neq B$. Choose $C \in \{B, B^c\}$ such that $h(C) \not\subset C$. Since L is a Boolean logic and $C^c \cap h(C) \neq \emptyset$, there is a $D \in L \setminus \{\emptyset\}$ such that $D \subset h(C)$ and $D \cap C = \emptyset$. We put $A = h^{-1}(D)$. □

Definition 4.3. Let (X, L) be a concrete logic. We say that a set $K \subset L$ is *dense* in L, if for each $A \in L \setminus \{\emptyset\}$ there is a $B \in K \setminus \{\emptyset\}$ such that $B \subset A$.

Proposition 4.4. *Every non-rigid Boolean logic with dense center admits a non-trivial automorphism h such that h^2 is the identity.*

Proof: Suppose that (X, L) is a Boolean logic with dense center and with a non-trivial automorphism g. According to Lemma 4.2, there is an $A \in L \setminus \{\emptyset\}$ such that $A \cap g(A) = \emptyset$. Since the center of L is dense in L, there is a central element $C \in L \setminus \{\emptyset\}$ such that $C \subset A$, hence $C \cap g(C) = \emptyset$. The mapping $h : L \to L$ defined by $h(B) = g(B \cap C) \cup g^{-1}(B \cap g(C)) \cup (B \cap (C \cup g(C))^c)$ for all $B \in L$ is the required automorphism. □

Corollary 4.5. *There are groups that are not representable as the automorphism groups of Boolean logics with dense centers.*

Since every Boolean algebra is a Boolean logic with dense center, the last two statements can be viewed as generalizations of the known results for Boolean algebras. Let us note that any atom of a Boolean logic is a central element, hence the condition of density of the center is fulfilled in atomic Boolean logics.

5. An open question.

The following question has naturally arisen in paragraph 4.

Problem 5.1. Can every group be represented as the automorphism group of a Boolean logic?

We conjecture that the answer is no but for now we have not been able to prove it.

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Department of Mathematics, Faculty of Electrical Engineering, Technical University of Prague, 166 27 Praha 6, Czechoslovakia

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