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ℕ-compact frames

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Abstract. We investigate notions of ℕ-compactness for frames. We find that the analogues of equivalent conditions defining ℕ-compact spaces are no longer equivalent in the frame context. Indeed, the closed quotients of frame 'ℕ-cubes' are exactly 0-dimensional Lindelöf frames, whereas those frames which satisfy a property based on the ultrafilter condition for spatial ℕ-compactness form a much larger class, and better embody what 'ℕ-compact frames' should be. This latter property is expressible without reference to maximal ideals or filters. We construct the co-reflections for both of the classes, (the 'ℕ-compactifications'), which both restrict to the spatial ℕ-compactification.

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0. Introduction.

Recall that a **frame** (locale, complete Heyting algebra) is a complete lattice L which is meet-continuous: $u \wedge \bigvee S = \bigvee u \wedge s (s \in S)$, for any $u \in L$ and $S \subseteq L$. We write the bottom as 0 and the top as e. A frame homomorphism preserves finite meets and arbitrary joins. The canonical example of a frame is the lattice $\mathfrak{O}(X)$ of open subsets of a topological space X, and indeed frames may be thought of as topological spaces in which the lattice of open subsets is taken as the primitive notion. There are certain advantages to this approach; frames tend to be better behaved than topological spaces, particularly under the taking of (co)products, (see Proposition 2.5 for example), and often there are constructive arguments available for a frame result where there are none for the analogous spatial result, (e.g. the Tychonoff Theorem, see [Ke], [Ve], [Jo] but see also [Sc]).

We recall the definition of an ℕ-compact space:

Definition 0.1. A topological space X is ℕ-**compact** if it is homeomorphic to a closed subspace of $\mathbb{N}^I$ for some index set I, where $\mathbb{N}$ is the discrete space of natural numbers.

The ℕ-compact spaces are important tools in topology; they are the 0-dimensional analogues of real-compact spaces, and play a similar role. Moreover, ℕ-compact spaces play a significant role in the study of the groups (and rings) $C(X, \mathbb{Z})$. Such groups occur as (particularly simple) examples of groups of global sections of sheaves on frames and so in order to 'lift' the results about $C(X, \mathbb{Z})$ to the more general context, one must know what an ℕ-compact frame is. This sort of lifting can be done;

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we consider some applications to Abelian Group theory in an upcoming paper, and mention a couple of results below.

However, unlike cases such as compactness and paracompactness, it is not immediately obvious what an '$\mathbb{N}$-compact frame' is. There are several equivalent statements which define $\mathbb{N}$-compact spaces, and the equivalences fail in the frame context. We list a few of these for reference, in the form of a theorem expressing their equivalence.

Recall that ζX denotes the universal 0-dimensional compactification of a topological space X, first described in [Ba]. A **clopen ultrafilter** on X is an ultrafilter in $B\mathfrak{O}(X)$, the lattice of complemented (clopen) elements of $\mathfrak{O}(X)$. Such an ultrafilter F has the **countable intersection property** if $\cap S \neq \emptyset$ for any countable subset $S \subseteq F$, and is **fixed** if $\cap F \neq \emptyset$. The symbol $\mathbb{N}^*$ denotes the one-point compactification of $\mathbb{N}$ and a subspace X of Y is $C(-,\mathbb{Z})$-embedded if any continuous function from X to $\mathbb{Z}$ extends to Y. A cardinal κ is **ω-measurable** if there is an ultrafilter in $\mathcal{P}(\kappa)$ which has the countable intersection property, but is not fixed.

Theorem 0.2. *Suppose X is a 0-dimensional sober space. Then these are equivalent:*

1. *X is $\mathbb{N}$-compact.*
2. *If X is a dense $C_{\mathbb{Z}}$-embedded subspace of a 0-dimensional sober space Y, then $X = Y$.*
3. *For any point $x \in \beta_0 X \setminus \beta_0[X]$ there is continuous function $\beta_0 X \xrightarrow{h} \mathbb{N}^*$ such that $h \upharpoonright \beta_0[X] \subseteq \mathbb{N}$ and $h(x) = \infty$.*
4. *Every clopen ultrafilter with the countable intersection property is fixed.*
5. *Any ring homorphism $C(X,\mathbb{Z}) \xrightarrow{h} \mathbb{Z}$ is the evaluation map at some point $x_0 \in X$; i.e. $h(f) = f(x_0)$ for all $f \in C(X,\mathbb{Z})$.*

The $\mathbb{N}$-compact spaces were introduced by Engelking and Mrówka in [En, **Mr**], where they were defined as in 0.1. Subsequent work has established the conditions equivalent to the definition. The equivalences (1) $\leftrightarrow$ (2) $\leftrightarrow$ (5) are due to Mrówka and Engelking in [En, **Mr**] and [Mr] respectively, and (1) $\leftrightarrow$ (3) $\leftrightarrow$ (4) to Herrlich in [He] and Chew in [Ch]. $\square$

Remark. It follows immediately from the definitions that a discrete space is $\mathbb{N}$-compact iff it is of non-ω-measurable cardinality. It is consistent (with ZFC) that ω-measurable cardinals do not exist. For later reference we note that the cardinals $\aleph_1$ and $2^{\aleph_0}$ are not ω-measurable, see [Je]. We note the following for future reference:

Proposition 0.3. *The $\mathbb{N}$-compactification of a 0-dimensional Hausdorff space X is denoted νX, and may be constructed in these two ways:*

1. *Embed the space via the evaluation map into $\mathbb{N}^{C(X,\mathbb{N})}$ and take the closure of its image.*
2. *Form the space of all clopen ultrafilters with the countable intersection property, with the usual ultrafilter space topology.*

The definition 0.1 of an $\mathbb{N}$-compact space has an obvious interpretation for frames, and these are our 'Stone-$\mathbb{N}$-compact' frames defined in Section 2 below, following the route hitherto taken for realcompact frames (cf. [Ma, **Ve**]). While this is the canonical translation of the definition into the language of frames, the proof of Theorem 2.6 shows that there is a radical departure from the spatial situation; all such '$\mathbb{N}$-compact' frames are Lindelöf. For similar reasons, the equivalences in Theorem 0.2 break down for frames. We shall show in Section 2 that our definition based on (4) of Theorem 0.2 captures the 'right' notion of what an $\mathbb{N}$-compact frame should be. In Section 3 we construct the co-reflections for both sorts of $\mathbb{N}$-compactness, and use these to establish some further results about the behaviour of $\mathbb{N}$-compact frames.

Where possible our arguments are constructive. We will discuss any use of (or independence from) choice principles in our arguments.

1. Preliminaries.

The connection with topology provides the primary motivation for the study of frames. If X is a topological space, then $\mathfrak{O}(X)$, the lattice of open subsets of X, is a **spatial** frame. With a continuous map $X \xrightarrow{\phi} Y$ we associate a frame morphism $\mathfrak{O}(Y) \xrightarrow{\mathfrak{O}(\phi)} \mathfrak{O}(X)$ which takes the open set U to $\phi^{-1}(U)$. To avoid this necessary twisting of maps, many authors prefer to work in the category of **locales**, the formal opposite of the category of frames. However in this paper we will stay entirely within $\mathbb{F}$rm, the category of frames and frame morphisms.

With every frame we can associate a topological space, called the **spectrum** of L, denoted ΣL, and defined to be the space of **completely prime filters**; those filters F in L such that $\bigvee S \in F$ iff $S \cap F \neq \emptyset$ for any $S \subseteq F$. The sets $\{F \in \Sigma L \mid u \in F\}$ for $u \in L$ are the open subsets of ΣL. The associations $\mathfrak{O}(-)$ and $\Sigma(-)$ are functorial, and together form an adjoint-on-the-right pair $\mathbf{Top} \underset{\mathfrak{O}}{\overset{\Sigma}{\rightleftarrows}} \mathbb{F}\mathbf{rm}$.

The **Boolean part** BL of a frame L is the sub-lattice consisting of all the complemented elements. The frame L is **0-dimensional** if it is generated as a frame by its Boolean part.

A frame L is **compact (Lindelöf)** if $\bigvee_L S = e$ for a subset $S \subseteq L$ implies that S has a finite (countable) subset with the same join.

If D is a distributive lattice, $\mathfrak{J}D$ is the frame of all ideals of D. The universal 0-dimensional compactification of a frame L is the frame of all ideals on the Boolean part, denoted $\mathfrak{J}BL$, with the adjunction given by the frame morphism $\mathfrak{J}BL \xrightarrow{j} L$ which takes an ideal to its join in L (see [Ba1]). The compactification functor takes a frame morphism $L \xrightarrow{\phi} M$ to $\mathfrak{J}B\phi$, which itself takes an ideal to the ideal generated by its image under ϕ.

We recall that a **nucleus** r on a frame L is a closure operator which preserves finite meets. The set of closed elements ($u = r(u)$) is then a frame, the **quotient frame** $L \mod r$, written $[L]_r$.

A **closed** nucleus is one of the form $r(u) = u \vee v$ for some $v \in L$. A nucleus is **dense** if $r(0) = 0$ and **codense** if $r(u) = e$ implies $u = e$. Similarly, a frame homomorphism ϕ is dense if $\phi(u) = 0$ implies $u = 0$ and codense if $\phi(u) = e$ implies

$u = e$. The following results are easy to establish.

Lemma 1.1. *Suppose $L \xrightarrow{\phi} M$ is a frame homomorphism. Then*

(i) *The map ϕ is* 1-1 *on the Boolean part of L if it is dense.*
(ii) *In the category of regular frames, ϕ is monic if it is dense.*
(iii) *If L is regular, ϕ is* 1-1 *iff it is codense.*

Further background and more detail can be found in [Jo1].

2. N-compact frames.

The definition of an $\mathbb{N}$-compact topological space has a natural translation into the language of frames. We recall that the I-indexed copower of a frame L is denoted $L^{(I)}$ and make the following definitions. (See [Jo1] for a description of frame coproducts.)

Definition 2.1. A frame L is **Stone-$\mathbb{N}$-compact** ('S-$\mathbb{N}$-compact') if it is a closed quotient of the frame $\mathfrak{O}(\mathbb{N})^{(I)}$ for some index set I.

Definition 2.2. A frame L is a **$C_{\mathbb{Z}}$-quotient** of a frame M, via the map $M \xrightarrow{\phi} L$, if any $\mathfrak{O}(\mathbb{Z}) \xrightarrow{\psi} L$ factors through ϕ as shown. (Here $\mathbb{Z}$ is the discrete space of integers.)

$$\begin{tikzcd} M \arrow[rr, "\phi"] & & L \\ & \mathfrak{O}(\mathbb{Z}) \arrow[ul, dotted] \arrow[ur, "\psi"'] & \end{tikzcd}$$

(Of course, this is the analogous property to one space being $C(-,\mathbb{Z})$-embedded in another.)

Definition 2.3. If L is a frame, and $I \in \mathfrak{I}BL$, we say that I is **σ-proper** if $\bigvee_L S \neq e_L$ for any countable $S \subseteq I$, and that I is **completely proper** if $\bigvee_L I \neq e_L$.

We quote the following result from [Sc].

Proposition 2.4. *The* 0*-dimensional Lindelöf coreflection of a frame L is the frame $[\mathfrak{I}BL]_{s_L}$, where $\mathfrak{I}BL \xrightarrow{s_L} \mathfrak{I}BL$ is the nucleus defined by*

$$s_L I = \{\bigvee_L S \mid S \subseteq I \textit{ countable}\},$$

with adjunction $[\mathfrak{I}BL]_{s_L} \xrightarrow{j} L$ the map taking an ideal to its join in L.

Since by Proposition 2.4 the category of 0-dimensional Lindelöf frames is coreflective in the category of frames, we obtain the following result, first proved in [Do, **St**].

Proposition 2.5. *The frame coproduct of a family of* 0*-dimensional Lindelöf frames is Lindelöf.*

Remark. We recall that a product of (0-dimensional) Lindelöf spaces need not be Lindelöf; the Sorgenfrei line is a counterexample, see [St, **Se**].

The following result is similar in nature to Theorem 2.1 of [Ma, **Ve**].

Theorem 2.6. *For a* 0*-dimensional frame* L*, the following are equivalent:*

1. *If* L *is a dense* $C_{\mathbb{Z}}$*-quotient of a* 0*-dimensional frame* M*, then* $L = M$ *(i.e. the map is an isomorphism).*
2. L *is* S-$\mathbb{N}$*-compact.*
3. L *is Lindelöf.*
4. *If* $I \in \mathfrak{I}BL$ *is* σ*-proper, then it is completely proper.*

Proof: $(1 \to 2)$ Denote by $\mathbb{Z}_L E$ the set of frame homomorphisms from $\mathfrak{O}(\mathbb{Z})$ to L. Then the canonical frame 'evaluation map' $\mathfrak{O}(\mathbb{Z})^{(\mathbb{Z}_L E)} \xrightarrow{F} L$ is surjective since L is 0-dimensional. It follows that L is a $C_{\mathbb{Z}}$-quotient of $\mathfrak{O}(\mathbb{Z})^{(Z_L E)}$ and therefore of its closure, with which, as a dense quotient, it must coincide. Since $\mathfrak{O}(\mathbb{Z}) \cong \mathfrak{O}(\mathbb{N})$, L is S-$\mathbb{N}$-compact.

$(2 \to 3)$ follows from Proposition 2.5 above, and the obvious fact that a closed quotient of a Lindelöf frame is again Lindelöf.

$(3 \to 4)$ Trivial.

$(4 \to 3)$ If $e = \bigvee_L u_\alpha$, consider I, the ideal in BL generated by $\{\downarrow(u_\alpha)\}_\alpha$. This is not completely proper and therefore not σ-proper. Thus there are $w_n \in I$ so that $\bigvee_L w_n = e$. Each w_n is dominated by a finite join $u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}$, so that some countable subset of the u_α's covers e.

$(3 \to 1)$ Suppose that L is a dense $C_{\mathbb{Z}}$-quotient of a 0-dimensional frame M, via $M \xrightarrow{\phi} L$. Since M is 0-dimensional and hence regular, it suffices to show by Lemma 1.1 that ϕ is co-dense.

Suppose that $\phi(u) = e$. Since $u = \bigvee_\alpha v_\alpha$ for some $v_\alpha \in BM, e = \phi(u) = \bigvee_\alpha \phi(v_\alpha)$. By hypothesis, there is a countable subfamily $\{v_{\alpha_n}\}_{n\in\mathbb{Z}}$ so that $\bigvee_{\mathbb{Z}} \phi(v_{\alpha_n}) = e$. We can suppose that this is an increasing list, and by subtracting off common intersections, produce countably many $w_n \in BM, (n \in \mathbb{Z})$, which are pairwise disjoint and have $\bigvee_{\mathbb{Z}} w_n = \bigvee_{\mathbb{Z}} v_{\alpha_n} \le u$.

Now let $\mathfrak{O}(\mathbb{Z}) \xrightarrow{\psi} L$ be the map determined by requiring $\psi(\{n\}) = \phi(w_n)$. By hypothesis there is a map $\mathfrak{O}(\mathbb{Z}) \xrightarrow{\overline{\psi}} M$ so that $\phi\overline{\psi} = \psi$. Since $\phi\overline{\psi}(\{n\}) = \psi(\{n\}) = \phi(w_n)$, we must have $\overline{\psi}(n) = w_n$, since the dense map ϕ is 1-1 on complemented elements (Lemma 1.1). Then $e = \bigvee_{\mathbb{Z}} \overline{\psi}(n) = \bigvee_{\mathbb{Z}} w_n \le u$, so that $u = e$. □

Remark. Note that our proof of the implication $4 \to 3$ used the Axiom of Countable Choice in choosing the covers of the w_n's, and in asserting that a countable union of finite sets is countable. We also made an implicit use of choice principles in the proof of the $2 \to 3$, in that we used Proposition 2.5. In fact it follows quickly form $3 \to 2$ (which holds in ZF), and Theorem 3.1 of [Sc] that one cannot prove that $2 \to 3$ holds in ZF, and indeed that it is equivalent in consistency strength to the Axiom of Countable Choice.

Example. The frame $\mathfrak{O}(\mathbb{Q}) \oplus \mathfrak{O}(\mathbb{Q})$ is a nonspatial frame ([Jo1]) which is Lindelöf (as it is countably generated) and is therefore S-$\mathbb{N}$-compact.

We have seen that although it is a natural notion, S-$\mathbb{N}$-compactness is somewhat restrictive. For when one admits the Axiom of Countable Choice ('CC'), all S-$\mathbb{N}$-compact frames are Lindelöf. Thus one can have an $\mathbb{N}$-compact space X (for instance ω_1 with the discrete topology), such that $\mathfrak{O}(X)$ is not S-$\mathbb{N}$-compact, and the notion hence fails to be a 'conservative' one. That is to say, the spatial notion is not preserved under the passage to frames, and the concept is not properly lifted (or co-lifted!) from the class of topological spaces to the larger class of frames. As we have mentioned, the statement that S-$\mathbb{N}$-compact frames are Lindelöf depends upon CC, so that in ZF it is *consistent* that there are more S-$\mathbb{N}$-compact frames than Lindelöf frames. But one could show nothing more in ZF about S-$\mathbb{N}$-compact frames than about Lindelöf frames.

Of course this all follows from the preservation of the Lindelöf property under frame coproducts, a desirable thing to have. But it demands a change in what one views as the fundamental notion of $\mathbb{N}$-compactness. There are other alternatives available; towards these we make the following definitions.

Definition 2.7. Let L be a frame. An ideal $I \in \mathfrak{I}BL$ is **super-σ-proper** if any proper ideal $I' \supseteq I$ is σ-proper.
We remark that the improper ideal is super-σ-proper, since the condition is vacuously fulfilled.

Definition 2.8. A 0-dimensional frame L is **Herrlich-$\mathbb{N}$-compact** ('H-$\mathbb{N}$-compact') if any proper $I \in \mathfrak{I}BL$ which is super-σ-proper is completely proper.

Remark. The definition looks less mysterious if we for the moment assume the Boolean Ultrafilter Theorem. Then a frame L is H-$\mathbb{N}$-compact iff every maximal ideal in BL which is σ-proper is completely proper. This then resembles the statement (4) of Theorem 0.2, and indeed we have

Lemma 2.9. *For a space X, $\mathfrak{O}(X)$ is* H-$\mathbb{N}$*-compact iff X is* $\mathbb{N}$*-compact.*

Proof: $(\to)$ Suppose that $\mathcal{F}$ is a ultrafilter with the countable intersection property in $\mathcal{B}\mathfrak{O}(X)$. Then $\mathcal{F}^* = \{U^* :\in \mathcal{F}\}$ is a maximal ideal in $\mathcal{B}\mathfrak{O}(X)$ which is σ-proper. So $\mathcal{F}^*$ is completely proper, implying that $\mathcal{F}$ is fixed.

$(\leftarrow)$ Suppose that $I \in \mathcal{JB}\mathfrak{O}(X)$ has the property of the definition. If $I' \supseteq I$ is some maximal ideal extending I, it is σ-proper by hypothesis, and therefore completely proper, by considerations like those in $(\to)$. This implies that I is completely proper. $\square$

Remark. Lemma 2.9 tells us that this notion of an $\mathbb{N}$-compact frame is a conservative one. Note that we need the Boolean Ultrafilter Theorem for the $\leftarrow$ implication. This is not too surprising however, since the definition of H-$\mathbb{N}$-compact frames is based on condition (4) of Theorem 0.2 which explicitly mentions ultrafilters.

Remark. $\mathbb{N}$-compact spaces are defined as in 0.1, but usually the property (4) of Theorem 2.6 is the easiest to understand and work with. Considering the complexity of the frame coproduct construction, it seems that the same is true for frames as well.

Remark. Note the similarity between Definition 2.8 and condition (4) of Theorem 2.6, and thus between the two notions of an $\mathbb{N}$-compact frame.

One can use the Lemma and the Boolean Ultrafilter Theorem to provide a formulation of spatial $\mathbb{N}$-compactness which is a cover condition. We have not seen this mentioned in the literature, but it may not be new.

Corollary 2.10. *A 0-dimensional space X is $\mathbb{N}$-compact iff for every cover S of X by clopen sets which has no finite subcovers, there is a larger (clopen) cover $T \supseteq S$ which also contains no finite subcovers, but does contain a countable subcover.*

Remark. That any Lindelöf frame is H-$\mathbb{N}$-compact is easy to see. Then by Theorem 2.6 any S-$\mathbb{N}$-compact frame is H-$\mathbb{N}$-compact. As a corollary of Theorem 3.9 we will obtain this result without any set theoretic assumptions.

Non-spatial examples of S-$\mathbb{N}$-compact and H-$\mathbb{N}$-compact frames are easy to find:

Lemma 2.11. *If B is a complete Boolean algebra, then*

(i) *B is S-$\mathbb{N}$-compact iff any antichain in B is countable.*
(ii) *B is H-$\mathbb{N}$-compact if any antichain in B is of non-measurable cardinality.*

Proof **(i)**: If $S \subseteq B$ is an antichain which is uncountable, then by adjoining another element of B if necessary we have a cover with no countable subcover, so that B is not Lindelöf and therefore not S-$\mathbb{N}$-compact. This is necessity. Towards sufficiency, we suppose that $(u_\alpha)_{\alpha\in\kappa}$ is a cover of e_B, for some cardinal κ. Define $v_\beta = \bigvee_{\alpha\leq\beta} u_\alpha$, and then set $w_\beta = v_{\beta+1} \wedge v_\beta^*$ for $\beta > 0$ and $w_0 = v_0$. Then the w_β are pairwise disjoint, so that there is a $\gamma \in \kappa$ with $w_\alpha = 0$ if $\alpha > \gamma$, for γ some countable ordinal. This implies that $v_\beta = v_\alpha$ if $\beta > \alpha$, so that there is a countable subcover of the cover $(u_\alpha)_{\alpha\in\kappa}$. By Theorem 2.6 ($3 \to 2$), B is S-$\mathbb{N}$-compact.

Proof **(ii)**: Suppose B is not H-$\mathbb{N}$-compact. Then there is a maximal ideal I in B so that I is σ-proper but not completely proper. Using Zorn's lemma, we can find a maximal antichain S in I, and we clearly must have $\bigvee S = e$. Let $\mathcal{F} \subseteq \mathbb{P}(S)$ be defined by requiring $X \in \mathcal{F}$ iff $\bigvee X \notin I$. Then $\mathcal{F}$ is a non-principal ultrafilter on S with the countable intersection property, so that $|S|$ must be measurable. □

We know of no counterexample to necessity for (ii), but are unable to show that it holds.

3. Frame N-compactifications.

We construct the S-$\mathbb{N}$-compactification and H-$\mathbb{N}$-compactification. Both of these restrict to the classical $\mathbb{N}$-compactification on spatial frames.

3.1. The S-N-compactification.

Since we showed in Theorem 2.6 that the S-$\mathbb{N}$-compact frames are exactly the 0-dimensional Lindelöf frames, the coreflection from $\mathbb{F}\mathrm{rm}$ to the subcategory of

0-dimensional Lindelöf frames mentioned in Proposition 2.4 is the S-ℕ-compactification. However, in [Sc] we showed that such a coreflection cannot be shown to exist in ZF; in fact asserting its existence is near to asserting the Axiom of Countable Choice. We can however easily construct the S-ℕ-compactification in ZF; we proceed just as we do in the spatial case; (1) of Proposition 0.3.

Given a 0-dimensional frame L, we form the evaluation morphism $\mathfrak{O}(\mathbb{Z})^{(Z_L E)} \xrightarrow{F} L$, which is a quotient mapping as L is 0-dimensional, and then the closure of this mapping, g (see [Jo1, p. 51]). We obtain the diagram =xydash10 =xyatip10 =xybtip10 =xybsql10 =xycirc10 cmat10 =xycmbt10

$$\begin{tikzcd} \mathfrak{O}(\mathbb{Z})^{(\mathbb{Z}_L E)} \arrow[rr, "F"] \arrow[dr, "g"] & & L \\ & \mathrm{cl}\, L \arrow[ur, "h"] & \end{tikzcd}$$

with h a dense map.

Proposition 3.1. *For a 0-dimensional frame L, the frame $\nu_S L = \mathrm{cl}\,(L)$ defined above is the* **Stone-ℕ-compactification**, *with coreflection map h.*

The proof of this fact proceeds as for the spatial case, and requires no choice principles. □

Theorem 3.2. *Let X be a 0-dimensional space. Then $\Sigma(\nu_S \mathfrak{O}(X)) \cong \nu X$.*

Proof: The spectrum functor transfers coproducts to products. □

Note that as a consequence of Proposition 3.1 the subcategory of S-ℕ-compact frames is closed under frame coproducts and closed quotients, as we would expect.

3.2. The H-N-compactification.

We construct the coreflection from the category of frames to the subcategory of H-ℕ-compact frames. Since a compact frame is H-ℕ-compact, there should be a frame map from $\mathfrak{I}BL$ to the universal H-ℕ-compactification of L (if this exists). We use this observation to construct the H-ℕ-compactification as a quotient of $\mathfrak{I}BL$.

For any frame L, define $\mathfrak{I}BL \xrightarrow{h} \mathfrak{I}BL$ by

$$hI = \{u \in BL \cap \downarrow\!(\bigvee_L I) \mid I \subseteq J,\ J \text{ super-}\sigma\text{-proper } \implies u \in J\}.$$

Lemma 3.3. *The map h is a nucleus.*

Proof: (i) Clearly $I \subseteq hI$.

(ii) We have only to show that $hI \cap hK \subseteq h(I \cap K)$, since h is order preserving. Towards this, fix $u \in hI \cap hK$. Then $u \leq \bigvee_L \wedge \bigvee_L K = \bigvee_L (I \cap K)$, so u satisfies the first criterion for membership in $h(I \cap K)$.

Suppose that $J \supseteq I \cap K$ is a super-σ-proper ideal. We must show that $u \in J$. Note that $J \vee I$ is an ideal containing J and is therefore itself super-σ-proper. Since

$J \vee I$ contains I, $u \in J \vee I$, as $u \in h(I)$. We can similarly show that $u \in J \vee K$, so that $u \in J = (J \vee K) \wedge (J \vee I)$, as desired.

(iii) Towards showing that $h^2 I \subseteq hI$, note first that if $u \in h^2 I$, then $u \leq \bigvee_L hI \leq \bigvee_L I$.

Now suppose $u \in h^2 I$, and $I \subseteq J$, J super-σ-proper. Then $hI \subseteq J$, by definition of hI. As $u \in h^2 I, u \in J$, and altogether we have $u \in hI$. □

We will eventually define the H-$\mathbb{N}$-compactification of L to be $[\mathfrak{I}BL]_h$. The coreflection map $[\mathfrak{I}BL]_h \xrightarrow{j_L} L$ will be the join map, defined by $j_L I = \bigvee_L I$, a frame homomorphism by an easy argument. When we must be careful to distinguish among nuclei, we will subscript them appropriately.

Lemma 3.4. *If L is* H-$\mathbb{N}$-*compact then j_L is an isomorphism.*

Proof: If $u \in L$ then $\downarrow(u) \cap BL$ is h_L-closed, so that j_L is onto. We thus need only to show that j_L is co-dense, by Lemma 1.1.

Suppose that $I \in [\mathfrak{I}BL]_h$ and $j_L I = \bigvee_L I = e$, so that I is not completely proper. If I were proper, since it is h-closed, there would be a proper super-σ-proper ideal $J \supseteq I$. But since L is H-$\mathbb{N}$-compact such a J would be completely proper, which is impossible as the sub-ideal I is not. Thus I is not proper, so that j_L is co-dense.

□

Lemma 3.5. *The frame $[\mathfrak{I}BL]_h$ is* H-$\mathbb{N}$-*compact.*

Proof: We first show that

$$B\,[\mathfrak{I}BL]_h \;\cong\; BL \quad \text{via,}$$

$$J \xrightarrow{\alpha} \bigvee_L J$$

$$\downarrow(u) \cap BL \xleftarrow{\beta} u$$

Note that the range of α is indeed as claimed, since if $J \in B\,[\mathfrak{I}BL]_h$ with complement J^*, then $J \cap J^* = 0_{BL}$, so that

$$\alpha J \wedge \alpha J^* = \bigvee_L J \wedge \bigvee_L J^* = \bigvee_L J \cap J^* = 0_L,$$

and $h(J \vee J^*) = E_{BL}$, so that

$$e_L = \bigvee_L (J \vee J^*) = \bigvee_L J \vee \bigvee_L J^* = \alpha J \vee \alpha J^*.$$

One can show in similar fashion that
(i) α, β are both Boolean algebra homomorphisms, and
(ii) $\alpha\beta = \mathrm{id}_L$.

It is also easy to see that $\beta\alpha = \mathrm{id}_{B[\mathfrak{I}BL]_h}$. For suppose $J \in B[\mathfrak{I}BL]_h$. Then $\bigvee_L J$ is complemented (see the paragraph above), and we must show that $\bigvee_L J \in J$. Towards this, assume that $J \subseteq K$ and K is super-σ-proper. Then $K \vee J^* = E_{\mathfrak{I}BL}$, so that $v \vee \bigvee_L J^* = e$ for some $v \in K$. Since K is an ideal, it follows that $\bigvee_L J$ is in K, since $\bigvee_L J^* = (\bigvee_L J)^*$.

Now towards our goal of showing that $[\mathfrak{I}BL]_h$ is H-$\mathbb{N}$-compact, suppose that $\mathrm{I} \in \mathfrak{I}B[\mathfrak{I}BL]_h$ is a super-σ-proper ideal. We must show that I is completely proper.

First note that the image of I under α is an ideal in BL, $\alpha[\mathrm{I}]$.

Claim 3.6. $\alpha[\mathrm{I}]$ is h-closed.

PROOF: It is enough to show that $\alpha[\mathrm{I}]$ is super-σ-proper.

Towards this, suppose that $\alpha[\mathrm{I}] \subseteq K$ and that K is proper. We must show that K is σ-proper. First note that $\mathrm{I} \subseteq \beta[K]$, and since $\beta[K]$ is proper, it is σ-proper. Let $\{u_n \mid n \in \omega\}$ be a countable subset of K. We know that

$$\bigvee_{[\mathfrak{I}BL]_h} \beta(u_n) \neq E_{[\mathfrak{I}BL]_h} \qquad \text{(Inequality 3.1)}$$

since $\beta[K]$ is σ-proper. Now if it were the case that $\bigvee_L u_n = e_L$, we could reason as follows.

$$\begin{aligned}
\bigvee_{[\mathfrak{I}BL]_h} \beta(u_n) &= (\bigvee_{\mathfrak{I}BL} \beta(u_n)) \\
&= \{v \in BL \cap \downarrow \bigvee_L \bigvee_{\mathfrak{I}BL} \beta(u_n) \mid \bigvee_{\mathfrak{I}BL} \beta(u_n) \subseteq H,\ H \text{ super-}\sigma\text{-proper} \Rightarrow v \in H\} \\
&= \{v \in BL \mid \bigvee_{\mathfrak{I}BL} \beta(u_n) \subseteq H,\ H \text{ super-}\sigma\text{-proper} \Rightarrow v \in H\}
\end{aligned}$$

so that because of Inequality 3.1, there must be some proper super-σ-proper ideal H which contains $\bigvee_{\mathfrak{I}BL} \beta(u_n)$. But such an H could not be σ-proper, as $\bigvee_L u_n = e_L$. This is a contradiction, so that we must have $\bigvee_L u_n \neq e_L$. Hence K is σ-proper, so that $\alpha[\mathrm{I}]$ is super-σ-proper, and thus h-closed.

We must finish by observing that $\mathrm{I} \subseteq B\,[\mathfrak{I}BL]_h \cap \downarrow \alpha[\mathrm{I}]$ which implies that I is completely proper, since $\alpha[\mathrm{I}] \neq E_{[\mathfrak{I}BL]_h}$. We have shown that $[\mathfrak{I}BL]_h$ is H-$\mathbb{N}$-compact.

□

Towards showing that the map $[\mathfrak{I}BL]_h \xrightarrow{j_L} L$ is universal as a map from an H-$\mathbb{N}$-compact frame to L, we prove the following lemma. Recall the definition of the functor $\mathfrak{I}B$ from Section 1.

Lemma 3.7. *If $M \xrightarrow{\phi} L$ is a frame homomorphism, then*

$$\begin{aligned}
[\mathfrak{I}BM]_{h_M} &\xrightarrow{\overline{\phi}} [\mathfrak{I}BL]_{h_L} \quad \textit{defined by,} \\
I &\longmapsto h_L(\mathfrak{I}B\phi(I))
\end{aligned}$$

is a frame homomorphism.

PROOF: It is clear that $\overline{\phi}$ preserves finite meets and is thus order-preserving. To see that it transfers arbitrary joins, it is enough to see that, given a collection of elements $\{I_\alpha\}_\alpha$ of $[\mathfrak{I}BM]_{h_M}$, we have

$$\overline{\phi}\,(\bigvee_{[\mathfrak{I}BM]_h} I_\alpha) \subseteq \bigvee_{[\mathfrak{I}BL]_h} \overline{\phi} I_\alpha\,.$$

(We will suppress mention of an index set for the indices α to avoid complicating the notation.) We begin by noting that

$$\begin{aligned}\bigvee_{[\mathfrak{I}BL]_h} \overline{\phi} I_\alpha &= \bigvee_{[\mathfrak{I}BL]_h} h_L \mathfrak{I}B\phi(I_\alpha) = \\ &= h_L(\bigvee_{\mathfrak{I}BL} h_L \mathfrak{I}B\phi(I_\alpha)) = h_L(\bigvee_{\mathfrak{I}BL} \mathfrak{I}B\phi(I_\alpha))\end{aligned}$$

so it is enough to show that

$$h_L\,(\mathfrak{I}B\phi\;(\bigvee_{[\mathfrak{I}BM]_h} I_\alpha)) \subseteq h_L\,(\bigvee_{\mathfrak{I}BL} \mathfrak{I}B\phi\;(I_\alpha)) \qquad \text{(Inequality 3.2)}$$

Fix v in the left-hand side of the Inequality 3.2. From the definition of h, we see that we have two criteria to verify in order to see that v is in the right-hand side. Towards the first, we have

$$\begin{aligned} v &\in h_L(\mathfrak{I}B\phi(\bigvee_{[\mathfrak{I}BM]_h} I_\alpha)) = \\ &= h_L(\mathfrak{I}B\phi(h_M \bigvee_{\mathfrak{I}BM} I_\alpha)), \quad \text{so that,} \\ v &\le \bigvee_L \mathfrak{I}B\phi(h_M \bigvee_{\mathfrak{I}BM} I_\alpha) = \bigvee_L \phi[h_M \bigvee_{\mathfrak{I}BM} I_\alpha] = \\ &= \phi(\bigvee_M h_M(\bigvee_{\mathfrak{I}BM} I_\alpha)) = \phi(\bigvee_M \bigvee_{\mathfrak{I}BM} I_\alpha) \text{ since } h_M(-) \subseteq \downarrow \bigvee_M(-) = \\ &= \bigvee_L \phi[\bigvee_{\mathfrak{I}BM} I_\alpha] = \bigvee_L \mathfrak{I}B\phi(\bigvee_{\mathfrak{I}BM} I_\alpha) = \bigvee_L \bigvee_{\mathfrak{I}BL} \mathfrak{I}B\phi(I_\alpha), \end{aligned}$$

so v satisfies the first criterion for membership in the right-hand side of Inequality 3.2. To see that it satisfies the second, suppose that $\bigvee_{\mathfrak{I}BL} \mathfrak{I}B\phi(I_\alpha) \subseteq H$, for H some super-σ-proper element of $\mathfrak{I}BL$. We must show that $v \in H$. If we can show that $\phi\,[h_M(\bigvee_{\mathfrak{I}BM} I_\alpha)] \subseteq H$, then $\mathfrak{I}B\phi\,(h_M(\bigvee_{\mathfrak{I}BM} I_\alpha)) \subseteq H$, so that $v \in H$, by hypothesis on v.

Claim 3.8. $\phi\,[h_M(\bigvee_{\mathfrak{I}BM} I_\alpha)] \subseteq H$.

Proof: If H is improper, we are done. Otherwise, fix $u \in h_M(\bigvee_{\mathfrak{I}BM} I_\alpha)$ and let $K = \bigvee_{\mathfrak{I}BM}\{L \mid \mathfrak{I}B\phi(L) \subseteq H\}$. Then K is proper since H is. We assert that if $K' \in \mathfrak{I}BM$ is a proper ideal which contains K, then $\mathfrak{I}B\phi(K') \vee H$ is proper. For if not, there are elements $w \in K'$ and $p \in H$ so that $\phi w \vee p = e_L$. Then it follows that $\phi(w^*) \leq p$ and hence that $\phi(w^*)$ is in H (where w^* denotes the complement of w.) Then $\downarrow(w^*) \subseteq K \subseteq K'$, so that both w and w^* are in K', so that $e \in K'$, contradicting the propriety of K'.

Thus $\mathfrak{I}B\phi(K') \vee H$ is proper for any such K', and since it contains H it is σ-proper. But this implies that K' is also σ-proper and hence that K is super-σ-proper. Now the definition of K and the hypothesis on H imply that $\bigvee_{\mathfrak{I}BM} I_\alpha \subseteq K$, since $\mathfrak{I}B\phi(\bigvee_{\mathfrak{I}BM} I_\alpha) = \bigvee_{\mathfrak{I}BL} \mathfrak{I}B\phi(I_\alpha)$, and so by hypothesis on u, we have $u \in K$. Thus $\phi u \in \phi[K] \subseteq H$, as desired. (Claim) $\square$ (Lemma) $\square$

We can now prove the

Theorem 3.9. *For an arbitrary frame L the map $[\mathfrak{I}BL]_h \xrightarrow{j_L} L$ is universal as a map from* H-ℕ-*compact frames to L.*

Proof: Suppose that we are given an H-ℕ-compact frame M and a frame homomorphism $M \xrightarrow{\phi} L$. We can form the diagram =xydash10 =xyatip10 =xybtip10 =xybsql10 =xycirc10 cmat10 =xycmbt10

$$\begin{tikzcd}
{[\mathfrak{I}BL]_{h_L}} \arrow[r, "j_L"] & L \\
{[\mathfrak{I}BM]_{h_M}} \arrow[u, "\overline{\phi}"] \arrow[r, "j_M"] & M \arrow[u, "\phi"'] \arrow[ul, dotted, "\phi'"']
\end{tikzcd}$$

and by considering the form of $\overline{\phi}$, see without trouble that the outer square commutes. Since j_M is an isomorphism, we can find a map ϕ' making the upper triangle commute. This map is unique since j_L is dense and therefore monic. $\square$

Definition 3.10. The full subcategory of H-ℕ-compact frames will be denoted H-ℕ-𝔽rm. The coreflection from 𝔽rm to H-ℕ-𝔽rm supplied by Theorem 3.9 we denote by ν_H.

Corollary 3.11. *The subcategory* H-ℕ-𝔽rm *is closed under frame coproducts and closed quotients.*

Proof: Any coreflective subcategory is closed under all colimits, and hence by Theorem 3.9 H-ℕ-𝔽rm is closed under frame coproducts. Towards the second assertion, we first show that it holds for 'clopen'-quotients; those of the form $\uparrow(u)$ for some complemented u.

Let L be an H-ℕ-compact frame and $u \in BL$. If I is a proper super-σ-proper ideal in $B(\uparrow(u))$, then $I' = \{v \in BL \mid v \vee u \in I\}$ is a proper super-σ-proper ideal in BL. By hypothesis I' is completely proper, and as it contains I, I is also.

Now if L is H-ℕ-compact and $u \in L$ is any element, we know that $u = \bigvee v_\alpha$ for v_α some elements in BL. It follows that the frame $\uparrow(u)$ is the colimit in 𝔽rm of the diagram with vertices the frames $\uparrow(v_\alpha)$ and maps the canonical $\uparrow(v_\alpha) \to \uparrow(v_\beta)$ obtained when $v_\alpha \leq v_\beta$. Since H-ℕ-𝔽rm is closed under colimits, $\uparrow(u)$ is H-ℕ-compact. $\square$

Corollary 3.12. *Any* S-$\mathbb{N}$-*compact frame is* H-$\mathbb{N}$-*compact.*

Proof: By Corollary 3.11 we need only show that $\mathfrak{O}(\mathbb{N})$ is H-$\mathbb{N}$-compact. This follows easily (and in ZF) from the fact that any cover of $\mathfrak{O}(\mathbb{N})$ has a countable refinement. □

Theorem 3.13. *The* H-$\mathbb{N}$-*compactification is conservative, so that* $\nu_H\mathfrak{O}(X) \cong \mathfrak{O}(\nu X)$ *for any* 0-*dimensional Hausdorff space* X.

Proof: If we can show that $\nu_H\mathfrak{O}(X)$ is spatial, then the co-universal properties of $\nu_H\mathfrak{O}(X) \xrightarrow{j} \mathfrak{O}(X)$ and the natural map $\mathfrak{O}(\nu X) \to \mathfrak{O}(X)$ together imply the existence of the isomorphism. Since $\nu_H\mathfrak{O}(X)$ is regular, we can see that it suffices to show that any proper element I is dominated by a maximal element. Now one of the following holds:
(i) $\bigvee_{\mathfrak{O}(X)} I = e_L$. In this case, since I is $h_{\mathfrak{O}(X)}$-closed, there must be a proper super-σ-proper $J \in \mathfrak{I}B\mathfrak{O}(X)$ so that $I \subseteq J$. Then J can be expanded to a maximal element of $\mathfrak{I}B\mathfrak{O}(X)$, which is then also super-σ-proper, and hence $h_{\mathfrak{O}(X)}$-closed.
(ii) $\bigvee_{\mathfrak{O}(X)} I \neq e_L$. In this case there is a maximal element P of $\mathfrak{O}(X)$ such that $\bigvee_{\mathfrak{O}(X)} I \leq P$. Then $\downarrow(P) \cap B\mathfrak{O}(X)$ is $h_{\mathfrak{O}(X)}$-closed and maximal in $\mathfrak{I}B\mathfrak{O}(X)$, and is hence a maximal element of $\nu_H\mathfrak{O}(X)$ containing I. □

Now it is clear that the S-$\mathbb{N}$-compactification of a frame will in general differ from the H-$\mathbb{N}$-compactification, since the first of these will always be a Lindelöf frame. However we can show that after a spatial reflection, the two compactifications coincide;

Lemma 3.14. *For any frame* L, $\Sigma\nu_S L = \Sigma\nu_H L$.

Proof: We know that $\nu_S L = [\mathfrak{I}BL]_{s_L}$ and $\nu_H L = [\mathfrak{I}BL]_{h_L}$, where s_L is the nucleus of Proposition 2.4. It is not difficult to show that the maximal elements of $[\mathfrak{I}BL]_{s_L}$ and $[\mathfrak{I}BL]_{h_L}$ are maximal in $\mathfrak{I}BL$, and it is easy to see that the s_L-closed maximal ideals are exactly the h_L-closed maximal ideals, so that the spectrums coincide. The topologies coincide since they both have a base consisting of sets of the form $\{P \text{ maximal } \mid u \notin P\}$, for $u \in BL$. □

Corollary 3.15. *For any frame* L, $\Sigma\nu_S L\,(= \Sigma\nu_H L)$ *is* $\mathbb{N}$-*compact.*

Proof: We know that $\nu_S L$ is a closed quotient of $\mathfrak{O}(\mathbb{N})^{(I)}$ for some index set I. It follows that $\Sigma\nu_S L$ is a closed subspace of $\Sigma\mathfrak{O}(\mathbb{N})^{(I)} = \mathbb{N}^I$, and is hence $\mathbb{N}$-compact. □

4. H-N-compact frames and sheaves.

We mention a couple of results which will be proved in an upcoming paper.

As we noted in the introduction, $\mathbb{N}$-compact spaces play an important role in the study of the groups and rings $C(X,\mathbb{Z})$. (Se especially [Ed, **Oh**] and [Mr2], and as an example, consider (5) of Theorem 0.2.) In [Mr2], Mrówka proved the following result. A subspace $K \subseteq X$ is said to be a **support** for a group homomorphism $C(X,\mathbb{Z}) \xrightarrow{h} \mathbb{Z}$ if $f \mid K = 0$ implies that $h(f) = 0$.

Proposition 4.1. *A space X is $\mathbb{N}$-compact iff any group homomorphism from $C(X,\mathbb{Z})$ to $\mathbb{Z}$ has compact support.*

In a forthcoming paper we extend Mrówka's result up to the class of groups of global sections of sheaves on frames, with H-$\mathbb{N}$-compact frames playing the role of $\mathbb{N}$-compact spaces. We can then use that theorem to prove the following proposition, a frame theoretic analogue of statement 5 of Theorem 0.2. Here the group $\mathbb{Z}_L E$ is the group of all frame homomorphisms from $\mathfrak{O}(\mathbb{Z})$ to the frame L, the analogue of the group of continuous functions $C(X,\mathbb{Z})$.

Definition 4.2. For a frame L, a ring homomorphism $\mathbb{Z}_L E \xrightarrow{h} \mathbb{Z}$ is **evaluation at a prime element** p of L if $h(\xi) = n$ iff $\xi(\{n\}) \nleq p$.

Remark. One can easily check that for an arbitrary prime p, a map defined in this way is indeed a ring homomorphism. For a spatial frame $L = \mathfrak{O}(X)$ these correspond to the homomorphisms $C(X,\mathbb{Z}) \to \mathbb{Z}$ which are the evaluation maps for various points of X.

Theorem 4.3. *A 0-dimensional frame L is* H-$\mathbb{N}$-*compact iff any ring homomorphism $\mathbb{Z}_L E \longrightarrow \mathbb{Z}$ is the evaluation map for some prime element of L.*

Thus we see that the frame analogues of the statements in Theorem 0.2 fall into two equivalence classes; $1 \leftrightarrow 2$ (Theorem 2.6) and $4 \leftrightarrow 5$ (Theorem 4.3). The statement 3 does not seem to have a natural frame counterpart.

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