Comment.Math.Univ.Carolin. 32,1 (1991)105–113

Revealed automorphisms

Jiří Sgall, Antonín Sochor

Abstract. We study automorphisms in the alternative set theory. We prove that fully revealed automorphisms are not closed under composition. We also construct some special automorphisms. We generalize the notion of revealment and Sd-class.

Keywords: alternative set theory, automorphism, revealed, fully revealed, revealment

Classification: Primary 03E70; Secondary 03H15

In the alternative set theory (AST; see [V] for its exposition) the study of automorphisms is one of the very interesting subjects as this aspect differs very much from the classical case. It is possible to prove inside the theory that a great variety of nontrivial automorphisms of the universe exists, and their study is a very useful tool for understanding the structure of the universe.

This subject has many connections with the study of automorphisms of recursively saturated models of Peano arithmetic. It was proved in [P–S 1984] that countable recursively saturated models of PA are exactly the restrictions of countable models of AST to the natural numbers; also the methods used for studying both kinds of object are very similar.

We thank R. Kaye for many discussions on the automorphisms of recursively saturated models of PA.

In the first section, we give some preliminaries. The second section is devoted to the study of the initial segments of fixed points of automorphisms. In the last section, we prove our main result, namely we prove that the fully revealed automorphisms are not closed under composition.

We use extensively the results on automorphisms and basic equivalences given in [V, Chapter V, Section 1]; all references to [V] are references to this section. E(C) denotes the basic equivalence with parameters C, which is denoted by $\overset{\circ}{\underset{C}{=}}$ in [V]. $\mathrm{FSL}_{\mathrm{C}}$ denotes the language of finite set-theoretical formulae with parameters from the class C. We use also the fact that it is possible to define a satisfaction class for $\mathrm{FSL}_{\mathrm{C}}$ and that the partial satisfaction relation (for formulae $\varphi < n, n \in \mathrm{FN}$) is a Sd-class.

1. Revealments.

In this section, we generalize the notion of revealment. We prove that for any class there exists a fully revealed class satisfying the same formulae from the language FL_{C} for C countable (not only for $C = \emptyset$ as in [S–V 1980]). Our results are straightforward generalizations of the theorems from [S–V 1979] and [S–V 1980].

Definition 1.1. Let F be a similarity. Y is an F-revealment of X if Y is fully revealed and for every normal formula φ in the language FL and $a_0, \dots, a_k \in \operatorname{dom}(F)$ there holds

$$\varphi(X, a_0, \dots, a_k) \Leftrightarrow \varphi(Y, F(a_0), \dots, F(a_k)).$$

Theorem 1.2. *For any countable similarity F there exists an endomorphism $G \supseteq F$ such that the endomorphic universe $G''V$ has a standard extension.*

Proof: Let H be any endomorphism such that $H''V$ has a standard extension. Let $H_0 \supseteq F \circ H^{-1}$ be an automorphism (it exists, since $F \circ H^{-1}$ is countable). Let $G = H_0 \circ H$. Then $G''V = H_0''(H''V)$ is an endomorphic universe with a standard extension; $F = F \circ \mathrm{Id} = F \circ H^{-1} \circ H \subseteq H_0 \circ H = G$. □

Theorem 1.3. *Every class X has an F-revealment for any countable similarity F.*

Proof: Let G be an endomorphism from Theorem 1.2. Then $\operatorname{Ex}(G''X)$ is an F-revealment of X (even a G-revealment), since for $\varphi \in \mathrm{FL}$ we have

$$\varphi(X, a_0, \dots, a_k) \Leftrightarrow \varphi^{G''V}(G''X, G(a_0), \dots, G(a_k)) \Leftrightarrow$$
$$\Leftrightarrow \varphi(\operatorname{Ex}(G''X), G(a_0), \dots, G(a_k)).$$

□

2. Initial segments of automorphisms.

Let us define two natural characteristics of an automorphism.

Definition 2.1. Let F be an automorphism. Let us denote

$$\operatorname{cut}(F) = \{\alpha;\ (\forall\, \beta < \alpha)(F(\beta) = \beta)\}$$
$$\operatorname{Cut}(F) = \{\alpha;\ (\forall\, \beta < \alpha)(F(\beta) = \beta \Rightarrow \beta \in \operatorname{cut}(F))\}.$$

It is easy to see that both $\operatorname{cut}(F)$ and $\operatorname{Cut}(F)$ are initial segments of N, both are proper classes, $\mathrm{FN} \subseteq \operatorname{cut}(F) \subseteq \operatorname{Cut}(F), F''\operatorname{cut}(F) = \operatorname{cut}(F)$ and $F''\operatorname{Cut}(F) = \operatorname{Cut}(F)$. They have also some closure properties, namely for $\alpha \in \operatorname{cut}(F)$ and $\beta \in \operatorname{Cut}(F)$ it holds that $2^\alpha \in \operatorname{cut}(F)$ and $\alpha \cdot \beta \in \operatorname{Cut}(F)$. These properties resemble the properties of the characteristics μ and ν of an ultrafilter (see [S–V 1981]). From the next lemma it follows (by the result of [S–V 1981]) that each pair $\operatorname{cut}(F), \operatorname{Cut}(F)$ is $\mu(\mathfrak{M}), \nu(\mathfrak{M})$ of some ultrafilters $\mathfrak{M}$, but the converse does not hold.

Lemma 2.2. *Let F be an automorphism.*

(i) $\operatorname{cut}(F)$ *is no π-class.*
(ii) *Let* $\operatorname{Cut}(F)$ *be a π-class. Then* $\operatorname{cut}(F) = \mathrm{FN}$.

PROOF: **(i)** Let us suppose $\mathrm{cut}\,(F)$ is a π-class. By prolongation there exists a decreasing function f with $\mathrm{dom}\,(F) \in \mathrm{cut}\,(F) \setminus \mathrm{FN}$ such that $\mathrm{cut}\,(F) = \bigcap\{f(n); n \in \mathrm{FN}\}$ and $F(f(n)) \neq f(n)$ for $n \in \mathrm{FN}$. Let $g = F(f)$. Then $g(n) \neq f(n)$ for $n \in \mathrm{FN}$ and thus also for some $\alpha \in \mathrm{cut}\,(F) \setminus \mathrm{FN}$ there holds $F(f(\alpha)) = (F(f))(F(\alpha)) = g(F(\alpha)) = g(\alpha) \neq f(\alpha)$. But this is a contradiction, since $f(\alpha) \in \mathrm{cut}\,(F)$.
(ii) Let us suppose that $\mathrm{Cut}\,(F)$ is a π-class and $\mathrm{cut}\,(F) \neq \mathrm{FN}$. Then there exists a decreasing function f with $\mathrm{dom}\,(F) \in \mathrm{cut}\,(F) \setminus \mathrm{FN}$ such that $\mathrm{Cut}\,(F) = \bigcap\{f(n); n \in \mathrm{FN}\}$ and $F(f(n)) = f(n)$ for $n \in \mathrm{FN}$. From (i) it follows that $\mathrm{cut}\,(F) \subset \mathrm{Cut}\,(F)$ and thus we can suppose that $\mathrm{rng}\,(f) \cap \mathrm{cut}\,(F) = \emptyset$. Let $g = F(f)$. Then $g(n) = f(n)$ for $n \in \mathrm{FN}$ and thus $F(f(\alpha)) = g(\alpha) = f(\alpha)$ also for some $\alpha \notin \mathrm{FN}$. But this is a contradiction, since $f(\alpha) \in \mathrm{Cut}\,(F) \setminus \mathrm{cut}\,(F)$. □

It is an interesting problem to characterize all possible values of $\mathrm{cut}\,(F)$ and $\mathrm{Cut}\,(F)$. Next lemma gives another necessary condition on them.

Similarly as in PA, we can define the notion of a strong segment: a segment $I \subseteq N$ is strong, iff for every f the class $f'' I \setminus I$ is not cofinal in $N \setminus I$. If I is strong, then it satisfies the axioms of PA.

Lemma 2.3. *Let F be an automorphism such that* $\mathrm{cut}\,(F) \neq \mathrm{Cut}\,(F)$. *Then* $\mathrm{cut}\,(F)$ *is strong.*

PROOF: Let F be given, $\alpha \in \mathrm{Cut}\,(F) \setminus \mathrm{cut}\,(F)$ and let f be arbitrary. We take $g = F(f)$ and β as the minimum of the set

$$\{\alpha\} \cup \{f(\gamma); \gamma < f(\gamma) \ \&\ f(\gamma) \neq g(\gamma)\}.$$

Note that $\beta \notin \mathrm{cut}\,(F)$ (because $\gamma < f(\gamma) \in \mathrm{cut}\,(F)$ implies $f(\gamma) = g(\gamma)$). If for some $\delta \in \mathrm{cut}\,(F)$ there holds $f(\delta) \notin \mathrm{cut}\,(F)$, then either $f(\delta) \geq \alpha$ or $f(\delta) \neq F(f(\delta)) = g(\delta)$. In both cases it follows that $f(\delta) \geq \beta$. Hence $f''\,\mathrm{cut}\,(F) \setminus \mathrm{cut}\,(F)$ is contained in $N \setminus \beta$ and thus it is not cofinal in $N \setminus \mathrm{cut}\,(F)$. We have proved that $\mathrm{cut}\,(F)$ is strong. □

A kind of converse to this lemma was proved by R. Kaye (unpublished) for the automorphisms of countable recursively saturated models of PA—he proved that for any strong cut I elementary in a model M there exists an automorphism F of M such that $I = \mathrm{cut}\,(F) = \{x; F(x) = x\}$.

In the rest of this section, we shall construct several different types of automorphisms. The first type satisfies $\mathrm{FN} = \mathrm{cut}\,(F) = \mathrm{Cut}\,(F)$ and the second one satisfies $\mathrm{FN} = \mathrm{cut}\,(F) \subset \mathrm{Cut}\,(F)$.

Next issue concerns fully revealed automorphisms. We know that an automorphism is revealed, if and only if $\mathrm{FN} \subset \mathrm{cut}\,(F)$, but we have no similar characterization of fully revealed automorphisms. If we take the revealments of the automorphisms above, we get automorphisms with $\mathrm{FN}^* = \mathrm{cut}\,(F) = \mathrm{Cut}\,(F)$ in the first case and with $\mathrm{FN}^* = \mathrm{cut}\,(F) \subset \mathrm{Cut}\,(F)$ in the second case.

First, we are going to construct an automorphism with $\mathrm{FN} = \mathrm{cut}\,(F) = \mathrm{Cut}\,(F)$. We need two lemmas:

Lemma 2.4. *Let $F \preccurlyeq \mathrm{FN}$ be a similarity. Then*

$$(\forall\, \alpha \notin \mathrm{FN}\,)\ (\exists\, \beta \in \alpha - \mathrm{FN}\,)\ (F \cup \{\langle \beta, \beta\rangle\} \text{ is a similarity}).$$

Proof: See [S–1985, p. 510]. □

Lemma 2.5. *Let $F \preccurlyeq \mathrm{FN}$ be a similarity. Then*

$$(\forall\, \alpha \notin \mathrm{FN}\,)\ (\exists\, \beta \in \alpha)\ (\exists\, \gamma \neq \beta)\ (F \cup \{\langle \gamma, \beta\rangle\} \text{ is a similarity}).$$

Proof: If the lemma does not hold for some α, then for every $\beta \in \alpha$, the class $\{\beta\}$ is a monad in the equivalence $E(\mathrm{dom}\,(F))$, but this is a contradiction, because $E(\mathrm{dom}\,(F))$ is a compact equivalence (see [V]). □

Theorem 2.6. *Any at most countable similarity can be prolonged into an automorphism F such that* $\mathrm{FN} = \mathrm{cut}\,(F) = \mathrm{Cut}\,(F)$.

Proof: Let $\mathcal{I}$ be the system of all similarities. Let

$$\begin{aligned}
\mathcal{D}_{\langle x,0\rangle} &= \{F \in \mathcal{I};\ x \in \mathrm{dom}\,(F) \cap \mathrm{rng}\,(F)\}\\
\mathcal{D}_{\langle \alpha,1\rangle} &= \{F \in \mathcal{I};\ \alpha \notin \mathrm{FN} \Rightarrow (\exists\, \beta \in \alpha)(F(\beta) \neq \beta)\}\\
\mathcal{D}_{\langle \alpha,2\rangle} &= \{F \in \mathcal{I};\ \alpha \notin \mathrm{FN} \Rightarrow (\exists\, \beta \in \alpha - \mathrm{FN}\,)(F(\beta) = \beta)\}.
\end{aligned}$$

Let $\mathcal{A}^c$ be the subsystem of all countable classes in a system $\mathcal{A}$. By [V], the systems $\mathcal{D}^c_{\langle x,0\rangle}$ are dense in $\mathcal{I}^c$ (i.e. for every $F \in \mathcal{D}^c_{\langle x,0\rangle}$ there is $G \in \mathcal{I}^c$ with $F \subseteq G$), the systems $\mathcal{D}^c_{\langle \alpha,1\rangle}$ and $\mathcal{D}^c_{\langle \alpha,2\rangle}$ are dense in $\mathcal{I}^c$ by Lemma 2.5 and Lemma 2.4. By [S–V 1989, Theorem 1], we have a similarity which is an element of every mentioned system. Thus it is an automorphism (systems $\mathcal{D}_{\langle x,0\rangle}$), $\mathrm{FN} = \mathrm{cut}\,(F)$ (systems $\mathcal{D}_{\langle \alpha,1\rangle}$) and $\mathrm{FN} = \mathrm{Cut}\,(F)$ (systems $\mathcal{D}_{\langle \alpha,2\rangle}$). □

Now we are going to construct an automorphism with $\mathrm{FN} = \mathrm{cut}\,(F) \subset \mathrm{Cut}\,(F)$, which is much more difficult.

Lemma 2.7. *Let F be a similarity. If $\mathrm{dom}\,(F)$ is closed under Def, then also $\mathrm{rng}\,(F)$ is closed under Def.*

Proof: Let $y \in \mathrm{Def}\,(\mathrm{rng}\,(F))$. Then there is φ and $a_1, \dots, a_k \in \mathrm{dom}\,(F)$ such that y is the unique z for which $\varphi(z, F(a_1), \dots, F(a_k))$. Let x be the unique z for which $\varphi(z, a_1, \dots, a_k)$; it exists and is unique, since F is a similarity. We have $x \in \mathrm{Def}\,(\mathrm{dom}\,(F)) = \mathrm{dom}\,(F)$, and so $\varphi(F(x), F(a_1), \dots, F(a_k))$. But then $F(x) = y$, because y was the unique one with the property, thus $y \in \mathrm{rng}\,(F)$. □

Lemma 2.8. *Let $C \preccurlyeq \mathrm{FN}$, let $A \neq \emptyset$ be a figure in the equivalence $E(C)$, and let $H_i, i \in \mathrm{FN}$, be $\mathrm{Sd_C}$-functions. If there are $b_i, i \in \mathrm{FN}$, such that $(\forall\, x \in A)(\exists\, i)(H_i(x) = b_i)$, then for some i we have $b_i \in \mathrm{Def}\,(C)$.*

Proof: We can assume that A is a monad in the equivalence E(C), since A is nonempty and the assumption of the lemma follows for each monad included in A.

Let $A_i = \{x; H_i(x) = b_i\}$, note that A_i are Sd_{V}-classes. Let $B_i, i \in \mathrm{FN}$, be an enumeration of all Sd_{C}-classes containing A. We have

$$\bigcap\{B_i; i \in \mathrm{FN}\} \subseteq A \subseteq \bigcup\{A_i; i \in \mathrm{FN}\}.$$

By prolongation, it holds for some $k \in \mathrm{FN}$ that $\bigcap\{B_i; i < k\} \subseteq \bigcup\{A_i; i < k\}$. Let $B = \bigcap\{B_i; i < k\}$. B is a Sd_{C}-class and thus there exists $x \in B \cap \mathrm{Def}\,(C)$ (see [V]). For some i we have $x \in A_i$ and thus $b_i = H_i(x) \in \mathrm{Def}\,(C)$. □

Lemma 2.9. *Let $F \preccurlyeq \mathrm{FN}$ be a similarity with domain closed under* Def , $a \notin \mathrm{dom}\,(F)$. *Then there is a similarity $G \preccurlyeq \mathrm{FN}$ such that* $\mathrm{dom}\,(G)$ *is closed under* Def , $F \subseteq G, a \in \mathrm{dom}\,(G)$ *and* $G(b) = b$ *implies* $b \in \mathrm{dom}\,(F)$.

Proof: We will extend F on the domain $\mathrm{Def}\,(\mathrm{dom}\,(F) \cup \{a\})$. Let $\mathrm{dom}\,(F) = \{a_i; i \in \mathrm{FN}\}, A = \{x; F \cup \{\langle x, a\rangle\}$ is a similarity$\}, C = \mathrm{rng}\,(F)$. By [V], the class A is a nonempty figure in the equivalence E(C).

Let φ be a set-theoretical formula such that there is a unique z such that $\varphi(z, a, a_0, \ldots, a_k)$. We denote this z by b_φ. If there is a unique y such that $\varphi(y, x, F(a_0), \ldots, F(a_k))$, we denote it by $H_\varphi(x)$. H_φ is a Sd_{C}-function and from the definition of A it follows that $A \subseteq \mathrm{dom}\,(H_\varphi)$. Let $G_x = F \cup \{\langle H_\varphi(x), b_\varphi\rangle; \varphi \in \mathrm{FSL}\}, A_\varphi = \{x \in A; H_\varphi(x) = b_\varphi\}$.

If $x \in A$, then G_x is a similarity with the domain $\mathrm{Def}\,(\mathrm{dom}\,(F) \cup \{a\})$. To finish the proof, we need to show that there exists $x \in A$ such that for every φ such that $b_\varphi \notin \mathrm{dom}\,(F)$, there holds $x \notin A_\varphi$. Suppose that there is no such x. Let $\varphi_i, i \in \mathrm{FN}$, be an enumeration of all φ such that $b_\varphi \notin \mathrm{dom}\,(F)$ and A_φ is nonempty. By Lemma 2.8 (with $b_i = b_{\varphi_i}, H_i = H_{\varphi_i}$) we can choose φ such that $b_\varphi \in \mathrm{Def}\,(C) \setminus \mathrm{dom}\,(F)$ and A_φ is nonempty. Let $x \in A_\varphi$. Then $H_\varphi(x) = b_\varphi \in \mathrm{Def}\,(C) = \mathrm{Def}\,(\mathrm{rng}\,(F)) = \mathrm{rng}\,(F)$. Thus for some $j \in \mathrm{FN}$ we have $\varphi(b_\varphi, a, a_0, \ldots, a_k)\,\&\, \varphi(b_\varphi, x, F(a_0), \ldots, F(a_k))\,\&\, F(a_j) = b_\varphi$, hence $\varphi(F(a_j), x, F(a_0), \ldots, F(a_k))$. Because G_x is a similarity, we have $b_\varphi = a_j$ and $b_\varphi \in \mathrm{dom}\,(F)$, a contradiction.

We proved that there is $x \in A$ such that $G = G_x$ has all required properties. □

Theorem 2.10. **(i)** *Any at most countable similarity F with domain closed under* Def *can be prolonged into an automorphism G such that $G(b) = b$ implies $b \in \mathrm{dom}\,(F)$.*
(ii) *Any at most countable similarity F can be prolonged into an automorphism G such that* $\mathrm{FN} = \mathrm{cut}\,(G) \subset \mathrm{Cut}\,(G)$.

Proof: **(i)** Let $\mathcal{I}$ be the system of all similarities with domain closed under Def , for which $(\forall\, b)(G(b) = b \Rightarrow b \in \mathrm{dom}\,(F))$. Let

$$\mathcal{D}_{\langle x,0\rangle} = \{G \in \mathcal{I}; x \in \mathrm{dom}\,(G)\},$$
$$\mathcal{D}_{\langle x,1\rangle} = \{G \in \mathcal{I}; x \in \mathrm{rng}\,(G)\}.$$

The systems $\mathcal{D}^c_{\langle x,0\rangle}$ and $\mathcal{D}^c_{\langle x,1\rangle}$ are dense in $\mathcal{I}^c$ by Lemma 2.9 used in the second case for inverse functions. By [S–V 1989, Theorem 1], we have a similarity G which is an element of every mentioned system. Thus it is an automorphism and from the definition of the system $\mathcal{I}$ it follows that it has the required property.
(ii) First, we extend the similarity F on the domain $\mathrm{Def}\,(\mathrm{dom}\,(F))$ (the extension is unique) and then we use (i). □

3. Revealed and fully revealed automorphisms.

In this section, we construct two fully revealed automorphisms such that their composition is not fully revealed. Consequently, we have also a revealed automorphism which is not fully revealed (since revealed automorphisms are closed under composition); this result was already proved by A. Sochor and C. Marchini (unpublished). The idea of this proof is similar to that of the main result of [S–1985], where it is proved that the fully revealed classes are not closed under intersection.

We define the notion of an I-Sd class which will be useful in our proof.

Definition 3.1. Let $I \subseteq N$. A relation R is I-Sd, if there exists a Sd_{V}-relation $S \supseteq R$ and $\beta \in I$ such that $(\forall\, x)(S''\{x\} \overset{\wedge}{\preccurlyeq} \beta)$.

Observation. **(i)** Let S be a Sd_{V}-class which is I-Sd. Then for any u the class $S \upharpoonright u$ is a set.

(i) Let R be an α-Sd relation such that its domain is a semiset. Then $(\exists\, r \supseteq R)(\forall\, x)(r''\{x\} \overset{\wedge}{\preccurlyeq} \alpha)$.

The next theorem is not used in the proof of the main theorem of this section, but it gives us some information about α-Sd automorphisms.

Theorem 3.2. *Let F be an automorphism such that* $\mathrm{cut}\,(F) \subset \mathrm{Cut}\,(F)$, *let* $\beta \notin \mathrm{cut}\,(F)$. *Then* $F \upharpoonright \beta$ *is not* FN*-Sd.*

Proof: We will prove by induction (w.r.t. n) that $F \upharpoonright \beta$ cannot be n-Sd if $\beta \notin \mathrm{cut}\,(F)$. Let us suppose that n is the smallest α such that for some $\beta \notin \mathrm{cut}\,(F)$ the class $F \upharpoonright \beta$ is α-Sd. Let us fix such β and a corresponding Sd_{V}-relation $S \supseteq F \upharpoonright \beta$. By assumption $\mathrm{Id} \upharpoonright \mathrm{cut}\,(F) \subseteq S$. But then also, for some $\gamma \in \mathrm{Cut}\,(F) \setminus \mathrm{cut}\,(F)$, we have $\mathrm{Id} \upharpoonright \gamma \subseteq S$, since S is Sd_{V}. By assumption $\gamma \in \mathrm{Cut}\,(F)$, we have $F \cap (\mathrm{Id} \upharpoonright \gamma) = \mathrm{Id} \upharpoonright \mathrm{cut}\,(F)$, so for $T = (S \upharpoonright (\gamma \setminus (\gamma/2))) \setminus \mathrm{Id}$ there holds $F \upharpoonright (\gamma \setminus (\gamma/2)) \subseteq T$ and $T''\{x\} \overset{\wedge}{\prec} S''\{x\} \overset{\wedge}{\preccurlyeq} n$ for every x. But $\langle \xi, \zeta \rangle \in F$ implies $\langle F(\gamma) - \xi, \gamma - \zeta \rangle \in F$ (since $F(\gamma - \zeta) = F(\gamma) - F(\zeta) = F(\gamma) - \xi$) and so $F \upharpoonright (\gamma/2) \subseteq \{\langle F(\gamma) - \xi, \gamma - \zeta \rangle; \langle \xi, \zeta \rangle \in T\}$; this is an Sd_{V}-class which witnesses that $F \upharpoonright (\gamma/2)$ is $(n-1)$-Sd, a contradiction. □

Now we are going to construct an automorphism with a special property, which will be useful in our construction.

Lemma 3.3. *For every countable similarity F, there exists an automorphism $G \supseteq F$ such that*

$$(\forall\, \alpha)(\forall\, \beta \notin \mathrm{FN})(G \upharpoonright (\alpha \cdot \beta) \ \text{ is not } \ G(\alpha) - Sd).$$

Proof: Let $\mathcal{I}$ be the system of all similarities. Let

$$\mathcal{D}_{\langle \alpha, \beta, r \rangle} = \{G \in \mathcal{I};\ \alpha \in \mathrm{dom}\,(G) \cap \mathrm{rng}\,(G)\, \&$$
$$\&\, ((\beta \notin \mathrm{FN}\ \&\, (\forall\, x)(r''\{x\} \overset{\wedge}{\preccurlyeq} G(\alpha))) \Rightarrow \neg(G \upharpoonright (\beta \cdot \alpha) \subseteq r))\}.$$

If we prove that the systems $\mathcal{D}^c_{\langle\alpha,\beta,r\rangle}$ are dense on $\mathcal{I}^c$, the proof is finished by [S–V 1989, Theorem 1].

Let a countable similarity G and $\langle\alpha,\beta,r\rangle$ be given. By [V], we can prolong the similarity G so that $\alpha \in \operatorname{dom}(G) \cap \operatorname{rng}(G)$ and $\{\beta, \alpha\cdot\beta\} \subseteq \operatorname{dom}(G)$. Now let $\beta \notin \mathrm{FN}$ and $(\forall\, x)(r''\{x\} \overset{\wedge}{\preccurlyeq} G(\alpha))$.

Let us take the indiscernibility equivalence $E(\operatorname{rng}(G))$. We claim that there is a "big" monad under $G(\alpha\cdot\beta)$, more precisely, there is a monad $A \subseteq G(\alpha\cdot\beta)$ such that $(\exists\, a \subseteq A)(G(\alpha) \overset{\wedge}{\prec} a)$. This follows easily from the fact that $G(\alpha\cdot\beta) = G(\alpha)\cdot G(\beta) \notin G(\alpha)\cdot \mathrm{FN}$ and from the compactness of $E(\operatorname{rng}(G))$.

Let us take a monad A with this property and γ such that $(\forall\, \delta \in A)(G \cup \{\langle\delta,\gamma\rangle\}$ is a similarity). From $A \subseteq G(\alpha\cdot\beta)$ follows $\gamma \in \alpha\cdot\beta$. By assumption on r and A, there is a $\delta \in A$ such that $\delta \notin r''\{\gamma\}$. The similarity $G \cup \{\langle\delta,\gamma\rangle\}$ has the required property. $\square$

Let $2^\alpha_0 = 2, 2^\alpha_{n+1} = 2^{2^\alpha_n}, 2^\alpha_{\mathrm{FN}} = \bigcup\{2^\alpha_n; n \in \mathrm{FN}\}$.

Corollary. *For any $\alpha \notin \mathrm{FN}$ there exists a fully revealed automorphism F such that $F(\alpha) = \alpha$ and $(\forall\, \beta \notin 2^\alpha_{\mathrm{FN}})(F \restriction \beta$ is not 2^α_{FN}-Sd).*

Proof: Let $\alpha \notin \mathrm{FN}$. If we use the lemma for the similarity $\{\langle\alpha,\alpha\rangle\}$, we get an automorphism G such that

$$(*) \qquad (\forall\, \nu)(G \restriction 2^\alpha_{\nu+1} \text{ is not } 2^\alpha_\nu\text{-Sd})$$

(actually, the property in Lemma 3.3 is much stronger, but the property $(*)$ is expressed by a normal formula). Let us take its $\{\langle\alpha,\alpha\rangle\}$-revealment F (by Theorem 1.3 it exists). F is an automorphism and it still has the property $(*)$, because it is given by a normal formula with a parameter α, and this guarantees the property in the corollary. $\square$

Theorem 3.4. *There exist two fully revealed automorphisms such that their composition is not fully revealed.*

Corollary. *There exists a revealed automorphism which is not fully revealed.*

Proof of Theorem 3.4: Let F be a fully revealed automorphism from the corollary of Lemma 3.3. Let $\mathcal{I}$ be the system of all functions G such that $G \cup \mathrm{Id} \restriction 2^\alpha_{\mathrm{FN}}$ is a similarity. Let

$$\mathcal{D}_{\langle x,0\rangle} = \{G \in \mathcal{I};\ x \in \operatorname{dom}(G) \cap \operatorname{rng}(G)\},$$
$$\mathcal{D}_{\langle \beta,1\rangle} = \{G \in \mathcal{I};\ \beta \notin 2^\alpha_{\mathrm{FN}} \Rightarrow (\exists\, \gamma < \beta)(G(\langle F(\gamma),\gamma\rangle) \notin F)\}.$$

We will prove in Lemma 3.5 and Lemma 3.6 that the systems $\mathcal{D}^c_{\langle x,0\rangle}$ and $\mathcal{D}^c_{\langle \beta,1\rangle}$ are dense in $\mathcal{I}^c$. By [S–V 1989, Theorem 1], we have a function G which is an element of every mentioned system. The systems $\mathcal{D}_{\langle x,0\rangle}$ guarantee that G is an automorphism. The systems $\mathcal{D}_{\langle \beta,1\rangle}$ together with the condition that $G \cup \mathrm{Id} \restriction 2^\alpha_{\mathrm{FN}}$ is a similarity guarantee that

$$\operatorname{cut}(G) \subseteq 2^\alpha_{\mathrm{FN}} \,\&\, (\forall\, \beta \notin 2^\alpha_{\mathrm{FN}})(\exists\, \gamma < \beta)(G(\langle F(\gamma),\gamma\rangle) \notin F).$$

Now let us consider the automorphisms F^{-1} and $G''F$. They are both fully revealed (as automorphic image of a fully revealed class is fully revealed), but as we shall see, their composition is not, since $\operatorname{cut}(F^{-1}\circ(G''F)) = 2^{\alpha}_{\mathrm{FN}}$.

Let $\beta \in 2^{\alpha}_{\mathrm{FN}}$. From $F(\alpha) = \alpha$ it follows that $F(\beta) \in 2^{\alpha}_{\mathrm{FN}}$, thus $\langle F(\beta),\beta\rangle = G(\langle F(\beta),\beta\rangle) \in G''F$ and $\beta \in \operatorname{cut}(F^{-1}\circ(G''F))$. Let $\beta \notin 2^{\alpha}_{\mathrm{FN}}$, let $\gamma < G^{-1}(\beta)$ be such that $G(\langle F(\gamma),\gamma\rangle) \notin F$ (its existence is guaranteed by the property of G). It follows that $G''F(G(\gamma)) \neq F(G(\gamma))$, hence $G(\gamma) \notin \operatorname{cut}(F^{-1}\circ(G''F))$ and also $\beta \notin \operatorname{cut}(F^{-1}\circ(G''F))$, since $G(\gamma) < \beta$; we are done. □

Lemma 3.5. *Let G be a countable function such that $G \cup \mathrm{Id} \restriction 2^{\alpha}_{\mathrm{FN}}$ is a similarity. Then for every z there exists t such that $G \cup \mathrm{Id} \restriction 2^{\alpha}_{\mathrm{FN}} \cup \{\langle t,z\rangle\}$ is a similarity.*

Proof: Since 2^{α}_{FN} codes all its subsets, we may restrict ourselves to the formulae with one parameter from 2^{α}_{FN}. Let z be given, let $\operatorname{dom}(G) = \{a_i; i \in \mathrm{FN}\}$. Let

$$\begin{aligned} A_n = \{y;\ (\forall\, \varphi \in \mathrm{FSL}, \varphi < n)(\forall\, \delta \in 2^{\alpha}_{n}) \\ (\varphi(z,\delta,a_0,\dots,a_k) \Leftrightarrow \varphi(y,\delta,G(a_0),\dots,G(a_k)))\}. \end{aligned}$$

By an argument from [Ve–1982] the classes A_n are nonempty: Consider the formula

$$\Phi(x,z,a_0,\dots,a_k) \Leftrightarrow (\forall\, \varphi \in \mathrm{FSL}, \varphi < n)(\forall\, \delta \in 2^{\alpha}_{n})(\langle\varphi,\delta\rangle \in x \Leftrightarrow \varphi(z,\delta,a_0,\dots,a_k)).$$

There exists $x \subseteq n \times 2^{\alpha}_{n}$ satisfying $\Phi(x,z,a_0,\dots,a_k)$, this x is coded in 2^{α}_{FN}. It follows that there exists y satisfying $\Phi(x,y,G(a_0),\dots,G(a_k))$ (since $\mathrm{Id} \restriction 2^{\alpha}_{\mathrm{FN}} \cup G$ is similarity) and this y is an element of A_n. Hence the classes A_n form a decreasing sequence of nonempty Sd_{V}-classes. It follows that they have a common element t which has the required property. □

Lemma 3.6. *Let F be as above, let G be a countable function such that $G \cup \mathrm{Id} \restriction 2^{\alpha}_{\mathrm{FN}}$ is a similarity, let $\beta \notin 2^{\alpha}_{\mathrm{FN}}$. Then there exist $\gamma < \beta$ and $b \notin F$ such that $G \cup \mathrm{Id} \restriction 2^{\alpha}_{\mathrm{FN}} \cup \langle b, \langle F(\gamma),\gamma\rangle\rangle$ is a similarity.*

Proof: Let $C = \operatorname{dom}(G)$. Let

$$\begin{aligned} A_{n,\gamma} &= \{y;\ (\forall\, \varphi \in \mathrm{FSL}_{\mathrm{C}}, \varphi < n)(\forall\, \delta \in 2^{\alpha}_{n})(\varphi(\langle F(\gamma),\gamma\rangle,\delta) \Leftrightarrow \varphi(\langle y,\gamma\rangle,\delta))\},\\ A_n &= \{\gamma < \beta; 2 \preccurlyeq A_{n,\gamma}\}. \end{aligned}$$

$A_{n,\gamma}$ and A_n are revealed classes, since they are defined by a normal formula from the only parameter F which is fully revealed.

Suppose that $A_n = \emptyset$. Then for every $\gamma < \beta$ there holds $A_{n,\gamma} = \{F(\gamma)\}$. Let for $a \subseteq n \times 2^{\alpha}_{n}$ be $H_a(\gamma)$ defined as the least y (if it exists) such that

$$(\forall\, \varphi \in \mathrm{FSL}_{\mathrm{C}}, \varphi < n)(\forall\, \delta \in 2^{\alpha}_{n})(\varphi(\langle y,\gamma\rangle,\delta) \Leftrightarrow \langle\varphi,\delta\rangle \in a).$$

Let $r = \{\langle H_a(\gamma),\gamma\rangle; a \subseteq n \times 2^{\alpha}_{n}, \gamma < \beta\}$. We have $F \restriction \beta \subseteq r$ and $(\forall\, x)(r''\{x\} \overset{\wedge}{\preccurlyeq} n \cdot 2^{\alpha}_{n})$, which is a contradiction with the property of F. We have proved that $A_n \neq \emptyset$.

Consequently, A_n is a decreasing sequence of nonempty revealed classes. Let $\gamma \in \bigcap\{A_n; n \in \mathrm{FN}\}$. Now $A_{n,\gamma}$ is a decreasing sequence of revealed classes each of which has at least two elements. (one of them is $F(\gamma)$). Thus there exists $y \neq F(\gamma)$ such that $y \in \bigcap\{A_{n,\gamma}; n \in \mathrm{FN}\}$. Let us take by Lemma 3.5 $\langle z, x\rangle$ such that $G \cup \mathrm{Id} \restriction 2^\alpha_{\mathrm{FN}} \cup \{\langle\langle z, x\rangle, \langle F(\gamma), \gamma\rangle\rangle\}$ is a similarity. If $\langle z, x\rangle \notin F$, then we put $b = \langle z, x\rangle$ and we are done. In the other case, $G \cup \mathrm{Id} \restriction 2^\alpha_{\mathrm{FN}} \cup \{\langle\langle z, x\rangle, \langle y, \gamma\rangle\rangle\}$ is a similarity (by the definition of $A_{n,\gamma}$). By Lemma 3.5, we can take some b such that $G \cup \mathrm{Id} \restriction 2^\alpha_{\mathrm{FN}} \cup \{\langle\langle z, x\rangle, \langle y, \gamma\rangle\rangle, \langle b, \langle F(\gamma), \gamma\rangle\rangle\}$ is a similarity; it follows that $b = \langle t, x\rangle$ for some t. The assumption $F(\gamma) \neq y$ implies that $t \neq z$ and hence $b \notin F$. $\square$

We would like to know whether there exists some nontrivial normal subgroup of the group of all automorphisms besides the subgroup of all revealed automorphisms. Fully revealed automorphisms are out of the question, since they are not closed under composition. However, it is an open problem whether the group generated by all fully revealed automorphisms is different from the group of revealed automorphism.

Note that this problem closely corresponds to the problem of finding normal subgroups of $\mathrm{Aut}(M)$, where M is countable recursively saturated model of PA. This problem is more interesting, if $M \equiv \mathbb{N}$, which corresponds to the axiom of elementarily equivalence in AST, which is equivalent with the statement $\mathrm{Def} = \mathrm{FV}$.

References

[P–S 1984] Pudlák P., Sochor A., *Models of the alternative set theory*, JSL **49** (1984), 570–585.

[S–1985] Sochor A., *Notes on revealed classes*, Comment. Math. Univ. Carolinae **26** (1985), 477–498.

[S–V 1979] Sochor A., Vopěnka P., *Endomorphic universes and their standard extensions*, Comment. Math. Univ. Carolinae **20** (1979), 605–629.

[S–V 1980] Sochor A., Vopěnka P., *Revealments*, Comment. Math. Univ. Carolinae **21** (1980), 97–118.

[S–V 1981] Sochor A., Vopěnka P., *Ultrafilters of sets*, Comment. Math. Univ. Carolinae **22** (1981), 698–699.

[S–V 1989] Sochor A., Vopěnka P., *Constructions of classes by transfinite induction in AST*, Comment. Math. Univ. Carolinae **30** (1989), 155–161.

[Ve 1982] Vencovská A., *Constructions of endomorphic universes and similarities*, Comment. Math. Univ. Carolinae **23** (1982), 557-577.

[V] Vopěnka P., *Mathematics in the alternative set theory*, Teubner-Texte, Leipzig, 1979.

KFMPV, MFF UK, Malostranské nám. 25, 188 00 Praha 1, Czechoslovakia

Mathematical Institute of ČSAV, Žitná 25, 115 67 Praha 1, Czechoslovakia

(*Received* August 16, 1990)