Non-perfect rings and a theorem of Eklof and Shelah

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Abstract. We prove a stronger form, A^+, of a consistency result, A, due to Eklof and Shelah. A^+ concerns extension properties of modules over non-left perfect rings. We also show (in ZFC) that A does not hold for left perfect rings.

Keywords: perfect ring, Ext, uniformization

Classification: 16A50, 16A51

Recently, a significant extension of the theory of Whitehead modules from domains to arbitrary non-left perfect rings has been performed by Paul C. Eklof and Sharon Shelah ([3]). In [3, Theorem 2.1 and Corollary 2.2], they proved that the assertion

A: "for any non-left perfect ring R and any uncountable cardinal κ such that $\mathrm{cf}\,(\kappa) = \aleph_0$ and $\kappa \geq \mathrm{card}\,(R)$ there is a non-projective κ^+-free module M such that $\mathrm{card}\,(M) = \kappa^+$ and $\mathrm{Ext}_R\,(M, N) = 0$ whenever N is a module with $\mathrm{card}\,(N) < \kappa$"

is consistent with ZFC + GCH. Their proof consists of two parts: the set theoretic one showing consistency of the existence of certain ω-trees and the algebraic one inferring A from the existence of the trees.

Independently, using consistency of a uniformization principle due to Shelah, we proved a weaker form of A is consistent in the particular case of von Neumann regular rings ([5, Lemma 2.4]). In the present paper, we show our approach can be modified to obtain a simple proof of the consistency of A. Moreover, we show that a stronger form of A, denoted by A^+, is consistent, namely the expression "κ^+-free" can be replaced by "strongly κ^+-free" (see Corollary 1.6 below). The point here is that we use the definition of Ext via Hom-groups rather than via exact sequences. We also work directly with the defining relations of modules rather than with the tree-module structures.

The result of Eklof and Shelah is the best possible: we show in ZFC that for any left perfect ring R there is a proper class C consisting of pairwise non-isomorphic modules such that $\mathrm{Ext}_R\,(M, N) \neq 0$ for all $N \in C$ and all non-projective modules M (Theorem 1.10).

Let M be a module. Then $\mathrm{gen}\,(M)$ denotes the minimum of cardinalities of R-generating subsets of M. Further, M is said to be κ-free provided for each submodule $N \subseteq M$ with $\mathrm{gen}\,(N) < \kappa$ there is a free module $P \subseteq M$ such that $N \subseteq P$ and $\mathrm{gen}\,(P) < \kappa$. Moreover, M is strongly κ-free provided foe each submodule $N \subseteq M$ with $\mathrm{gen}\,(N) < \kappa$ there is a free module $P \subseteq M$ such that $N \subseteq P, \mathrm{gen}\,(P) < \kappa$ and M/P is κ-free. A sequence $(M_\alpha \mid \alpha < \kappa)$ is said to be a κ-filtration of M, if for all $\alpha < \kappa$, M_α is a submodule of $M_{\alpha+1}$ such that $\mathrm{gen}\,(M_\alpha) < \kappa$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit $\alpha < \kappa$, and $M = \bigcup_{\alpha<\kappa} M_\alpha$.

Let R be a ring. Then R is said to be completely reducible provided R is a ring direct sum of a finite number of full matrix rings over skew fields.

Homomorphisms of (left R-)modules are written as acting on the right. Further concepts and notation can be found e.g. in [1] and [2].

Definition 1.1. Let R be a non-left perfect ring. By [1, Theorem 28.4], there exist elements $a_i \in R, i < \aleph_0$, such that $(a_0 \dots a_i R \mid i < \aleph_0)$ is a strictly decreasing chain of principal right ideals of R. Let κ be an infinite cardinal and E be a subset of κ^+ such that $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}\,(\alpha) = \aleph_0\}$. Let $(n_\nu \mid \nu \in E)$ be a ladder system, i.e. for each $\nu \in E$, let $(n_\nu(i) \mid i < \aleph_0)$ be a strictly increasing sequence of non-limit ordinals less that ν such that $\sup_{i<\aleph_0} n_\nu(i) = \nu$.

Let $(R_\alpha \mid \alpha < \kappa)$ be a system of free modules defined as follows: $R_\alpha = R$ provided $\alpha \in \kappa^+ \setminus E$, and $R_\alpha = R^{(\aleph_0)}$ provided $\alpha \in E$. For $\alpha \in \kappa^+ \setminus E$, denote by 1_α the canonical generator of R_α, and for $\alpha \in E$ let $\{1_{\alpha,i} \mid i < \aleph_0\}$ be the canonical basis of R_α. Note that by [1, Lemmas 28.1 and 28.2], for every $\nu \in E$, the module

$$S_\nu = \sum_{i<\aleph_0} R(-1_{\nu,i} + a_i \cdot 1_{\nu,i+1})$$

is a free submodule of R_ν such that R_ν/S_ν is not projective. Put $P = \oplus \sum_{\alpha<\kappa^+} R_\alpha$ and $Q = \sum_{\alpha\in E} Q_\alpha$, where $Q_\alpha = \sum_{i<\aleph_0} Rg_{\alpha i}$ and $g_{\alpha i} = (1_{n_\alpha(i)} - 1_{\alpha,i} + a_i \cdot 1_{\alpha,i+1}) \in P$, for all $\alpha \in E$ and $i < \aleph_0$. Finally, put $M = P/Q \in R\text{-}\mathrm{mod}$.

Lemma 1.2. (i) $\mathrm{gen}\,(M) = \kappa^+$.

(ii) *If E is a stationary subset of κ^+, then M is not projective.*

(iii) *If E is non-reflecting (i.e. $E \cap \sigma$ is not stationary in σ for all limit ordinals $\sigma < \kappa^+$), then M is strongly κ^+-free.*

Proof: (i) This follows easily from the fact that $\{1_\alpha + Q \mid \alpha \in \kappa^+ \setminus E\} \cup \{1_{\alpha,i} + Q \mid \alpha \in E, i < \aleph_0\}$ is an R-generating subset of M.

(ii) Put $M_0 = 0$ and, for each $0 < \alpha < \kappa^+$, $M_\alpha = (\oplus \sum_{\beta<\alpha} R_\beta + Q)/Q$. Then $(M_\alpha \mid \alpha < \kappa^+)$ is a κ^+-filtration of M.

Assume M is projective. By [1, Corollary 26.2] there exist modules $(P_\alpha \mid \alpha < \kappa^+)$ such that $\mathrm{gen}\,(P_\alpha) \le \aleph_0$ for all $\alpha < \kappa^+$ and $M = \oplus \sum_{\alpha<\kappa^+} P_\alpha$. Put $N_0 = 0$ and, for each $0 < \alpha < \kappa^+, N_\alpha = \oplus \sum_{\beta<\alpha} P_\beta$. Clearly, $(N_\alpha \mid \alpha < \kappa^+)$ is a κ^+-filtration of M. Since the set $C = \{\alpha < \kappa^+ \mid M_\alpha = N_\alpha\}$ is closed and cofinal in κ^+, there exists $\nu \in E \cap C$. Of course, $D = C \cap \{\alpha < \kappa^+ \mid \nu < \alpha\}$ is also closed and cofinal in κ^+, whence there is some $\mu \in E \cap D$. Then $X = N_\mu/N_\nu$ is a projective module. On the other hand, put $Y = \oplus \sum_{\nu<\alpha<\mu} R_\alpha$. Then $X = M_\mu/M_\nu = M_{\nu+1}/M_\nu + (Y + M_\nu)/M_\nu$. By 1.1, $(Y + M_\nu) \cap M_{\nu+1} \subseteq M_\nu$, whence $M_{\nu+1}/M_\nu \simeq R_\nu/S_\nu$ is a non-projective direct summand of X, a contradiction.

(iii) First, we prove by induction on $\nu < \kappa^+$ that for any $\emptyset \ne A \subseteq E$ such that $\sup\,(A) = \nu$ there is a sequence $(p_a \mid a \in A)$ such that $p_a < \aleph_0$ for all $a \in A$, and $\{\{n_a(i) \mid p_a < i < \aleph_0\} \mid a \in A\}$ is a set of disjoint subsets of ν

(cp. with [3, p. 15]). For $\nu = \min(E)$, put $p_a = 0$. If $\nu > \min(E)$, there is a closed and cofinal subset $C \subseteq \nu$ such that $C \cap E \cap \nu = \emptyset$ and $0 \in C$. Let f be a strictly increasing function $f : \mathrm{card}(\nu) \to C$. For each $\alpha < \mathrm{card}(\nu)$, put $B_\alpha = \{\beta \mid f(\alpha) < \beta < f(\alpha+1)\}$. If $A \cap B_\alpha \neq \emptyset$, then by induction there are $(q_a \mid a \in A \cap B_\alpha)$ such that $\{\{n_a(i) \mid q_a < i < \aleph_0\} \mid a \in A \cap B_\alpha\}$ is a set of disjoint subsets of $f(\alpha+1)$. For $a \in A \cap B_\alpha$, put $s_a = \min\{i < \aleph_0 \mid f(\alpha) < n_a(i)\}$. Since A is a disjoint union of the sets $A \cap B_\alpha, \alpha < \mathrm{card}(\nu)$, it suffices to put $p_a = \max(q_a, s_a)$, for all $a \in A \cap B_\alpha$ and $\alpha < \mathrm{card}(\nu)$. To complete the proof, we show that for all $\alpha < \kappa^+$, the module $M_\alpha = (\oplus \sum_{\beta<\alpha} R_\beta + Q)/Q$ is free, and for all $\alpha < \beta < \kappa^+$, the module $M_\beta/M_{\alpha+1}$ is free. Put $A = E \cap \alpha$. By 1.1 and the construction of $(p_a \mid a \in A)$, we see that $\{1_{a,i} + Q \mid a \in A \,\&\, p_a < i < \aleph_0\} \cup \{1_b + Q \mid b < \alpha \,\&\, b \notin A \,\&$ non-$(\exists\, a \in A \exists\, i < \aleph_0 : p_a < i \,\&\, b = n_a(i))\}$ is a free R-basis of the module M_α. Finally, put $A = E \cap \beta$. For each $a \in A$ such that $a > \alpha$, let $r_a < \aleph_0$ be such that $p_a \leq r_a$ and $\alpha < n_a(i)$ for all $r_a < i < \aleph_0$. Then by 1.1, $\{1_{a,i} + M_{\alpha+1} \mid a \in A \,\&\, a > \alpha \,\&\, r_a < i < \aleph_0\} \cup \{1_b + M_{\alpha+1} \mid \alpha < b < \beta \,\&\, b \notin A \,\&$ non-$(\exists\, a \in A \exists\, i < \aleph_0 : a > \alpha \,\&\, r_a < i \,\&\, b = n_a(i))\}$ is a free R-basis of the module $M_\beta/M_{\alpha+1}$. □

Lemma 1.3. *Let κ be a cardinal such that* $\mathrm{cf}(\kappa) = \aleph_0$. *Consider the following assertion*
UP_κ: *"there exist a non-reflecting stationary subset E of κ^+ satisfying $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \aleph_0\}$ and a ladder system $(n_\nu \mid \nu \in E)$ such that for each cardinal $\lambda < \kappa$ and each sequence $(h_\nu \mid \nu \in E)$ of mappings from $\aleph_0$ to λ there is a mapping $f : \kappa^+ \to \lambda$ such that $\forall\, \nu \in E\ \exists\, j < \aleph_0\ \forall\, j < i < \aleph_0 : f(n_\nu(i)) = h_\nu(i)$".*
Then the assertion "UP_κ holds for every uncountable cardinal κ such that $\mathrm{cf}(\kappa) = \aleph_0$*" is consistent with* ZFC + GCH.

PROOF: By [4, §2] or [3, §2]. □

Lemma 1.4. *Let κ be a cardinal such that* $\mathrm{cf}(\kappa) = \aleph_0$ *and* $\mathrm{card}(R) \leq \kappa$. *Assume* UP_κ *holds. Let $M = P/Q$ be the module corresponding to the E and $(n_\nu(i) \mid \nu \in E)$ from* UP_κ *by 1.1. Then* $\mathrm{Ext}_R(M, N) = 0$ *for all $N \in R$-*mod *such that* $\mathrm{card}(N) < \kappa$.

PROOF: Since P is a free module, we have $\mathrm{Ext}_R(M, N) = \mathrm{Hom}_R(Q, N)/\tau \circ \mathrm{Hom}_R(P, N)$, τ being the inclusion of Q into P. Hence, we are to prove that every $x \in \mathrm{Hom}_R(Q, N)$ is a restriction of some $y \in \mathrm{Hom}_R(P, N)$, i.e. $x = \tau y$. Take $x \in \mathrm{Hom}_R(Q, N)$. Let $b : N \to \lambda$ be a bijection of N onto $\lambda = \mathrm{card}(N)$. Using the notation of 1.1, for each $\nu \in E$, we define $h_\nu : \aleph_0 \to \lambda$ by $h_\nu(i) = b(g_{\nu i} x)$ for all $i < \aleph_0$. By UP_κ, there exists $f : \kappa^+ \to \lambda$ such that $\forall\, \nu \in E \exists\, j_\nu < \aleph_0\ \forall\, j_\nu < i < \aleph_0 : h_\nu(i) = f(n_\nu(i))$. Define $y \in \mathrm{Hom}_R(P, N)$ as follows: Take $\alpha < \kappa^+$.
(I) If $\alpha = n_\nu(i)$ for some $\nu \in E$ and $j_\nu < i < \aleph_0$, put $1_\alpha y = b^{-1} f(\alpha)$;
(II) If α does not satisfy (I) and $\alpha \notin E$, put $1_\alpha y = 0$;
(III) If $\alpha \in E$, put $1_{\alpha,i} y = 0$ provided $i > j_\alpha$. For $0 \leq i \leq j_\alpha$, define $1_{\alpha,i} y$ by induction on i (downwards): If there exist $\nu \in E$ and $k > j_\nu$ such that $n_\alpha(i) = n_\nu(k)$, put $1_{\alpha,i} y = b^{-1} f(n_\alpha(i)) - g_{\alpha i} x + a_i \cdot 1_{\alpha,i+1} y$. If there are no $\nu \in E$ and $k > j_\nu$ such that $n_\alpha(i) = n_\nu(k)$, put $1_{\alpha,i} y = -g_{\alpha i} x + a_i \cdot 1_{\alpha,i+1}$.

It remains to prove that $g_{\alpha i}x = g_{\alpha i}y$ for all $\alpha \in E$ and $i < \aleph_0$. Put $\beta = n_\alpha(i)$. Of course, $g_{\alpha i}y = 1_\beta y - 1_{\alpha,i}y + a_i \cdot 1_{\alpha,i+1}y$. We distinguish the following three cases:
(1) $i > j_\alpha$. Then $1_\beta y = b^{-1}f(\beta) = b^{-1}h_\alpha(i) = g_{\alpha i}x$ and $1_{\alpha,i}y = 1_{\alpha,i+1}y = 0$, whence $g_{\alpha i}y = g_{\alpha i}x$;
(2) $i \leq j_\alpha$, but there exist $\nu \in E$ and $k > j_\nu$ such that $\beta = n_\nu(k)$. Then $1_\beta y = b^{-1}f(\beta)$ and $1_{\alpha,i}y = b^{-1}f(\beta) - g_{\alpha i}x + a_i \cdot 1_{\alpha,i+1}y$, whence $g_{\alpha i}y = g_{\alpha i}x$;
(3) $i \leq j_\alpha$ and there are no $\nu \in E$ and $k > j_\nu$ such that $\beta = n_\nu(k)$. Then $1_\beta y = 0$ and $1_{\alpha,i}y = -g_{\alpha i}x + a_i \cdot 1_{\alpha,i+1}y$ whence $g_{\alpha i}y = g_{\alpha i}x$, q.e.d. □

Theorem 1.5. *Let κ be a cardinal such that* $\mathrm{cf}\,(\kappa) = \aleph_0$ *and* UP_κ *holds. Let R be a non-left perfect ring with* $\mathrm{card}\,(R) \leq \kappa$. *Then there is a non-projective strongly κ^+-free module M such that* $\mathrm{card}\,(M) = \kappa^+$ *and* $\mathrm{Ext}_R\,(M,N) = 0$ *for all* $N \in R\text{-}\mathrm{mod}$ *with* $\mathrm{card}\,(N) < \kappa$.

Proof: By 1.2 and 1.4. □

Corollary 1.6. *Consider the following assertion*
A^+: "for any non-left perfect ring R and any uncountable cardinal κ such that $\mathrm{cf}\,(\kappa) = \aleph_0$ *and* $\kappa \geq \mathrm{card}\,(R)$ *there is a non-projective strongly κ^+-free module M such that* $\mathrm{card}\,(M) = \kappa^+$ *and* $\mathrm{Ext}_R\,(M,N) = 0$ *for all* $N \in R\text{-}\mathrm{mod}$ *with* $\mathrm{card}\,(N) < \kappa$*".*
Then A^+ is consistent with ZFC + GCH.

Proof: By 1.3 and 1.5. □

The following proposition shows (in ZFC) that the extension properties of "small" non-projective modules may depend strongly on the particular structure of the non-left perfect ring R.

Proposition 1.7. (i) *Let $R = k[y,D]$ be the ring of all differential polynomials in one indeterminate y over a universal differential field k with the differentiation D. Then R is not left perfect, but* $\mathrm{Ext}_R\,(M,N) \neq 0$ *for all non-injective modules N and all finitely generated non-projective modules M.*
(ii) *Let R be a simple countable non-completely reducible von Neumann regular ring. Then R is not left perfect, but* $\mathrm{Ext}_R\,(M,N) \neq 0$ *for all non-projective modules M such that* $\mathrm{gen}\,(M) \leq \aleph_0$ *and all non-zero modules N such that* $\mathrm{gen}\,(N) \leq \aleph_0$. *However, there exist a simple non-projective module S and a non-injective module N such that* $\mathrm{Ext}_R\,(S,N) = 0$.
(iii) *Let R be a self-injective non-left perfect ring (e.g. let R be the maximal left quotient ring of a non-completely reducible von Neumann regular ring). Then there exists a non-projective module M such that* $\mathrm{gen}\,(M) = \aleph_0$ *and* $\mathrm{Ext}_R\,(M,N) = 0$ *for all finitely generated modules N.*

Proof: (i) By [6, Theorem 9.3].
(ii) By [6, Theorem 10.4].
(iii) Let $a_i, i < \aleph_0$ be as in 1.1. Let $1_i, i < \aleph_0$ be the canonical basis of $F = R^{(\aleph_0)}$ and let $G = \sum_{i<\aleph_0} R(1_i - a_i \cdot 1_{i+1}) \subseteq F$. Put $M = F/G$. By [1, Lemmas 28.1 and 28.2], F and G are free modules , M is not projective, and $\mathrm{gen}\,(M) = \aleph_0$. If $\mathrm{gen}\,(N) < \aleph_0$, we have $N \simeq R^{(n)}/X$ for some $n < \aleph_0$ and a submodule $X \subseteq R^{(n)}$.

As the sequence $0 \to G \to F \to M \to 0$ is exact, we get $0 = \mathrm{Ext}_R(G, X) \to \mathrm{Ext}^2_R(M, X) \to \mathrm{Ext}^2_R(F, X) = 0$, whence $\mathrm{Ext}^2_R(M, X) = 0$. Since the sequence $0 \to X \to R^{(n)} \to N \to 0$ is exact and R is left self-injective, we have $0 = \mathrm{Ext}_R(M, R^{(n)}) \to \mathrm{Ext}_R(M, N) \to \mathrm{Ext}^2_R(M, X) = 0$, whence $\mathrm{Ext}_R(M, N) = 0$.

□

Theorem 1.8. *Let R be a left perfect ring.*

(i) *For any non-projective module M there is a simple module S_M such that* $\mathrm{Ext}_R(M, S_M) \neq 0$.

(ii) *There exists a module N such that* $\mathrm{Ext}_R(M, N) \neq 0$ *for all non-projective modules M.*

Proof: (i) Since R is left perfect, there exists a projective cover of M, i.e. a projective module P and a non-zero superfluous submodule $K \subseteq P$ such that $M \simeq P/K$. By [1, Theorem 28.4], there exists a maximal submodule L of K. Put $S_M = K/L$. Let $x \in \mathrm{Hom}_R(K, S_M)$ be the projection of K onto K/L. Assume there exists $y \in \mathrm{Hom}_R(P, S_M)$ such that $\tau y = x$, τ being the inclusion of K into P. Then $\mathrm{Ker}(y)$ is a maximal submodule of P and by [1, Proposition 9.13], $K \subseteq \mathrm{Rad}(P) \subseteq \mathrm{Ker}(y) \subset P$. Thus $\tau y = 0$, a contradiction.

Hence $\mathrm{Hom}_R(K, S_M)/\tau \circ \mathrm{Hom}_R(P, S_M) = \mathrm{Ext}_R(M, S_M) \neq 0$.

(ii) Denote by V a representative set of the class of all simple modules. Put $N = \oplus \sum_{S \in V} S$. Then $\mathrm{Ext}_R(M, N) \simeq \mathrm{Ext}_R(M, S_M) \dotplus X$, for an abelian group X. Thus, by (i), $\mathrm{Ext}_R(M, N) \neq 0$. □

Definition 1.9. Let R be a ring. Define $W = \{N \in R\text{-}\mathrm{mod} \mid \mathrm{Ext}_R(M, N) \neq 0$ for all non-projective $M \in R\text{-}\mathrm{mod}\}$.

Theorem 1.10. *Let R be a ring. Consider the following assertions:*

(i) *R is left perfect;*

(ii) *$W \neq \emptyset$;*

(iii) *There exists a proper class C such that $C \subseteq W$ and no two distinct elements of C are isomorphic.*

Then (i) *implies* (ii), *and* (ii) *is equivalent to* (iii). *The implication* (iii) $\Rightarrow$ (i) *is independent of ZFC + GCH.*

Proof: (i) implies (ii) by 1.8 (ii). If $N \in W$, then also $\{N^{(\kappa)} \mid \kappa \geq \mathrm{card}(N)\} \subseteq W$ and $N^{(\kappa)} \not\simeq N^{(\lambda)}$ for all cardinals $\kappa \neq \lambda \geq \mathrm{card}(N)$. Hence (ii) is equivalent to (iii). By 1.6, the implication (iii) $\Rightarrow$ (i) is consistent with ZFC + GCH. On the other hand, by [6, Theorem 10.8 (ii)], (non-(i) & (ii)) is consistent with ZFC + GCH.

□

Remark 1.11. Let R be a left perfect ring. Denote by I the class of all injective modules. Clearly, always $W \subseteq R\text{-}\mathrm{mod} \setminus I$. Despite 1.10 (iii), almost never $W = R\text{-}\mathrm{mod} \setminus I$. Indeed, if R is left non-singular, then $W = R\text{-}\mathrm{mod} \setminus I$, if and only if either $R = S$ or $R = T$ or $R = S \boxplus T$, where S is a completely reducible ring and there exists a skew field K such that T is Morita equivalent to the upper triangular matrix ring of degree two over K (see [6, Theorems 3.4 and 8.1]).

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(*Received* November 5, 1990)