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Compactifications and uniformities on sigma frames

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Abstract. A bijective correspondence between strong inclusions and compactifications in the setting of σ-frames is presented. The category of uniform σ-frames is defined and a description of the Samuel compactification is given. It is shown that the Samuel compactification of a uniform frame is completely determined by the σ-frame consisting of its uniform cozero part, and consequently, any compactification of any frame is so determined.

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1. Background.

A *σ-frame* is a lattice L which has countable joins and satisfies the (countable) distribution law: $x \wedge \bigvee x_n = \bigvee x \wedge x_n$ ($n \in I$, countable) for $x, x_n \in L$. A *σ-frame morphism* $h : L \longrightarrow M$ is a lattice morphism preserving countable joins. The resulting category is denoted σ**Frm**. An element a of L is said to be *rather below* b, written $a \prec b$, if there exists $s \in L$, called the separating element, such that $a \wedge s = 0$ and $b \vee s = e$. A σ-frame L is *regular* if for each $a \in L$, there is a sequence (a_n) in L with $a_n \prec a$ and $a = \bigvee a_n$. L is *normal* if for each pair $a,\ b$ in L with $a \vee b = e$, there exists $u,\ v$ in L such that $a \vee u = e = b \vee v$ and $u \wedge v = 0$. Banaschewski [2] shows that every regular σ-frame is normal, and hence the rather below relation interpolates. The full subcategory of regular σ-frames is denoted **Reg**σ**Frm**, and is coreflective in σ**Frm**. A σ-frame morphism $h : L \longrightarrow M$ is *dense* if $h(x) = 0$ implies $x = 0$. In **Reg**σ**Frm** if $h : L \longrightarrow M$ is dense then h is monic. An element $c \in L$ is *compact* if for any countable $X \subseteq L$ with $c \leq \bigvee X$, there exists finite $E \subseteq X$ with $c \leq \bigvee E$. L is *compact* if e is compact (in other words every countable cover has a finite subcover, where a countable $X \subseteq L$ is a *cover* if $\bigvee X = e$). An ideal $J \subseteq L$ is *regular* if for each $x \in J$ there exists $y \in J$ with $x \prec y$, and J is said to be *countably generated* if there exists a sequence (x_n) in J such that for each $a \in J$, $a \leq x_n$ for some n. The full subcategory of compact regular σ-frames, denoted **KReg**σ**Frm**, is coreflective in **Reg**σ**Frm** with the coreflection functor given by $K_\sigma L$, the σ-frame consisting of all countably generated regular ideals, and the coreflection map given by join. This gives the Stone–Čech compactification of a σ- frame as shown by Banaschewski and Gilmour [2]. Extending the notions above to allow for arbitrary joins gives the definition of a frame, and the resulting

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category **Frm** [7]. An element a in a frame L is *completely below* b in L, written $a \prec\prec b$, if there is a family $\{x_i \mid i \in \mathbf{Q} \cap [0,1]\}$ of elements of L satisfying $x_0 = a$, $x_1 = b$ and $i \leq j$ implies $x_i \prec x_j$. L is *completely regular* if for each $a \in L$, $a = \bigvee\{x \mid x \prec\prec a\}$. An ideal $J \subseteq L$ is *completely regular* if for each $x \in J$ there exists $y \in J$ with $x \prec\prec y$. The full subcategory of compact regular frames, denoted **KRegFrm**, is coreflective in **Frm** with the coreflection functor given by KL, the frame consisting of all completely regular ideals, and the coreflection map given by join. This gives the Stone–Čech compactification of a frame [3]. Obviously every frame is a σ-frame, thus there is a forgetful functor $\mathcal{U} : \sigma\mathbf{Frm} \longrightarrow \mathbf{Frm}$. This has a left adjoint, the covariant functor $\mathcal{H}$ defined by letting $\mathcal{H}L$ be the frame of all σ-ideals of L where an ideal $J \subseteq L$ is a *σ- ideal* if it is closed under countable joins. Madden and Vermeer [8] show that this adjoint pair restricts to an equivalence of **RegσFrm** and the category of regular Lindelöf frames.

2. Strong inclusions and compactifications.

For any σ-frame L, a binary relation $\triangleleft$ on L is called a *strong inclusion* if the following five conditions are satisfied:
For any elements x, y, a and b in L

- (SI1) If $x \leq a \triangleleft b \leq y$ then $x \triangleleft y$.
- (SI2) $\triangleleft \subseteq L \times L$ is a sublattice. That is $0 \triangleleft 0$, $e \triangleleft e$, $x, y \triangleleft a, b$ implies $x \triangleleft a \wedge b$, $x \vee y \triangleleft a$.
- (SI3) If $x \triangleleft a$ then $x \prec a$.
- (SI4) If $x \triangleleft y$ then there exists z with $x \triangleleft z \triangleleft y$.
- (SI5) If $x \triangleleft y$ then there exists $a, b \in L$ with $b \triangleleft a$, $b \vee y = e$ and $a \wedge x = 0$.

The pair $(L, \triangleleft)$ is called a *proximal σ-frame* if $\triangleleft$ is a strong inclusion and moreover

- (SI6) each element of L is a countable join of elements strongly included (or strongly below) it. $\triangleleft$ is said to be *compatible* with L.

A σ-frame morphism $h : (L, \triangleleft) \longrightarrow (M, \triangleleft')$ is said to be *proximal* if h preserves strong inclusions. That is, $h \times h\, [\triangleleft] \subseteq \triangleleft'$. These form the objects and morphisms of a category denoted **ProxσFrm**. For a regular σ–frame L, the rather below relation $\prec$ is a strong inclusion and hence $(L, \prec)$ is a proximal σ-frame. The only conditions that need checking are (SI4) and (SI5) which follow from the result that regularity implies normality for σ-frames [2]. The condition given in (SI3) shows immediately that every proximal σ-frame is regular and, in fact, the rather below relation $\prec$ is the "finest" strong inclusion compatible with a σ–frame, in the sense that it contains any other such relation.

A σ-frame is said to be *compact* if any cover has a finite subcover. Any compact regular σ-frame has a unique compatible strong inclusion which is precisely the rather below relation: suppose $(L, \triangleleft)$ is a compact proximal σ-frame and take $a \prec b$. Using the separating element,the fact that b is a countable join of elements strongly below it and (SI5), gives that $a \triangleleft b$. A σ-frame morphism $h : M \longrightarrow L$ is called a *compactification* of L if M is compact regular and h is dense and surjective. For any given strong inclusion $\triangleleft$ on a σ-frame L , an ideal I of L is said to be *strongly regular* if for each element x in I, there exists an element y in I with $x \triangleleft y$. Now, consider $C_\sigma L$, the set of all countably generated strongly regular ideals. Using the properties of these ideals, it follows that $C_\sigma L$ is a σ-frame and, as every strongly regular ideal is necessarily regular, $C_\sigma L$ is a sub σ-frame of $K_\sigma L$, the compact regular σ-frame of all countably generated regular ideals, and so is also compact [2].

Proposition 2.1. *For a proximal σ–frame $(L, \triangleleft)$, the join map $\rho_L : C_\sigma L \longrightarrow L$ is a compactification.*

Proof: First it is necessary to show that $C_\sigma L$ is regular as a σ-frame: Take an ideal J in $C_\sigma L$, then J is generated by some sequence $a_1 \triangleleft a_2 \triangleleft a_3 \ \ldots$. For each n, let J_n be the ideal generated by the sequence $a_n = a_{n_0} \triangleleft a_{n_1} \triangleleft \ldots a_{n+1}$ obtained by repeated interpolation. Then $J = \bigvee J_n$. Moreover, for each n, applying (SI5) to $a_{n+1} \triangleleft a_{n+2}$ gives elements b_n, c_n in L with $c_n \triangleleft b_n$, $c_n \vee a_{n+2} = e$ and $b_n \wedge a_{n+1} = 0$. Let I_n be the ideal generated by the sequence $c_n = c_{n0} \triangleleft c_{n_1} \triangleleft \ldots b_n$ again obtained by repeated interpolation. Then I_n is an element of $C_\sigma L$, $I_n \cap J_n = 0$ and $I_n \vee J_{n+2} = L$. Hence $J_n \prec J_{n+2} \subseteq J$ and consequently J is a countable join of elements rather below it. Now each element of L is a countable join of a sequence of elements strongly below one another, that is, for a in L, $a = \bigvee a_n$ with $a_1 \triangleleft a_2 \triangleleft \ldots$. Let J be the ideal generated by this sequence, then $\rho_L\,(J) = \bigvee a_n = a$ and hence ρ_L is surjective. Since ρ_L is the restriction of $\kappa_L : K_\sigma L \longrightarrow L$ to the sub σ-frame of countably generated strongly regular ideals, it is a well defined dense σ-frame morphism. □

For any proximal map $h : (L, \triangleleft) \longrightarrow (M, \triangleleft')$, h preserves the strong inclusion and hence for each J in $C_\sigma L$, the ideal denoted by $[h(J)]$ and generated by $h(a_1) \triangleleft' h(a_2) \triangleleft' \ldots$ if $a_1 \triangleleft a_2 \triangleleft \ldots$ generates J, is an element of $C_\sigma M$. It is straightforward to see that $[h(J)]$ is not dependant on the choice of the generating sequence for J. This shows that the assignment $C_\sigma : \mathbf{Prox\sigma Frm} \longrightarrow \mathbf{KReg\sigma Frm}$ is functorial.

Proposition 2.2. $\rho_L : C_\sigma L \longrightarrow (L, \triangleleft)$ *is a proximal map.*

Proof: Since $C_\sigma L$ is compact, the unique strong inclusion is given by the rather below relation $\prec$. Take $I \prec J$ in $C_\sigma L$ with I and J generated by $a_1 \triangleleft a_2 \triangleleft \ldots$ and $b_1 \triangleleft b_2 \triangleleft \ldots$ respectively. Let $a = \bigvee a_n$ and $b = \bigvee b_n$. Now, there exists K in $C_\sigma L$ with $K \cap I = \{ 0 \}$ and $K \cup J = L$. Hence there exists $c \in K$, $d \in J$ such that $c \vee d = e$, and moreover, $c \wedge a_n = 0$ for each n, since $a_n \in I$. Thus $a_n \prec d$ for each n, and hence $a = \bigvee a_n \leq d$. But $d \triangleleft b$ since J is strongly regular, consequently $a \triangleleft b$. □

Since dense implies monic in $\mathbf{Reg\sigma Frm}$, ρ_L is a surjective monomorphism. If L is compact, both L and $C_\sigma L$ have unique strong inclusions given by the rather below relation and so ρ_L will be an isomorphism in $\mathbf{Prox\sigma Frm}$.

Proposition 2.3. *The full subcategory* $\mathbf{KReg\sigma Frm}$ *is coreflective in* $\mathbf{Prox\sigma Frm}$.

Proof: Since σ-frame morphisms preserve the rather below relation, $\mathbf{KReg\sigma Frm}$ is a full subcategory of $\mathbf{Prox\sigma Frm}$. Consider a proximal morphism $h : (M, \prec) \longrightarrow (L, \triangleleft)$ with $M \in \mathbf{KReg\sigma Frm}$. Since h preserves countable joins, $\rho_L \circ C_\sigma\, h =$

$h \circ \rho_M$. But M is compact, so ρ_M is an isomorphism, thus $\rho_L \circ C_\sigma h \circ (\rho_M)^{-1} = h$. That is, h factors via ρ_L. This factorisation is unique since ρ_L is dense and hence monic. □

Proposition 2.4. *Given any compactification $h : M \longrightarrow L$ in* **RegσFrm**, *then the binary relation derived from h, defined by $x \lhd y$ if and only if there exists $a, b \in M$ with $a \prec b$ and $h(a) = x$, $h(b) = y$, is a strong inclusion compatible with L.*

Proof: (SI1) Take $x' \le x \lhd y \le y'$ in L, then since h is surjective and by the definition of $\lhd$, there exists $a, a', b, b' \in M$ with $a \prec b$ and $h(a) = x$, $h(a') = x'$, $h(b) = y$ and $h(b') = y'$. Now $x' = x' \wedge x = h(a') \wedge h(a) = h(a' \wedge a)$ and similarly $y' = h(b' \vee b)$. Since $a' \wedge a \le a \prec b \le b \vee b'$, it follows that $x' \lhd y'$.

(SI2) $0 \lhd 0$ and $e \lhd e$ since $0 \prec 0$, $e \prec e$ and h preserves the top and the bottom. Take $x, x' \lhd y$, then there exists $a, a', b \in M$ with $a, a' \prec b$ and $h(a) = x$, $h(a') = x'$ and $h(b) = y$. Now $x \vee x' = h(a \vee a')$ and $a \vee a' \prec b$, hence $x \vee x' \lhd y$. A similar argument shows that if $x \lhd y, y'$ then $x \lhd y \wedge y'$.

(SI3) If $x \lhd y$ with $x = h(a)$, $y = h(b)$ and $a \prec b$ in M, then since h preserves the rather below relation, $x \prec y$.

(SI4) Take $x \lhd y$ as above. Since the rather below relation interpolates in **RegσFrm** there exists c with $a \prec c \prec b$ and hence $x \lhd h(c) \lhd y$.

(SI5) Take $x \lhd y$ as above. Since $a \prec b$ in M, there exists $c \prec d$ with $d \wedge a = 0$ and $c \vee b = e$. Now $h(c) \lhd h(d)$ and $h(d) \wedge x = h(d) \wedge h(a) = 0$. Also $h(c) \vee y = h(c) \vee h(b) = e$.

(SI6) Since h is surjective, for each $a \in L$, $a = h(x)$ for some $x \in M$. By the regularity of M, $x = \bigvee x_n$ with $x_n \prec x$. Hence $h(x_n) \lhd h(x)$ and $a = \bigvee h(x_n)$. □

Proposition 2.5. *For any proximal σ-frame $(L, \lhd)$, the strong inclusion derived from the compactification $\rho_L : C_\sigma L \longrightarrow L$ is precisely $\lhd$.*

Proof: Let $\lhd'$ represent the strong inclusion derived from the compactification ρ_L . That is, $x \lhd' y$ if and only if there exists $I \prec J$ in $C_\sigma L$ with $\rho_L(I) = x$ and $\rho_L(J) = y$. Take $x \lhd y$ in L and interpolate to get $x \lhd z \lhd y$. Suppose that $x = \bigvee x_n$, $z = \bigvee z_n$ and $y = \bigvee y_n$ with $x_1 \lhd x_2 \ldots$ etc.. Assume, without loss of generality that $z \le y_n$ for each n. Let I, J and K be the ideals generated by $(x_n), (y_n)$ and (z_n). By (SI5), there exists $b \lhd a$ in L with $b \vee z = e$ and $a \wedge x = 0$. Interpolate repeatedly between $b \lhd a$ and generate an ideal $P \in C_\sigma L$. Then $P \cap I = \{ 0 \}$ and $P \bigvee J = L$, hence $\prec J$. Since $\rho_L(I) = x$ and $\rho_L(J) = y$, $x \lhd' y$. Take $x \lhd' y$ with $I \prec J$ in $C_\sigma L$ with $\rho_L(I) = x$ and $\rho_L(J) = y$. Suppose that I is generated by (x_n). There exists an ideal K such that $K \cap I = \{ 0 \}$ and $K \bigvee J = L$ and hence there exist elements c in K and d in J with $c \vee d = e$. Moreover, $c \wedge x_n = 0$ for each n, so $x_n \prec d$ for all n. Thus $x = \bigvee x_n \le d$ and since $d \lhd y$, $x \lhd y$. □

Proposition 2.6. *Given any compactification $h : M \longrightarrow L$, the strong inclusion $\lhd$ derived from h gives rise to a compactification $\rho_L : C_\sigma L \longrightarrow L$ which is isomorphic to h.*

Proof: Take any compactification $h : M \longrightarrow L$. Then h is dense and surjective, and M is compact. Let $\lhd$ be defined as above and consider the associated compactification $\rho_L : C_\sigma L \longrightarrow L$. Since $\rho_M : C_\sigma M \longrightarrow M$ is an isomorphism, each element a of M is the image of a unique element of $C_\sigma M$, denoted by J_a. The map $g : M \longrightarrow C_\sigma L$ given by $g(a) = [h(J_a)]$ is clearly a well-defined σ-frame morphism. Take K in $C_\sigma L$, then K is generated by some sequence, say $b_1 \lhd b_2 \lhd \dots$ in L. By the definition of $\lhd$ and the denseness of h, there exists a sequence $x_1 \prec x_2 \prec \dots$ with $b_i = h(x_i)$ for each i. Let $x = \bigvee x_n$, then $h(x) = \bigvee h(x_n) = \bigvee b_n = \bigvee K$. In fact, K is generated by $h(x_1) \lhd h(x_2) \lhd \dots$ and hence $K = [h(J_x)]$ where J_x is the unique ideal associated with x and generated by (x_n). Thus $K = g(x)$ which shows that g is surjective. Suppose $g(a) = \{ 0 \}$, that is, $[h(J_a)] = \{ 0 \}$, then $h(x) = 0$ for all $x \in J_a$. But h is dense, thus $x = 0$ for all $x \in J_a$, hence $J_a = \{ 0 \}$. Now $a = \bigvee J_a$, so $a = 0$ and therefore g is dense and hence monic. Since g is a σ-frame morphism, g preserves the rather below relation which gives the unique strong inclusions on the compact σ-frames $C_\sigma L$ and M. Thus g is an isomorphism. □

3. Uniform sigma frames.

For any σ-frame L, a *cover* of L is a sequence (a_n) in L with $\bigvee a_n = e$. One cover $A = (a_n)$ is said to *refine* another cover $B = (b_n)$, written $A \leq B$, if for each n there exists m with $a_n \leq b_m$. Two covers A and B are *equivalent* if $A \leq B$ and $B \leq A$. For an element x of L, and a cover A of L, the *star of* x w.r.t. A is denoted by Ax and is given by $\bigvee\{a \in A \mid a \wedge x \neq 0\}$. The *star of* A, denoted A^* is the cover given by $\{Aa \mid a \in A\}$. The *meet* of two covers A and B, written $A \wedge B$, is a cover defined pairwise as $\{a \wedge b \mid a \in A, b \in B\}$.$A$ is said to *star refine* B, written $A \leq^* B$, if $A^* \leq B$.

It is easy to see that for any element x in L and covers A, B, C and D of L:

(1) $x \prec Ax$.
(2) If $A \leq^* B$ then $A(Ax) \leq Bx$.
(3) If $A \leq^* B$ and $C \leq^* D$ then $A \wedge C \leq^* B \wedge D$.

A *uniformity* is a collection μ of covers of L such that the following two conditions are satisfied:

(U1) μ is a filter with respect to $\leq$.
(U2) For each cover A in μ, there exists a cover B in μ with $B \leq^* A$.

For elements a and b of L, a is said to be *uniformly below* b (relative to μ), written $a \lhd_\mu b$, or $a \lhd b$ when the context is clear, if there exists a cover A in μ such that $Aa \leq b$. A uniformity μ is said to be *compatible* with L if

(U3) each element a of L is a countable join of elements uniformly below it.

In such a case, (L, μ) is called a *uniform σ-frame.* A σ–frame morphism $h : (L, \mu) \longrightarrow (M, \nu)$ is *uniform* if $h[A] \in \nu$ whenever $A \in \mu$. For such an h, $h(a) \lhd_\nu h(b)$ whenever $a \lhd_\mu b$. These are the objects and morphisms of a category **UniσFrm** .

By the altering of axiom (U3) and allowing for covers and joins to be indexed over any set, the definition of the category **UniFrm** of uniform frames and uniform frame morphisms as defined by Frith is obtained [4]. It is not difficult to show that this description is equivalent to that formulated by Pultr [9].

Proposition 3.1. *For any uniformity μ on a σ-frame L, $\lhd_\mu$ is a compatible strong inclusion.*

PROOF: (SI1), (SI2) and (SI3) follow as simple consequences of the properties of covers. That the uniformly below relation interpolates (SI4) follows by using the star-refinement property (U2) of a uniformity together with (2) above. Property (SI6) follows immediately from the compatibility of μ with L. Thus it only remains to verify that $\lhd_\mu$ satisfies (SI5): suppose $a \lhd b \lhd c$ with respect to μ, and $A \in \mu$ is such that $Aa \leq b$. By (U2), there exists B in μ with $B \leq^* A$ and $a \lhd Ba \lhd Aa \leq b \lhd c$. Now let $x = \bigvee \{ z \in B \mid z \wedge a = 0 \}$ and $y = \bigvee \{ z \in B \mid z \wedge Ba = 0 \}$, then x separates $a \prec b$, $y \vee c = e$ and $y \lhd x$. $\square$

Since a uniform σ-frame morphism preserves the uniformly below relation, the assignment of $\lhd_\mu$ to μ defines a functor from **UniσFrm** to **ProxσFrm**. Conversely, each proximal σ-frame has the natural compatible uniformity generated by the collection β of all those finite covers A of L for which there is a finite cover B such that if $b \in B$ then there exists $a \in A$ with $b \lhd a$ (written $B \lhd A$). This assignment too, is functorial since a proximal σ–frame morphism preserves strong inclusions:

Proposition 3.2. *For any proximal σ-frame $(L, \lhd)$ there exists a uniformity $\mu_\lhd$ compatible with L such that the associated uniformly below relation is precisely $\lhd$.*

PROOF: To show this collection β of covers generates a uniformity, it suffices to check that any such cover consisting of two elements is star- refined by another such cover: suppose $\{a, b\}$ is such a cover, say refined by $\{c, d\}$. That is, $c \lhd a$ and $d \lhd b$. Interpolate to get $c \lhd c_1 \lhd c_2 \lhd a$ and $d \lhd d_1 \lhd d_2 \lhd b$ and then apply (SI5) to $c_1 \lhd c_2$ and $d_1 \lhd d_2$. This gives elements $y \lhd x$ with $y \vee c_2 = e$, $x \wedge c_1 = 0$ and $t \lhd s$ with $t \vee d_2 = e$ and $s \wedge d_1 = 0$. Hence $\{y, c_2\} \lhd \{x, a\}$,$\{t, d_2\} \lhd \{s, b\}$ and $\{c, d\} \lhd \{c_1, d_1\}$ and then the cover $C = \{x, a\} \wedge \{s, b\} \wedge \{c_1, d_1\}$ has the required property and star-refines $\{a, b\}$. Suppose $a \lhd b$, then for $a \lhd c \lhd b$ there exists $y \lhd x$ with $y \vee c = e$ and $x \wedge c = 0$, hence $\{x, b\} \in \beta$ and $\{x, b\}a = b$, thus a is uniformly below b and consequently $\mu_\lhd$ is compatible with L. Conversely, if a is uniformly below b then there exists A in β with $Aa \leq b$. Take a finite cover $B \lhd A$, then since B is a cover and by (SI2), $a \leq Ba \lhd Aa \leq b$, and hence by (SI1), $a \lhd b$. $\square$

It was previously noted that the rather below relation is a strong inclusion, hence every regular σ-frame has a compatible uniformity. Moreover, it follows from 3.1 that a uniform σ-frame is regular, thus a σ-frame is regular if and only if it is uniformizable.

A uniform σ-frame (L, μ) is said to be *precompact* if each uniform cover has a finite uniform refinement. Thus the uniformity $\mu_\lhd$ of a proximal σ-frame is precompact and contains any other compatible precompact uniformity which induces the same strong inclusion $\lhd$. In fact, it is straightforward to show that $\mu_\lhd$ is the unique compatible precompact uniformity inducing $\lhd$. This shows that the category of proximal σ-frames is isomorphic to the full subcategory of **UniσFrm** consisting of precompact uniform σ-frames. This is the analogue of the result obtained by

Frith [4] for proximal frames and precompact uniform frames which in turn corresponds to this result of Smirnov for spaces. Using proofs analogous to those used in the fundamental results of Ginsburg and Isbell [5] for uniform spaces, and by considering the uniformity generated by all finite uniform covers, it can be shown that precompact uniform σ-frames are coreflective in **UniσFrm** with the coreflection map given by the identity. Similarly in the frame setting, by considering the uniformity generated by the collection of all countable uniform covers, the full subcategory of separable uniform frames can be shown to be coreflective in **UniFrm**, where a frame is defined to be *separable* if every uniform cover has a countable uniform refinement. It also follows that, as in the case of spaces, every Lindelöf uniform frame is separable.

The uniformly below relation $\triangleleft_\mu$ on a uniform σ-frame(L, μ) as a strong inclusion in the setting of **UniσFrm** gives rise to the compactification $\rho_L : R_\sigma L \longrightarrow L$ where $R_\sigma L$ is the collection of all countably generated uniformly regular ideals of L. From 2.3 above, it follows that the full subcategory **KRegσFrm** is coreflective in **UniσFrm**. Banaschewski obtains the analogous result for frames by considering RL, the collection of all the uniformly regular ideals. For spatial uniform frames, that is, uniform spaces, this gives rise to the Samuel compactification. Thus the above coreflections are called the Samuel compactification of the uniform σ-frame, and uniform frames repectively. It should be noted that any compactification arising from a compatible strong inclusion may be viewed as the Samuel compactification of the uniform σ-frame $(L, \mu_\triangleleft)$ associated with that strong inclusion. In this way, the Stone–Čech compactification is associated with the rather below relation. It is, by (SI3), the largest of the Samuel compactifications of a given σ-frame in the sense that it contains all other compactifications as sub σ-frames.

4. The cozero part of a uniform frame.

An ideal is called a *σ–ideal* if it is closed under countable joins. For any uniform σ-frame (L, μ), consider $\mathcal{H}L$ the Lindelöf frame of all σ-ideals of L [7] and let $\mathcal{H}\mu$ be generated by $\{\downarrow A \mid A \in \mu\}$ where $\downarrow A = \{\downarrow a \mid a \in A\}$. It is straight forward to show that $\mathcal{H}\mu$ is a separable uniformity compatible with $\mathcal{H}L$ and, moreover this assignment of a uniform frame to a uniform σ-frame is functorial. The underlying σ-frame of a completely regular frame L need not be regular (for example a space which is completely regular but not perfectly normal) and is in fact, only regular if it equals the cozero part of L, the largest regular sub σ-frame of L. Thus a structured version of the functor Coz, rather than the forgetful functor is considered: An element a of a uniform frame (L, μ) is *uniformly cozero* if $a = h\ ((0, 1])$ for some uniform frame morphism $h : \Omega[0, 1] \longrightarrow (L, \mu)$where $\Omega[0, 1]$ is the frame of all the open sets of the unit interval. Let $Coz_u L$ denote the set of all the uniformly cozero elements of L. Obviously $Coz_u\ L$ is a subset of $Coz\ L$, the σ-frame of all elements which are cozero elements for some frame morphism of L. The following series of results using the Samuel compactification RL of the uniform frame (L, μ), show that $Coz_u\ L$ is a regular σ-frame which generates L as a frame:

Lemma 4.1. *For a uniform frame (L, μ), $Coz_u\ L = \rho_L\ (Coz\ RL\ )$.*

Proof: For a compact regular frame, every cozero element is uniformly cozero, and thus since uniform frame maps preserve uniform cozero elements, $\rho_L\ (Coz\ RL\ )$

$\subseteq Coz_u\ L$. Take any $a \in Coz_u\ L$, say $a = h\ ((0,1])$, then h factors via RL since $\Omega\ [0,1]$ is compact and hence $Coz_u\ L \subseteq \rho_L\ (Coz\ RL\ )$. □

Since RL is compact and regular, and hence spatial, $CozRL$ is a regular σ-frame. The image of a regular σ-frame is regular, hence $\rho_L\ (Coz\ RL\ )$ is a regular σ-frame and thus so is $Coz_u\ L$. Since RL is spatial, it can be shown using properties of cozero sets in Lindelöf spaces, that $Coz\ RL$ consists of precisely the countably generated uniformly regular ideals.

Lemma 4.2. *The uniform cozero elements of a uniform frame (L, μ) are precisely those elements which are the join of a sequence of elements uniformly below each other.*

PROOF: Let $a \in Coz_u\ L$, then $a = \rho_L\ (J)$ for some $J \in Coz\ RL$. Now J is countably generated, say by $a_1 \triangleleft a_2 \triangleleft \dots$, thus $a = \bigvee J = \bigvee a_n$. Conversely, take $a = \bigvee a_n$ with $a_n \triangleleft a_{n+1}$. Let $J = \{ x \in L \mid x \leq a_n, \text{ for some } n \}$. Then J is a countably generated uniformly regular ideal, hence in $Coz\ RL$. Since $a = \bigvee J$, $a \in Coz_u\ L$. □

Lemma 4.3. *$Coz_u L$ generates L as a frame.*

PROOF: Let $J \in RL$. For each $b \in J$ choose a sequence $b \triangleleft b_1 \triangleleft b_2 \triangleleft \dots$ in J. Let J_b be the uniformly regular ideal generated by this sequence, then each $J_b \in Coz\ RL$ and J is the join of all the J_b's , thus $Coz\ RL$ generates RL as a frame. For each $a \in L$, since ρ_L is surjective, $a = \rho_L\ (J)$ for some J in RL. Thus $a = \rho_L\ (\bigvee J_b) = \bigvee \rho_L\ (J_b)$ and $\rho_L\ (J_b) \in Coz_u\ L$ for each J_b. Hence each element of L is the join of uniform cozero elements. □

Let $Coz_u\ \mu$ be the collection of all countable uniform covers consisting of uniformly cozero elements. That any uniformity has a basis of uniformly cozero covers follows immediately from the following:

Lemma 4.4. *If $a \triangleleft b$ in (L, μ) then there exists $c \in Coz_u\ L$ such that $a \leq c \leq b$.*

PROOF: Let $a \triangleleft b$ in (L, μ), then by repeated interpolation, there exists a sequence $a \triangleleft x_1 \triangleleft x_2 \triangleleft \dots \triangleleft b$. Put $c = \bigvee x_n$, then by the lemma above $c \in Coz_u\ L$. □

Proposition 4.5. *$Coz_u\ \mu$ is a uniformity compatible with $Coz_u\ L$. Moreover, $(Coz_u\ L, Coz_u\ \mu)$ generates the separable coreflection of (L, μ).*

PROOF: For any uniformity μ, the collection of countable uniform covers is a uniform basis, and any uniformity has a basis of uniform covers consisting of uniformly cozero elemets. Thus $Coz_u\ \mu$ contains a countable cozero basis and hence is a uniformity on a σ-frame. Compatibility follows since each element of $Coz_u\ L$ is a countable join of uniform cozero elements relative to μ and hence relative to $Coz_u\ \mu$. The frame uniformity generated by $Coz_u\ \mu$ is obviously separable, and by the very definition of $Coz_u\ \mu$, it is precisely the uniformity obtained by the separable coreflection.

□

Since uniform frame morphisms preserve uniformly cozero elements, a frame morphism restricts to a σ-frame morphism between the corresponding uniformly cozero parts. Thus Coz_u : **UniFrm** $\longrightarrow$ **UniσFrm** is functorial. For regular Lindelöf frames, the countable elements (where an element a is countable if whenever $a \leq \bigvee X$ for some set X, there exists a countable subset $Y \subseteq X$ with $a \leq \bigvee Y$) are precisely the cozero elements [7]. It follows from the results above that for a Lindelöf uniform frame, the countable elements are precisely the uniformly cozero elements. By considering the unit $\eta_L : (L, \mu) \longrightarrow Coz_u(\mathcal{H}L, \mathcal{H}\mu)$ defined by $\eta_L(x) = \downarrow x$ and the counit $\varepsilon_L : \mathcal{H}Coz_u\ (L, \mu) \longrightarrow (L, \mu)$ given by the join map, the following proposition can easily be checked:

Proposition 4.6. *$\mathcal{H}$ is left adjoint to Coz_u.*

For a Lindelöf uniform frame, as in the case of frames, $\mathcal{H}Coz_u\ (L, \mu)$ is isomorphic to (L, μ) and any uniform σ-frame (L, μ) is isomorphic to $Coz_u\ \mathcal{H}\ (L, \mu)$ (cf. [8] for frames). Hence these are the fixed objects of the adjunction. It is now possible to obtain an alternative description of the Samuel compactification RL of a uniform frame (L, μ) via the corresponding compactification in **UniσFrm** of its cozero part.

Proposition 4.7. *For any uniform frame (L, μ),*

$$CozRL \cong R_\sigma Coz_u L.$$

Proof: Since RL is compact, $Coz\ RL$ is a compact regular σ-frame and hence has a unique uniformity. The uniform frame morphism $\rho_L : RL \longrightarrow (L, \mu)$ restricts to a uniform σ-frame morphism $\rho_L : CozRL \longrightarrow (Coz_u L, Coz_u \mu)$. By the coreflection property this factors via $R_\sigma\ Coz_u\ L$, say by h. That is, $\rho_L = \rho_{Coz_u\ L} \circ h$. Now define $f : Coz_u\ L \longrightarrow Coz\ RL$ by $f(a) = \{x \in Coz_u\ L \mid x \lhd a\}$. Since each element of $Coz_u\ L$ is a countable join of elements of $Coz_u\ L$ uniformly below it, this is a countably generated uniformly regular ideal, hence f is a well-defined σ- frame morphism and moreover since $Coz\ RL$ is compact, $f : (Coz_u\ L, Coz_u\ \mu) \longrightarrow Coz\ RL$ is uniform. Also $f \circ \rho_L = id_{Coz_u\ L}$ and it follows that $f \circ \rho_{Coz_u\ L}$ is inverse to h. Hence $Coz\ RL \cong R_\sigma\ Coz_u\ L$. □

Applying the functor $\mathcal{H}$ to the isomorphism described above and observing that RL is Lindelöf and hence $RL \cong \mathcal{H}\ Coz\ RL$, one gets the following proposition which shows that the Samuel compactification of a uniform frame is completely determined by its uniformly cozero part:

Proposition 4.8. *For any uniform frame (L, μ),*

$$R(L, \mu) \cong \mathcal{H} R_\sigma Coz_u L.$$

As mentioned earlier, any compactification of a regular σ-frame may be viewed as the Samuel compactifition of the associated proximal, and hence precompact uniform σ-frame. The same observation can be made for frames, and thus the proposition above shows that any compactification of a completely regular frame is determined by a regular σ-frame, namely the one consisting of all the "uniformly" cozero elements.

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