A generalization of boundedly compact metric spaces

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Abstract. A metric space $\langle X, d\rangle$ is called a UC space provided each continuous function on X into a metric target space is uniformly continuous. We introduce a class of metric spaces that play, relative to the boundedly compact metric spaces, the same role that UC spaces play relative to the compact metric spaces.

Keywords: UC space, boundedly UC space, boundedly compact space, Atsuji space, uniform continuity on bounded setbounded sets, topology of uniform convergence on bounded setbounded sets, Attouch–Wets topology

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1. Introduction.

Many of the basic properties of compact metric spaces actually characterize a larger class of metric spaces, usually called UC *spaces* or *Atsuji spaces*. For example, a metric space X is a UC space provided any of the following equivalent conditions hold: (1) for each metric space Y, each continuous function from X to Y is uniformly continuous; (2) each open cover of X has a Lebesgue number; (3) each pair of disjoint closed subsets of X lie a positive distance apart. But there are many other intriguing characterizations, and these spaces have received a great deal of attention over the past forty years [MP], [At1], [Na], [Le], [Wa], [To], [Ra], [Hu], [Be1–3], [BHPV]. Recently, UC spaces have been studied in the context of stability of optimization problems by Revalski and Zhivkov [RZ]. Most of the characteristic properties of UC spaces are uniform/proximity properties, and it is appropriate to study them in this more general framework (see, eg., [At2], [DCN]). We will not work at this level of generality here.

It is natural to ask if there is a class of metric spaces that plays, relative to boundedly compact metric spaces, a role parallel to that played by the UC spaces for compact metric spaces. It is the aim of this note to display this class, which we call the *boundedly* UC *spaces*, in various ways.

2. Preliminaries.

Let $\langle X, d\rangle$ be a metric space. We denote the nonempty closed subsets of X by $\mathrm{CL}(X)$, and the set of its accumulation points by X'. Following [Be1], we call a sequence $\langle x_n\rangle$ of isolated points in X with distinct terms a *paired sequence of isolated points* provided $\lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = 0$.

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If A is a nonempty subset of X and $x \in X$, we shall write $d(x, A)$ for $\inf\{d(x,a) : a \in A\}$. The *ε-parallel body* of A is defined by the formula $S_\varepsilon[A] = \{x \in X : d(x, A) < \varepsilon\}$. We say that nonempty subsets A_1 and A_2 of X are *uniformly separated* provided for some $\varepsilon > 0$, we have $S_\varepsilon[A_1] \cap S_\varepsilon[A_2] = \emptyset$; put differently, the sets A_1 and A_2 are a positive distance apart. If A_1 and A_2 are nonempty subsets of X, we define the *excess* [CV] of A_1 over A_2 by the formula $e(A_1, A_2) = \sup\{d(x, A_2) : x \in A_1\}$. Hausdorff distance can then be defined on $\mathrm{CL}(X)$ by the formula $H_d(A_1, A_2) = \max\{e(A_1, A_2), e(A_2, A_1)\}$.

A metric space $\langle X, d\rangle$ is called *boundedly compact* provided each closed and bounded subset of X is compact. The following terminology is thus appropriate:

Definition. A metric space $\langle X, d\rangle$ is called *boundedly* UC (or *boundedly Atsuji*) provided each closed and bounded subset of X is UC.

3. Characterizations of boundedly UC spaces.

Theorem 3.1. *Let $\langle X, d\rangle$ be a metric space. The following are equivalent:*

(1) *X is boundedly* UC*;*
(2) *For each metric space Y and for each continuous function f from X to Y, f is uniformly continuous on bounded subsets of X;*
(3) *Each continuous real valued function on X is uniformly continuous on bounded subsets of X;*
(4) *Each bounded continuous real valued function on X is uniformly continuous on bounded subsets of X;*
(5) *Whenever A and B are disjoint nonempty closed subsets of X with B bounded, then A and B can be uniformly separated;*
(6) *Whenever A and B are disjoint nonempty closed and bounded subsets of X, then A and B can be uniformly separated;*
(7) *X' is boundedly compact, and each bounded paired sequence of isolated points has a cluster point;*
(8) *Whenever $\langle x_n\rangle$ is a bounded sequence in X with $\lim_{n\to\infty} d(x_n, \{x_n\}^c) = 0$, then $\langle x_n\rangle$ has a cluster point;*
(9) *Whenever B is a closed and bounded subset of X and $\{V_i : i \in I\}$ is a collection of open subsets of X with $B \subset \cup V_i$, then there exists $\delta > 0$ such that each subset of X of diameter less than δ which meets B lies entirely within some V_i.*

Proof: (1) $\Rightarrow$ (2). Let $f : X \to Y$ be continuous, and let B be a nonempty bounded subset of X. Since $\mathrm{cl}\, B$ is UC, $f \mid \mathrm{cl}\, B$ is uniformly continuous so that $f \mid B$ is uniformly continuous.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (4) This is trivial.

(4) $\Rightarrow$ (5). Let f be a Urysohn function separating A and B, say $f(A) = 1$ and $f(B) = 0$. The restriction of f to $S_1[B]$ is uniformly continuous; so, there exists $\delta < 1$ such that if $d(x_1, x_2) < \delta$ and $\{x_1, x_2\} \subset S_1[B]$, then $|f(x_1) - f(x_2)| < 1$. This implies that $S_{\delta/2}[B] \cap S_{\delta/2}[A] = \emptyset$. Else, if x were in the intersection, we could

choose $b \in B$ and $a \in A$ with $d(a,b) < \delta$. Since also $a \in S_1[B]$, we would have $|f(b) - f(a)| < 1$, a contradiction.

(5) $\Rightarrow$ (6). This is trivial.

(6) $\Rightarrow$ (7). Let B be a closed and bounded subset of X'. Suppose B is noncompact. Then there exists a sequence $\langle b_n \rangle$ of distinct terms in B with no cluster point. We can find a positive sequence $\langle \varepsilon_n \rangle$ such that for each $n, \varepsilon_n < 1/n$ and $\{S_{\varepsilon_n}[b_n] : n \in Z^+\}$ is a disjoint family. For each n choose x_n in $S_{\varepsilon_n}[b_n] - \{b_n\}$. Then $\{x_n : n \in Z^+\}$ and $\{b_n : n \in Z^+\}$ are disjoint closed bounded sets that cannot be uniformly separated, violating (6). Thus B must be compact. Each bounded sequence $\langle x_n \rangle$ of paired isolated points must have a cluster point, else $\{x_{2n} : n \in Z^+\}$ and $\{x_{2n-1} : n \in Z^+\}$ cannot be uniformly separated.

(7) $\Rightarrow$ (8). Let $\langle x_n \rangle$ be a bounded sequence as specified in (8). Choose a sequence $\langle y_n \rangle$ such that for each $n, x_n \neq y_n$ and $\lim d(x_n, y_n) = 0$. Let $\langle z_n \rangle$ be the sequence $x_1, y_1, x_2, y_2, \ldots$. If $\langle z_n \rangle$ has a constant subsequence, we are done, so we may pass to the case that all terms are distinct. Now $\langle z_n \rangle$ will either have a subsequence in X' or a tail that is a sequence of paired isolated points. Thus, (8) follows from (7).

(8) $\Rightarrow$ (9). Suppose no such number δ exists. For each $n \in Z^+$ choose $b_n \in B$ and a subset E_n of X of diameter less than $1/n$ containing b_n such that E_n sits in no single member of $\{V_i : i \in I\}$. Since no E_n can be a singleton, we see that $\lim_{n\to\infty} d(b_n, \{b_n\}^c) = 0$. By (8), this means that $\langle b_n \rangle$ has a cluster point b which lies in B, for B is closed. But for some $i \in I$, we have $b \in V_i$, and since V_i is open, we conclude that V_i contains E_n for a sufficiently large integer n, a contradiction.

(9) $\Rightarrow$ (1). Condition (9) guarantees, in particular, that each open cover (in the relative topology) of a closed and bounded subset B of X has a Lebesgue number. This implies that B, in its relative topology, is a UC space. $\square$

Evidently, the metric for a boundedly UC space $\langle X, d \rangle$ is a complete metric, for if $\langle x_n \rangle$ were a Cauchy sequence with distinct terms in X lacking a cluster point, then the condition (8) in Theorem 3.1 would fail for this sequence. It is interesting to note that the complete metric spaces themselves are just the *totally boundedly* UC spaces, i.e., those spaces for which each closed and totally bounded subset is UC. To see this, first suppose $\langle X, d \rangle$ is complete, and let B be a closed and totally bounded subset of X. If A_1 and A_2 are closed subsets of B that cannot be uniformly separated, then for each $n \in Z^+$, we can find $x_n \in A_1$ with $d(x_n, A_2) < 1/n$. But by total boundedness of A_1, $\langle x_n \rangle$ has a Cauchy subsequence which by completeness converges to some point of $A_1 \cap A_2$. Thus, the two sets are not disjoint.

Conversely, suppose $\langle X, d \rangle$ is totally boundedly UC but not complete. Let $\langle x_n \rangle$ be a Cauchy sequence in X with distinct terms without a cluster point. Then $\{x_n : n \in Z^+\}$ is a totally bounded subset of X with two disjoint closed subsets that cannot be uniformly separated, namely $\{x_{2n} : n \in Z^+\}$ and $\{x_{2n-1} : n \in Z^+\}$. This violates totally boundedly UC-ness of X.

It is well-known that a metrizable space admits a UC metric if and only if its set of accumulation points X' is compact [Ra]. We now show that a metrizable space admits a boundedly UC metric if and only if X' is locally compact and separable.

Theorem 3.2. *Let X be a metrizable space. Then X has a compatible metric d*

for which $\langle X, d\rangle$ is a boundedly UC *space if and only if X' is locally compact and separable.*

Proof: If $\langle X, d\rangle$ is a boundedly UC space, then by Theorem 3.1, X' is boundedly compact and is thus locally compact and separable. Conversely, if $X' = \emptyset$, then the zero-one metric is compatible and X so metrized is actually UC. Otherwise, X' is nonempty, locally compact and separable; so, by a result of Vaughan [Va], X' admits a boundedly compact metric d'. Since X' is closed, by a theorem of Hausdorff [Ha], d' can be extended to a compatible metric d for X. Following [Be3], we define another compatible metric ϱ on X as follows:

$$\varrho(x,y) = \begin{cases} 0 & \text{if } x = y \\ d(x,y) + \max\{d(x,X'), d(y,X')\} & \text{otherwise.} \end{cases}$$

Notice that ϱ restricted to $X' \times X'$ is just d', and a set is ϱ-bounded if and only if it is d-bounded. We claim that $\langle X, \varrho\rangle$ is boundedly UC.

To see this, let A and B be disjoint closed and bounded subsets of X. Suppose these sets are not uniformly separated with respect to ϱ. Take sequences $\langle a_n\rangle$ in A and $\langle b_n\rangle$ in B such that for each $n \in Z^+$, we have $\varrho(a_n, b_n) < 1/n$. Since $a_n \neq b_n$, we have $\varrho(a_n, X') < 1/n$ for each n, so that, for each n, we have $\varrho(a_n, X' \cap \operatorname{cl} S_1^\varrho[A]) < 1/n$. By the compactness of $X' \cap \operatorname{cl} S_1^\varrho[A]$, the sequence $\langle a_n\rangle$ must have a cluster point, whence $\langle b_n\rangle$ has the same cluster point. This contradicts $A \cap B = \emptyset$. □

4. Boundedly UC spaces and the Attouch–Wets hyperspace topology.

Connections between UC spaces and the familiar Hausdorff metric hyperspace topology have been examined in [Mi], [Be1], [BHPV]. Similar connections between boundedly UC spaces and the weaker Attouch–Wets hyperspace topology [AW], [AP], [Be5], [ALW], [BDC], [BL1], [Pe], [Ho] are outlined here. Indeed, it is the interplay between uniform continuity on bounded sets and this topology exhibited in [BDC] that motivated this note in the first place. Since some arguments are parallel those in the Hausdorff metric case, we do not give all details.

As the Hausdorff metric topology on $\mathrm{CL}(X)$ is the topology of uniform convergence of distance functionals for closed sets, the Attouch–Wets topology τ_{AW_d} is the topology of uniform convergence of distance functionals for closed sets on bounded subsets of X. One (metrizable) uniformity Ω_d for the Attouch–Wets topology τ_{AW_d} has as its base all sets of the form

$$V_d[B;\varepsilon] = \{(A_1, A_2) : \sup_{x\in B} |d(x, A_1) - d(x, A_2)| < \varepsilon\},$$

where B is an arbitrary bounded subset of X, and $\varepsilon > 0$. Another (weaker) compatible uniformity Σ_d has as a base all sets of the form

$$U_d[B;\varepsilon] = \{(A_1, A_2) : A_1 \cap B \subset S_\varepsilon[A_2] \text{ and } A_2 \cap B \subset S_\varepsilon[A_1]\},$$

where again B is a bounded subset of X, and $\varepsilon > 0$. For a proof that these two uniformities determine the same topology, the reader may consult [Be4, Lemma 3.1] or [AP, Proposition 2.1]. The main result of [BDC] says that compatible metrics d and ϱ give rise to the same Attouch–Wets topologies if and only if they determine the same bounded sets and the same set of functions that are uniformly continuous on bounded sets.

Theorem 4.1. *Let $\langle X, d\rangle$ be a metric space, and let $\mathcal{B}$ be the family of d-bounded subsets of X. The following are equivalent:*

(1) *$\langle X, d\rangle$ is boundedly* UC*;*
(2) *τ_{AW_d} is the finest topology on* $\mathrm{CL}(X)$ *among the Attouch–Wets topologies determined by compatible metrics ϱ for X that determine the same bounded sets $\mathcal{B}$;*
(3) *For each compatible metric ϱ for X determining the same collection of bounded sets $\mathcal{B}$ and for each $B \in \mathcal{B}$, $A \to e_\varrho(B, A)$ is a τ_{AW_d}-continuous functional on* $\mathrm{CL}(X)$*;*
(4) *For each compatible metric ϱ for X determining the same collection of bounded sets $\mathcal{B}$ and for each $x \in X, A \to \varrho(x, A)$ is a τ_{AW_d}-continuous functional on* $\mathrm{CL}(X)$.

Proof: (1) $\Rightarrow$ (2). This is immediate from Theorem 3.1 of [BDC] and Theorem 3.1 above.

(2) $\Rightarrow$ (3). By Theorem 5.4 of [BL2], the supremum of all Attouch–Wets topologies corresponding to all compatible metrics ϱ that determine the same bounded subsets as a given metric d is the weakest topology on $\mathrm{CL}(X)$ such that each functional $A \to e_\varrho(B, A)$ is continuous. Thus, if τ_{AW_d} is maximal, it coincides with this topology.

(3) $\Rightarrow$ (4). $\varrho(x, A) = e_\varrho(\{x\}, A)$.

(4) $\Rightarrow$ (1). By Theorem 5.1 of [BL2], the weakest topology τ on $\mathrm{CL}(X)$ such that each functional of the form $A \to \varrho(x, A)$ is continuous, has as a subbase all sets of the form

$$\{F \in \mathrm{CL}(X) : F \cap V \neq \emptyset\} \qquad (V \ \text{ open in } \ X),$$
$$\{F \in \mathrm{CL}(X) : F \cap B = \emptyset\} \qquad (B \in \mathcal{B}).$$

In particular, given $B \in \mathcal{B}$ and A a closed subset of X with $A \cap B = \emptyset$, the set $\{F \in \mathrm{CL}(X) : F \cap B = \emptyset\}$ contains some τ_{AW_d}-neighborhood $\mathcal{A}$ of A. In view of our second presentation of τ_{AW_d} as a uniform topology, we may assume $\mathcal{A}$ to be of the form

$$\mathcal{A} = \{F \in \mathrm{CL}(X) : F \cap B_0 \subset S_\varepsilon[A] \ \text{ and } \ A \cap B_0 \subset S_\varepsilon[F]\},$$

where $B_0 \in \mathcal{B}$ and ε is positive. We claim $S_\varepsilon[A] \cap B = \emptyset$. If not, take $b \in B$ with $d(b, A) < \varepsilon$. Then $A \cup \{b\} \in \mathcal{A}$, whereas $A \cup \{b\} \notin \{F \in \mathrm{CL}(X) : F \cap B = \emptyset\}$. This contradicts the choice of $\mathcal{A}$. Thus, A and B can be uniformly separated, so that $\langle X, d\rangle$ is boundedly UC. $\square$

One of the more intriguing characterizations of UC spaces involves function spaces. Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces, and equip $X \times Y$ with a metric ϱ compatible with the product uniformity, e.g., the box metric. We may then speak of the Hausdorff metric topology for $\mathrm{CL}(X \times Y)$ in an unambiguous way, since uniformly equivalent metrics determine the same hyperspace. Identifying elements of the set $C(X, Y)$ of continuous functions from X to Y with their graphs, we may speak of the Hausdorff metric topology $\tau_{\mathrm{H}_\varrho}$ for $C(X,Y)$. Now let τ_u be the topology of uniform convergence on $C(X,Y)$. With no assumptions we always have $\tau_u \supset \tau_{\mathrm{H}_\varrho}$ on $C(X,Y)$. Equality of the two topologies is characterized as follows [Be1, Theorem 1]:

Theorem. *Let $\langle X, d_X\rangle$ be a metric space. The following are equivalent:*

(1) *$\langle X, d_X\rangle$ is a* UC *space;*
(2) *For all metric spaces $Y, \tau_u = \tau_{\mathrm{H}_\varrho}$ on $C(X,Y)$;*
(3) *$\tau_u = \tau_{\mathrm{H}_\varrho}$ on $C(X,R)$.*

For boundedly UC spaces, it is natural to look at the (metrizable) topology of uniform convergence on bounded sets τ_{ub} in lieu of the topology of uniform convergence, and the Attouch–Wets topology $\tau_{\mathrm{AW}_\varrho}$ in lieu of the Hausdorff metric topology $\tau_{\mathrm{H}_\varrho}$. Assertion (a) of Theorem 4.1 of [BDC] shows that $\tau_{\mathrm{AW}_\varrho} \subset \tau_{ub}$ on $C(X,Y)$ always, but the example preceding shows that the reverse inclusion fails even if X is compact and $Y = R$. To get an analog of the theorem stated immediately above, we must require boundedness of d_Y.

Theorem 4.2. *Let $\langle X, d_X\rangle$ be a metric space. The following are equivalent:*

(1) *$\langle X, d_X\rangle$ is a boundedly* UC *space;*
(2) *For all bounded metric spaces $Y, \tau_{ub} = \tau_{\mathrm{AW}_\varrho}$ on $C(X,Y)$;*
(3) *$\tau_{ub} = \tau_{\mathrm{AW}_\varrho}$ on $C(X,[0,1])$.*

Proof: (1) ⇒ (2). This follows from assertions (a) and (b) together in Theorem 4.1 of [BDC].

(2) ⇒ (3). This is trivial.

(3) ⇒ (1). This is established using the construction in the proof of (c) ⇒ (a) in Theorem 1 of [Be1], using in turn condition 7 of Theorem 3.1, characterizing boundedly UC spaces, and the lemma preceding the proof of Theorem 1 of [Be1]. □

We refer the reader to a recent paper of Holá [Ho] for further results on the space $\langle C(X,Y), \tau_{\mathrm{AW}_\varrho}\rangle$. We mention in closing that the constructive approximation of real functions in Hausdorff distance has been a fundamental research area of the Bulgarian Academy of Sciences over the past 30 years (see, e.g., [Se]). Little has been done in this area for the relatively new Attouch–Wets convergence, which seems more appropriate for functions defined on unbounded intervals that are not periodic.

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