The compact extension property: the role of compactness

Jos van der Bijl*, Jan van Mill

Abstract. We consider separable metrizable topological spaces. Among other things we prove that there exists a non-contractible space with the compact extension property and we prove a new version of realization of polytopes for ANR's.

Keywords: absolute retract, the compact extension property, contractibility, simplicial complex, partial realization

Classification: 55M15, 57N17

1. Introduction.

We only consider separable metrizable spaces. A *map* is a continuous function and an *extension* is always supposed to be continuous. A space X has the *compact* (*neighborhood*) *extension property*, abbreviated C(N)EP, if for every space Y, every map $f : A \to X$, where A is a compact subset of Y, has an extension to (a neighborhood of A in) Y. The difference of this notion with the well known A(N)R-property lies in the requirement of compactness for A, instead of closedness of A in Y. The question by Kuratowski of whether the C(N)EP is strictly weaker than the A(N)R-property, motivated by his result that for finite-dimensional spaces the C(N)EP is equivalent with the A(N)R-property, was solved by J. van Mill in [12]. Further investigations around the C(N)EP are collected in [7], [3], [4] and [2]. We refer the reader to these for more information.

In these and other investigations it turned out that spaces with the C(N)EP often behave like A(N)R's. Many of the properties, that are well known from ANR-theory crop up in the neighborhood of the C(N)EP, and many constructions thereof can easily be adapted to serve for the study of the C(N)EP. However, there is one persistent difference. A property, known from ANR-theory, has sometimes to be adjusted in the sense, that it is now not applicable to the whole space under consideration, but only to an arbitrary compactum in the space. In this paper we show in which way compacta are essential. We treat our subject in two instances. The first one (§2) deals with (local) contractibility; the second one (§3) with realization of polytopes.

2. Contractibility.

We study properties of spaces with the C(N)EP that are related to contractibility. First, we prove a contractibility-like property for spaces with the CEP.

*Research supported by the Netherlands organization for scientific research (NWO).

Proposition 2.1. *If X has the* CEP*, then for every compact subset K of X there is a compact subset L of X, such that K is contractible in L.*

Proof: Take a space X with the CEP and a compact subset K of X. There is an embedding $j : K \to Q$ (Q is the Hilbert cube). Because X has the CEP, the map $j^{-1} : j(K) \to X$ has an extension $\xi : Q \to X$. Let $L = \xi(Q)$ and denote the straight-line contraction $(x,t) \mapsto t \cdot x$ by $G : Q \times I \to Q$. A contraction of K in L is given by $\xi \circ G \circ (j \times id) : K \times I \to L$. □

Analogous to the situation in ANR-theory, the CEP can be characterized in terms of the CNEP and the contractibility property of this proposition. We prove somewhat more. As usual, a space X is said to be C^∞ if for every $m \in \mathbb{N}$, every map $f : S^m \to X$ has an extension $g : B^{m+1} \to X$.

Theorem 2.2. *The following assertions are equivalent:*

(i) *X has the* CEP*,*
(ii) *X has the* CNEP *and X is C^∞,*
(iii) *X has the* CNEP *and every compact subset K of X is contractible in X.*

Proof: (iii) ⇒ (ii): This is easy.

(ii) ⇒ (i): Choose a space Y, a compact subspace A of Y and a map $f : A \to X$. According to [5, Theorem V.6.2] there exist a compact AR Z and an embedding $i : A \to Z$ such that $N = Z \setminus i(A)$ is a locally finite polytope with triangulation $\mathcal{T} = \{\sigma_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} diam\ \sigma_i = 0$. There is a map $\alpha : Y \to Z$ such that $i = \alpha \restriction A$. Identify $i(A)$ and A in the sequel and denote $f \circ i^{-1} : i(A) \to X$ by $\tilde{f}$. By definition, there is an extension $h : V \to X$ of $\tilde{f} : A \to X$, where V is a neighborhood of A in Z, $V = A \cup \widetilde{N}$, $\widetilde{N}$ is a subpolytope of N. Because X is C^∞, there is by [13, 5.2.14] an extension $k : N \to X$ of $h \restriction \widetilde{N}$. Define $\tilde{g} : Z \to X$ by

$$\tilde{g}(z) = \begin{cases} h(z) & (z \in V), \\ k(z) & (z \in N). \end{cases}$$

In each point of Z, $\tilde{g}$ is either locally h or locally k, so $\tilde{g}$ is continuous. The desired extension of $f : A \to X$ is given by $g = \tilde{g} \circ \alpha : Y \to X$.

(i) ⇒ (iii): See Proposition 2.1. □

Having deduced Proposition 2.1 we ask whether for spaces with the CEP we can prove contractibility of the whole space. In other words, we ask about the essentiality of the restriction to compact sets in the above. We provide the answer to this question. Our construction is a more sophisticated version of [6]. For more details see that paper. The proof of Theorem 2.3 shall give another example of a non-ANR with the CEP. Note that since the publication of [6] a cell-like dimension raising map has been found, [8]. For information on cell-like maps see [1].

Theorem 2.3. *There exists a topologically complete space with the* CEP*, which is not contractible.*

Proof: Let $f : X \to Y$ be a cell-like map from a finite-dimensional compact space X onto an infinite-dimensional space Y, [8]. We may assume $X \subset S^{n-1}$ for some

$n \in \mathbb{N}$. Find a totally disconnected G_δ-subset S of Y with $dim\, S > n$, [6, 3.1]. Consider the adjunction space $Z = B^n \cup_f Y$ and its subspaces

$$Z_A = (Int\, B^n) \cup A, \quad A \text{ closed in } S.$$

Note that in this situation, A is closed in Z_A. The quotient map $\check{f} : B^n \to Z$ is cell-like, and hence its restrictions

$$f_A : (Int\, B^n) \cup f^{-1}(A) \to Z_A$$

for $A \subset S$ closed are cell-like as well.

Claim 1. If $dim\, A < \infty$, then $Z_A \in$ AR.

Proof: Note that [6, 3.2] gives that $(Int\, B^n) \cup f^{-1}(A)$ is an AR. Furthermore, the singular set $S(f_A)$ of f_A is contained in A, so it is finite-dimensional. [1] finishes the argument.

Claim 2. For any closed $A \subset S$ the set Z_A has the CEP.

Proof: Observe that a space having the property that each compact subspace is contained in an AR, has the CEP. So pick $K \subset Z_A$ compact. Then $K \cap A$ is compact as well, and it is totally disconnected, so by [9, 6.2.9] we have $dim(K \cap A) \leq 0$. But then $Z_{A \cap K}$ is an AR by Claim 1. By $K \subset Z_{A\cap K} \subset Z_A$ we are done.

Claim 3. There is a closed subspace A of S such that Z_A is not contractible.

Proof: Suppose not. We shall prove $dim\, S \leq n$, which is a contradiction with the requirements on S. We use [13, 4.6.4]. Pick a closed subspace A of S and a map $\phi : A \to S^n$. The dimension of $Z_A \setminus A = Int\, B^n$ is not greater than n, so there is an extension $\tilde{\phi} : Z_A \to S^n$ of ϕ, [13, 4.6.3]. Now the contractibility of Z_A implies that $\tilde{\phi}$ is nullhomotopic, so ϕ is nullhomotopic. The Borsuk homotopy extension theorem [10, IV.2.2] gives an extension $\overline{\phi} : S \to S^n$ of ϕ and we are done.

It is easy to see that Z_A is topologically complete for every closed $A \subset S$. By Claim 2, the space Z_A with A as in Claim 3 is as required. □

After our considerations on global contractibility we prove a local-contractibility-like property for spaces with the CNEP.

Proposition 2.4. *If X has the* CNEP, *then for every compact subset K of X there is a compact subset L of X, such that for every element p of K and every open set U, containing p, there is an open set $O \subset U$, containing p, such that $O \cap K$ is contractible in $U \cap L$.*

Proof: Take a space X with the CEP and a compact subset K of X. There is an embedding $j : K \to Q$. Since X has the CNEP, the map $j^{-1} : j(K) \to K$ has an extension $\xi : W \to X$ to some open neighborhood W of $j(K)$ in Q. Take an open set V such that $j(K) \subset V \subset \overline{V} \subset W$ and let $L = \xi(\overline{V})$. Pick $p \in K$ and U open, containing p. The set $\xi^{-1}(U)$ is open in Q and contains $j(p)$. Choose a convex open set C with $j(p) \in C \subset \xi^{-1}(U) \cap V$ and consider the straight-line contraction $G : C \times I \to C$ of C onto $j(p)$. Determine an X-open set O with $O \cap K = j^{-1}(C)$. Then we have $O \cap K \subset U$, so without loss of generality $O \subset U$. Let $H : (O \cap K) \times I \to U \cap L$ be given by $H(x,t) = \xi G(j(x), t)$. □

Just as in the case of (global) contractibility we prove that restriction to compacta is essential.

Corollary 2.5. *There exists a topologically complete space X with the* CNEP, *which is not locally contractible.*

PROOF: Let Y be a space as in Theorem 2.3 and let X be the countably infinite product Y^∞ of copies of Y. Then X has the CEP and X is topologically complete. We shall prove that no non-empty open subset of X is contractible in X. Suppose that U is an arbitrary non-empty open subset of X, and $H : U \times I \to X$ is a contraction. Pick $x = (x_1, x_2, \dots) \in U$. There are a $k \in \mathbb{N}$ and open subsets $U_1, U_2, \dots, U_{k-1}$ of Y with $x_i \in U_i$ for every i, such that

$$U_1 \times U_2 \times \cdots \times U_{k-1} \times Y \times Y \times \cdots \subset U.$$

Operating in the k-th factor it is easy to construct a contraction of Y in itself, a contradiction. □

At the end of this section we may conclude that, as far as contractibility and local contractibility are concerned, the C(N)EP is a property working essentially on compacta. It gives knowledge about compacta in the space, but does not imply anything about behavior of the whole space.

Questions, that cannot be suppressed, are whether the assumption of contractibility or local contractibility allows us to conclude the A(N)R-property from the C(N)EP. The question about contractibility can be answered. It is possible to prove that the metric cone ΔX (see [13]) over a space X with the CNEP has the CEP. Now let X be a space as in Theorem 2.3. Then ΔX has the CEP and is contractible, but (see [13, 5.4.2]) it is not an ANR. A conditional answer to the same question was already provided in [2]. There we constructed, under the assumption that there exists a (topological) linear space which is not an AR, a linear (so, contractible) space with the CEP, that is not an AR. To be answered remains:

Question: Does there exist a locally contractible space X with the C(N)EP, which is not an ANR?

Note that the above mentioned considerations from [2] also provide a conditional positive answer to this question, for every linear space is locally contractible as well.

3. Realization of polytopes.

Speaking about polytopes and simplicial complexes (cf. [13]) we always mean countable and locally finite ones. Let X be a space and let $\mathcal{U}$ be an open cover of X. In addition, let $\mathcal{T}$ be a simplicial complex with the Whitehead topology and let $\mathcal{S}$ be a subcomplex of $\mathcal{T}$, containing all the vertices of $\mathcal{T}$. A *partial realization* of $\mathcal{T}$ in X relative to $\mathcal{S}$ and $\mathcal{U}$ is a map $f : |\mathcal{S}| \to X$ such that for every $\sigma \in \mathcal{T}$ there is $U \in \mathcal{U}$ with $f(\sigma \cap |\mathcal{S}|) \subset U$. If $\mathcal{S} = \mathcal{T}$, then f is called a *full realization*. For convenience, we may denote a partial realization of $\mathcal{T}$ in X relative to $\mathcal{S}$ and $\mathcal{U}$ by the quadruple $(\mathcal{T}, \mathcal{S}, f, \mathcal{U})$.

Consider the standard realization property for ANR's:

> $(*)$: for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$, such that for every simplicial complex $\mathcal{T}$, every partial realization $(\mathcal{T}, \mathcal{S}, f, \mathcal{V})$ in X can be extended to a full realization $(\mathcal{T}, \mathcal{T}, g, \mathcal{U})$ in X.

It is well known that for a space X, this property $(*)$ is equivalent to X being an ANR, cf. [10, §IV.4]. In [2] the authors derived a realization property for the CNEP, analogous to the above one. It reads as follows (in two versions):

> $(**)$: for every compact subset K of X there is a compact subset L of X such that for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$, such that for every simplicial complex $\mathcal{T}$, every partial realization $(\mathcal{T}, \mathcal{S}, f, \mathcal{V})$ in K can be extended to a full realization $(\mathcal{T}, \mathcal{T}, g, \mathcal{U})$ in L.
>
> $(***)$: for every compact subset K of X there is a compact subset L of X such that for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$, such that for every *finite* simplicial complex $\mathcal{T}$, every partial realization $(\mathcal{T}, \mathcal{S}, f, \mathcal{V})$ in K can be extended to a full realization $(\mathcal{T}, \mathcal{T}, g, \mathcal{U})$ in L.

In [2] we proved that $(**)$ and $(***)$ are both equivalent to X having the CNEP. In the realm of polytopes compactness precisely means finiteness. Superficially, the difference between the realization property $(*)$ for ANR's on the one hand, and properties $(**)$ and $(***)$ for spaces with the CNEP on the other hand is twofold. The first striking point is the occurrence of the two compacta K and L in both of $(**)$ and $(***)$. Second, in $(***)$ we moreover restrict the polytopes admitted to finite ones. It is therefore natural to consider a fourth realization property, in which we restrict the polytopes to finite ones, but we do not introduce the compacta K and L. We are thus led to:

> $(****)$: for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$, such that for every finite simplicial complex $\mathcal{T}$, every partial realization $(\mathcal{T}, \mathcal{S}, f, \mathcal{V})$ in X can be extended to a full realization $(\mathcal{T}, \mathcal{T}, g, \mathcal{U})$ in X.

Since "the difference" between the CNEP and the ANR-property is in the introduction of compacta in suitable places, it might be suspected that property $(****)$ is equivalent to the CNEP for X. We show that, surprisingly, $(****)$ is equivalent to X being an ANR. Our main tool is the possibility of chopping up a given polytope into finite pieces in an appropriate way. We start with that.

Lemma 3.1. *Let $\mathcal{T}$ be a (countable and locally finite) simplicial complex. Then there exist finite subcomplexes $\mathcal{T}_n (n = 0, 1, \dots)$, such that*

(i) $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$, *and*
(ii) $\mathcal{T}_n \cap \mathcal{T}_m = \emptyset$ *for* $|n - m| \geq 2$.

Proof: If $\mathcal{A}$ is a collection of simplexes of $\mathcal{T}$, then we denote by $\mathcal{A}^\flat$ the subcomplex of $\mathcal{T}$, formed by all faces of elements of $\mathcal{A}$. First suppose for simplicity that $\mathcal{T}$ is connected. We define an increasing sequence $\{\mathcal{S}_n\}_{n=-1}^{\infty}$ of finite subpolytopes of $\mathcal{T}$. Let $\mathcal{S}_{-1} = \emptyset$ and $\mathcal{S}_0 = \{\tau\}^\flat$, where $\tau \in \mathcal{T}$ is arbitrary. If $\mathcal{S}_{-1}, \dots, \mathcal{S}_{n-1}$ are chosen, then we put

$$\mathcal{S}_n = \{\sigma \in \mathcal{T} \mid \sigma \cap |\mathcal{S}_{n-1}| \neq \emptyset\}^\flat.$$

Note that by local finiteness of $\mathcal{T}$, this is finite. Having constructed the $\mathcal{S}_n$'s, we put $\mathcal{T}_n = (\mathcal{S}_n \setminus \mathcal{S}_{n-1})^\flat$ for $n \geq 0$. By connectedness, (i) holds; (ii) is easily verified. In the case that $\mathcal{T}$ is not connected, pick elements $\tau_0, \tau_1, \dots,$ of $\mathcal{T}$ such that every component of $\mathcal{T}$ is represented. Change the definition of the $\mathcal{S}_n$'s into $\mathcal{S}_0 = \{\tau_0\}^\flat$,

$$\mathcal{S}_n = (\{\sigma \in \mathcal{T} \mid \sigma \cap |\mathcal{S}_{n-1}| \neq \emptyset\} \cup \{\tau_n\})^\flat.$$

The proof is finished in the same way as before. □

Theorem 3.2. *An arbitrary space X is an* ANR *iff property* $(****)$ *holds.*

Proof: Every ANR has property $(*)$, so certainly the apparently weaker property $(****)$. We are left with proving that $(****)$ implies X being an ANR. This shall be done by deducing $(*)$ from $(****)$. Take an arbitrary open cover $\mathcal{U}$ of X. Refine $\mathcal{U}$ three times as follows: find open covers $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{V}$ such that

(i) $\mathcal{V} < \mathcal{W}_2 \overset{*}{<} \mathcal{W}_1 < \mathcal{U}$,
(ii) every partial realization in X of any finite simplicial complex with respect to $\mathcal{W}_1$ can be extended to a full realization in X with respect to $\mathcal{U}$, and
(iii) every partial realization in X of any finite simplicial complex with respect to $\mathcal{V}$ can be extended to a full realization in X with respect to $\mathcal{W}_2$.

Choose an arbitrary simplicial complex $\mathcal{T}$, a subcomplex $\mathcal{S}$ of $\mathcal{T}$, containing all the vertices of $\mathcal{T}$, and a partial realization $f : |\mathcal{S}| \to X$ of $\mathcal{T}$ in X with respect to $\mathcal{V}$. We shall extend f to a full realization of $\mathcal{T}$ with respect to $\mathcal{U}$. To that end apply Lemma 3.1 and determine finite subcomplexes $\mathcal{T}_0, \mathcal{T}_1, \dots$ of $\mathcal{T}$ such that

(iv) $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$, and
(v) $\mathcal{T}_n \cap \mathcal{T}_m = \emptyset$ for $|n - m| \geq 2$.

It is obvious that for every $n \geq 0$ the map $f \upharpoonright |\mathcal{S} \cap \mathcal{T}_n| : |\mathcal{S} \cap \mathcal{T}_n| \to X$ is a partial realization of the finite simplicial complex $\mathcal{T}_n$ with respect to $\mathcal{V}$. For even n, using (iii), we extend this map to a full realization $g_n : |\mathcal{T}_n| \to X$ with respect to $\mathcal{W}_2$. For odd n, consider the subcomplex

$$\widetilde{\mathcal{T}}_n = \mathcal{T}_n \cap (\mathcal{T}_{n-1} \cup \mathcal{S} \cup \mathcal{T}_{n+1})$$

of $\mathcal{T}_n$, and the map $h_n : |\widetilde{\mathcal{T}}_n| \to X$ given by

$$h_n(x) = \begin{cases} g_{n-1}(x) & (x \in |\mathcal{T}_{n-1}|), \\ f(x) & (x \in |\mathcal{S}|), \\ g_{n+1} & (x \in |\mathcal{T}_{n+1}|). \end{cases}$$

(It is easy to see that h_n is well defined.) □

Claim. The map h_n is a partial realization of $\mathcal{T}_n$ with respect to $\mathcal{W}_1$.

PROOF: Pick $\sigma \in \mathcal{T}_n$. We have

$$h_n(\sigma \cap |\widetilde{\mathcal{T}}_n|) = g_{n-1}(\sigma \cap |\mathcal{T}_{n-1}|) \cup f(\sigma \cap |\mathcal{S}|) \cup g_{n+1}(\sigma \cap |\mathcal{T}_{n+1}|).$$

By the properties of f there is an element $V \in \mathcal{V}$ with $f(\sigma \cap |\mathcal{S}|) \subset V$. Furthermore, $\sigma \cap |\mathcal{T}_{n-1}|$ consists of a number of simplexes τ, each of which intersects $\sigma \cap |\mathcal{S}|$ (for $\mathcal{S}$ contains all the vertices of $\mathcal{T}$), and for each of which there is a $W_\tau \in \mathcal{W}_2$ such that $g_{n-1}(\tau) \subset W_\tau$. The same holds for $\sigma \cap |\mathcal{T}_{n+1}|$. So we have $h_n(\sigma \cap |\widetilde{\mathcal{T}}_n|) \subset V \cup \bigcup_\tau W_\tau$ for certain members $W_\tau \in \mathcal{W}_2$ with $W_\tau \cap V \neq \emptyset$. By $\mathcal{V} < \mathcal{W}_2 \overset{*}{<} \mathcal{W}_1$ there exists $W \in \mathcal{W}_1$ with $h_n(\sigma \cap |\widetilde{\mathcal{T}}_n|) \subset W$.

Now by (ii), and finiteness of $\mathcal{T}_n$, we find a full realization $g_n : |\mathcal{T}_n| \to X$ with respect to $\mathcal{U}$, extending h_n, for every odd n. Define $g : |\mathcal{T}| \to X$ by $g(x) = g_n(x)(x \in |\mathcal{T}_n|)$. By (v), and the coincidence of g_n and g_{n+1} on $|\mathcal{T}_n \cap \mathcal{T}_{n+1}|$, g is well defined. It is continuous by the Whitehead topology. For every n the map g_n is a full realization of $\mathcal{T}_n$ with respect to $\mathcal{U}$, so g is a full realization of $\mathcal{T}$ with respect to $\mathcal{U}$. It clearly extends f. □

We conclude that with partial realization of polytopes, the compactness condition has nothing to do with the polytopes, but everything with the image set of the realization.

References

[1] Ancel F.D., *The role of countable dimensionality in the theory of cell-like relations*, Trans. A.M.S. **287** (1985), 1–40.

[2] van der Bijl J., van Mill J., *Linear spaces, absolute retracts and the compact extension property*, Proc. A.M.S. **104** (1988), 942–952.

[3] van der Bijl J., Dobrowolski T., Hart K.P., van Mill J., *Admissibility, homeomorphism extension and the AR-property in topological linear spaces*, submitted to Proc. A.M.S..

[4] van der Bijl J., *Extension of continuous functions with compact domain*, Ph.D. thesis, Vrije Universiteit, Amsterdam, 1991, to appear.

[5] Borsuk K., *Theory of retracts*, PWN, Warszawa, 1967.

[6] Curtis D.W., van Mill J., *The compact extension property*, Proc. Sixth Prague Topology Symposium, Heldermann, Berlin, 1988, pp. 115–119.

[7] Dobrowolski T., *On extending mappings into nonlocally convex metric spaces*, Proc. A.M.S. **93** (1985), 555–560.

[8] Dranishnikov A.N., *On a problem of P.S. Alexandrov*, Matem. Sbornik **135** (1988), 551–557 (Russian); =*Math. USSR Sbornik* **63** (1989), 539–545.

[9] Engelking R., *General Topology*, PWN, Warszawa, 1977.

[10] Hu S.-T., *Theory of Retracts*, Wayne State University Press, Detroit, 1965.

[11] Kuratowski K., *Sur quelques problèmes topologiques concernant le prolongement des fonctions continus*, Colloq. Math. **2** (1951), 186–191.

[12] van Mill J., *Another counterexample in ANR-theory*, Proc. A.M.S. **97** (1986), 136–138.

[13] van Mill J., *Infinite-dimensional topology: prerequisites and introduction*, North–Holland Publishing Company, Amsterdam, 1989.

FACULTEIT WISKUNDE EN INFORMATICA, VRIJE UNIVERSITEIT, DE BOELELAAN 1081^a, 1081 HV AMSTERDAM, THE NETHERLANDS

(*Received* February 21, 1991)