Absolutely terminal continua and confluent mappings

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Abstract. Interrelations between three concepts of terminal continua and their behaviour, when the underlying continuum is confluently mapped, are studied.

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1. Introduction.

A *continuum* means a compact connected metric space and a *mapping* means a continuous function.

A proper subcontinuum K of a continuum X is said to be a *terminal continuum* of X provided that if whenever A and B are proper subcontinua of X having union equal to X such that $A \cap K \neq \emptyset \neq B \cap K$, then either $X = A \cup K$ or $X = B \cup K$ ([1, Definition 1.1, p. 7]). A proper subcontinuum K of a continuum X is said to be an *absolutely terminal continuum* of X provided that K is a terminal continuum of each subcontinuum L of X which properly contains K (see [1, Definition 4.1, p. 34]; note that for this concept, the name of a terminal continuum is used in Fugate's paper [5, p. 461], and in Nadler's book [9, 1.54, p. 107]).

Obviously each absolutely terminal continuum is terminal, but not conversely. Namely, if K is a middle part of the limit segment of the $\sin(1/x)$-curve, i.e., if

$$X = \mathrm{cl}\{(x, \sin(1/x)) : 0 < x \leq 1\}$$

and $K = \{(0, y) : y \in [-1/2, 1/2]\}$, then K is terminal (see [1, Example 1.2 (a), p. 7]) but not absolutely terminal ([1, p. 34]) subcontinuum of X.

The proof of the following known result is left to the reader (see [5, p. 461]; compare [9, Definition 1.54 and Lemma 1.55, p. 107]).

Theorem 1. *The following conditions are equivalent for a proper subcontinuum K of a continuum X:*

(2) *K is an absolutely terminal continuum of X;*
(3) *for each two proper subcontinua A and B of X, if $A \cap K \neq \emptyset \neq B \cap K$, then either $A \subset B \cup K$ or $B \subset A \cup K$;*
(4) *for each two proper subcontinua A and B of X, if $K \subset A \cap B$, then either $A \subset B$ or $B \subset A$.*

For various structural properties of terminal and of absolutely terminal continua see [1]. The aim of this paper is to study mapping properties of these (and of related) continua.

A surjective mapping $f : X \to Y$ between continua X and Y is said to be:

–*confluent* (*weakly confluent*) *with respect to a subcontinuum* Q of Y provided that, for each (for some) component C of the inverse image $f^{-1}(Q)$, we have $f(C) = Q$;

–*confluent* (*weakly confluent*) provided that it is confluent (weakly confluent) with respect to each subcontinuum Q of Y;

–*semi-confluent* provided that for each subcontinuum Q of Y and for each two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

2. Absolutely terminal continua.

We start with the following result.

Theorem 5. *Let a subcontinuum K of a continuum X be given, and let a mapping $f : X \to Y$ satisfy the condition:*

(6) *for each subcontinuum Q of Y containing $f(K)$ there exists a component C of $f^{-1}(Q)$ such that $K \subset C$ and $f(C) = Q$.*

Then

(7) *if K is an absolutely terminal continuum of X, then $f(K)$ either equals Y or is an absolutely terminal continuum of Y.*

Proof: Assume $f(K) \neq Y$. To prove that it is an absolutely terminal continuum of Y we apply the condition (4) of Theorem 1. Let Q_1 and Q_2 be two proper subcontinua of Y such that $f(K) \subset Q_1 \cap Q_2$, and for $i \in \{1, 2\}$ let C_i denote the component of $f^{-1}(Q_i)$ with $K \subset C_i$ and $f(C_i) = Q_i$ (which exists by the condition (6)). Since K is an absolutely terminal continuum of X, applying the condition (4) of Theorem 1 we have either $C_1 \subset C_2$ or $C_2 \subset C_1$. Thus the former inclusion implies $Q_1 = f(C_1) \subset f(C_2) = Q_2$, and similarly the latter one gives $Q_2 \subset Q_1$. Applying the condition (6) of Theorem 1 once more, we see that $f(K)$ is an absolutely terminal continuum of Y. □

Corollary 8. *Let a subcontinuum K of a continuum X be given, and let a mapping $f : X \to Y$ be confluent with respect to each subcontinuum Q of Y which contains $f(K)$. Then the implication* (7) *holds true.*

Corollary 9. *Let a subcontinuum K of a continuum X be given, and let a mapping $f : X \to Y$ be confluent. Then the implication* (7) *holds true.*

Remarks 10. (a) Note that the condition (6) is stronger than weak confluence of f with respect to Q.

(b) Neither the condition (6) of Theorem 5 nor the assumption of Corollary 8 can be relaxed to weak confluence of f with respect to Q. Similarly, the assumption of confluence of f in Corollary 9 cannot be relaxed to semi-confluence. This can be seen by an example below.

Example 11. The mapping $f : [0,1] \to [0,1]$ defined by the formula

$$f(x) = \begin{cases} -x + 1/2 & \text{for } \ x \in [0,1/2], \\ 2x - 1 & \text{for } \ x \in [1/2,1] \end{cases}$$

is both weakly confluent and semi-confluent. The continuum $K = [0,1/4]$ is an absolutely terminal subcontinuum of the domain, while its image $f(K) = [1/4,1/2]$ is not even terminal.

A surjective mapping $f : X \to Y$ between continua X and Y is said to be *quasi-monotone* provided that, for each subcontinuum Q of Y with nonempty interior, the inverse image of $f^{-1}(Q)$ has finitely many components each of which is mapped onto Q. Note that neither a quasi-monotone mapping needs to be confluent, nor a confluent one needs to be quasi-monotone. However, if the domain space is a locally connected continuum, then the two classes of mappings coincide (see [2, IX, p. 215] and [10, Theorem 8.4, p. 153]).

The following result is known (see [4, Corollary 12]).

Theorem 12. *Let a surjective mapping $f : X \to Y$ of a continuum X be quasi-monotone. Then*

(13) *if a continuum K is terminal in X, then $f(K)$ either equals Y or it is terminal in Y.*

Therefore Theorem 12 and the above mentioned coincidence of quasi-monotone and of confluent mappings, if they are defined on locally connected continua, imply the next result.

Corollary 14. *Let a surjective mapping $f : X \to Y$ of a locally connected continuum X be confluent. Then the implication* (13) *holds true.*

The assumption of local connectivity of X is essential in Corollary 14, as it can be seen from an example below, where the considered mapping is even open (thus in particular it is confluent, see [10, 7.5, p. 148]; cf. [2, VI, p. 214]). Recall that a continuum is said to be *arclike* (or chainable), if for each positive number ε there exists an ε-chain covering it.

Example 15. Open retractions of arclike continua do not preserve terminality of subcontinua.

Proof: In fact, let S denote the $\sin(1/x)$-line defined by

$$S = \{(x, \sin(1/x)) : 0 < x \le 2/\pi\}, \tag{16}$$

let L stand for the limit segment of S, i.e.,

$$L = \{(0,y) : y \in [-1,1]\}, \tag{17}$$

and let $f : L \cup S \to L$ be the projection defined by $f((x,y)) = (0,y)$ for all $(x,y) \in L \cup S$. Observe that $L \cup S$ and L are continua and f is an open retraction.

Then $K = \{(0,y) : y \in [-1/2, 1/2]\}$ is a terminal continuum in $L \cup S$, while $f(K) = K \subset L$ is not terminal in L. □

Thus we see that confluent mappings preserve the concept of an absolutely terminal subcontinuum, and quasi-monotone mappings preserve the concept of a terminal subcontinuum, which is not preserved by confluent mappings. So, it is natural to ask if quasi-monotone mappings preserve absolute terminality. The answer is negative.

Example 18. Quasi-monotone retractions of arclike continua do not preserve absolute terminality of subcontinua.

Proof: Let S and L have the same meaning as in Example 15 (see (16) and (17)), and put

$$A = \{(x, x+1) : x \in [-1, 0]\}.$$

Note that $A \cup L \cup S$ and $L \cup S$ are arclike continua, and consider a retraction $f : A \cup L \cup S \to L \cup S$ defined by $f((x,y)) = (0,y)$ for all $(x,y) \in A$ and $f((x,y)) = (x,y)$ for all $(x,y) \in L \cup S$, and observe that f is quasi-monotone. Then $K = \{(x, x+1) : x \in [-1, -1/2]\} \subset A$ is an absolutely terminal subcontinuum of $A \cup L \cup S$, while its image $f(K) = \{(0,y) : y \in [0, 1/2]\} \subset L$ is a terminal but not an absolutely terminal subcontinuum of $L \cup S$. □

A surjective mapping $f : X \to Y$ between continua X and Y is said to be *hereditarily monotone* (*hereditarily quasi-monotone*, *hereditarily confluent*) provided that, for each subcontinuum C of X, the partial mapping $f \mid C : C \to f(C) \subset Y$ is monotone (quasi-monotone, confluent, respectively). Since each hereditarily confluent mapping is known to be quasi-monotone (see e.g. [8, Corollary 4.45, p. 26]), any such mapping serves as an example of a hereditarily quasi-monotone one. Note that the mapping f of Example 18 is not hereditarily quasi-monotone.

Question 19. Let a mapping $f : X \to Y$ between continua X and Y be hereditarily quasi-monotone. Is then the implication (7) true?

3. HU-terminality.

To avoid confusion or misunderstanding in the terminology, we will keep the name of a terminal continuum only in the sense used by Bennett and Fugate in [1, Definition 1.1, p. 7] (see Introduction above). Consequently, we are forced to use another name for the concept of a terminal continuum as defined by Gordh in [6]. Since he restricts his considerations to subcontinua of hereditarily unicoherent continua only, we rename his concept as HU-terminal. To recall the definition we need some additional notions.

A subcontinuum I of a continuum X is said to be *irreducible about a subset* $S \subset X$ provided no proper subcontinuum of I contains S. A continuum X is said to be *irreducible* provided there are two points a and b in X such that X is irreducible about $\{a, b\}$. A continuum X is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected.

A subcontinuum K of a hereditarily unicoherent continuum X is said to be a HU-*terminal continuum* of X provided that K is contained in an irreducible subcontinuum of X and for every irreducible subcontinuum I of X containing K there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$ (see [6, p. 458]).

A continuum T is called a *triod* provided there exists a proper subcontinuum Q of T such that $T \setminus Q$ is the union of three mutually disjoint sets. A continuum which contains no triod is said to be *atriodic*. It is well known that every arclike continuum is hereditarily unicoherent, irreducible and atriodic.

Interrelations between the two concepts of absolutely terminal and of HU-terminal continua are illustrated by the theorem below which is due to Gordh (see [6, Theorem 3.1, p. 463] and Theorem 1 above) and by the examples following it.

Theorem 20 (Gordh). *Let a hereditarily unicoherent continuum X be atriodic. Then a proper subcontinuum of X is* HU-*terminal if and only if it is absolutely terminal.*

The equivalence described in Theorem 20 need not be true in a continuum X which fails to be atriodic as it is shown in [6, Example 1, p. 463] (see also [1, Example 4.3, p. 35]), where X is the union of a simple triod T and of a spiral (i.e., a one-to-one image of the real half line) approximating T. Let Q denote a proper subcontinuum of T such that $T \setminus Q$ consists of three components. Then Q is HU-terminal while not absolutely terminal in X. To see an absolutely terminal and not HU-terminal subcontinuum K of a hereditarily unicoherent continuum T, take again as T a simple triod, i.e., the union of three straight line segments pa, pb and pc any two of which have the point p in common only. Let m denote the middle point of pc. Then $K = pa \cup pb \cup pm$ has the needed properties.

In connection with Theorem 20, the following question seems to be interesting.

Question 21. Let a hereditarily unicoherent continuum X have the property that each its proper subcontinuum is HU-terminal if and only if it is absolutely terminal. Must then X be atriodic?

Theorem 20 implies that, in the realm of atriodic hereditarily unicoherent continua, mappings having the property considered in Theorem 5 (thus in particular confluent mappings, see Corollary 9) preserve HU-terminality. However, if one is looking for a suitable class of mappings which preserve HU-terminality in the realm of all hereditarily unicoherent continua, then any condition expressed in terms of confluence seems to be rather inadequate for such invariance. Even as strong condition as hereditary monotonicity of the mapping does not imply the considered property. In fact, recall that a continuum is hereditarily unicoherent if and only if each monotone mapping defined on X is hereditarily monotone (see [8, (6.10), p. 53]; cf. [7, Corollary 3.2, p.126], and [3, Lemma 1, p. 932]). But the trivial example of a (hereditarily) monotone retraction of the simple triod $pa \cup pb \cup pc$ onto the arc $pa \cup pb$ applied to the singleton $\{c\}$ as a HU-terminal subcontinuum of the triod gives a negative solution.

Question 22. What mappings among hereditarily unicoherent continua map HU-terminal subcontinua of the domain onto HU-terminal subcontinua of the range?

Recall that the image of a hereditarily unicoherent continuum need not be hereditarily unicoherent, even if the considered mapping is open, as it follows from an example of an open mapping of a solenoid onto a circle (see [2, p. 218]).

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