Normal structure and weakly normal structure of Orlicz spaces

Shutao Chen, Yanzheng Duan

Abstract. Every Orlicz space equipped with Orlicz norm has weak sum property, therefore, it has weakly normal structure and fixed point property. A criterion of sum property also of normal structure for such spaces is given as well, which shows that every Orlicz space has isonormal structure.

Keywords: Orlicz space, normal structure

Classification: 46B20

Introduction.

T. Landes [4] shows that, under certain conditions, an Orlicz sequence space with Luxemburg norm has normal or weakly normal structure iff it is separable. For Orlicz spaces with Orlicz norm, we will discover that the results are much different.

We begin with some notations. A sequence (x_n) of a Banach space X is called limit affine, if the limit $\lambda(x) := \lim_n \|x_n - x\| > 0$ exists for every $x \in \mathrm{conv}(x_n)$, and λ is an affine function on $\mathrm{conv}(x_n)$. (x_n) is called growing, if $\lambda(x_n) \leq \lambda(x_{n+1})$ for all $n \in N$. X is said to have sum property, if it contains no growing limit affine sequence. X is said to have weak sum property, if it contains no growing weakly converging limit affine sequence. X is said to have normal structure, if it contains no limit affine sequence (x_n) with $\lambda(x_n) = \lambda(x_{n+1}) > 0$ for all $n \in N$. X is said to have weakly normal structure, if it contains no weakly converging limit affine sequence (x_n) with $\lambda(x_n) = \lambda(x_{n+1}) > 0$ for all $n \in N$. X is said to have isonormal structure, if it is isomorphic to a Banach space with normal structure. X is said to have fixed point property if every nonexpansive selfmapping on a weakly compact convex subset of X has a fixed point.

It is well known that sum property $\Rightarrow$ normal structure and that weak sum property $\Rightarrow$ weakly normal structure $\Rightarrow$ fixed point property.

Throughout this paper, we always denote by (G, Σ, μ) a complete, nonatomic, finite measure space. We say $M : R \to R^+$ to be an N-function, if it is a continuous, convex, even function satisfying $M(u) = 0$, iff $u = 0$ and $M(u)/u \to 0$ (resp. ∞) as $u \to 0$ (resp. ∞). If $M(u)$ is an N-function, then we denote by $p(u)$ its right-hand derivative and by $N(v)$ the conjugate N-function of $M(u)$, i.e., $N(v) := \max_u\{uv - M(u)\}$.

*Supported by the Special Scientific Research Fund of China for University Doctoral Instruction.

Let M be an N-function. For every μ-measurable function $x : G \to R$ we define $\varrho_M(x) = \int_G M(x(t))\,d\mu$, and

$$
\begin{aligned}
L_M =& \{x : \varrho_M(\beta x) < \infty \ \text{ for some } \ \beta > 0\},\\
E_M =& \{x : \varrho_M(\beta x) < \infty \ \text{ for all } \ \beta > 0\},\\
\|x\| =& \|x\|_M = \inf_{k>0} [1 + \varrho_M(kx)]/k, \quad x \in L_M\,.
\end{aligned}
\tag{1}
$$

Then the Orlicz space $(L_M, \|\cdot\|)$ and its subspace $(E_M, \|\cdot\|)$ are Banach spaces.

Main results.

Lemma 1. *Suppose $x_n \in L_M, \|x_n\| \le K, n \in N$ and $x_n(t) \to x(t)$ in measure as $n \to \infty$, then $x \in L_M$.*

Proof: Since $\|x_n/K\| \le 1$, by [1], $\varrho_M(x_n/K) \le \|x_n/K\| \le 1, n \in N$. Without loss of generality, we may assume $x_n(t)$ μ-a.e. on G (pass a subsequence, if necessary), then, by Fatou's lemma, $\varrho_M(x/K) \le \liminf_{n\varrho_M}(x_n/K) \le 1$, i.e., $x \in L_M$. □

Lemma 2. *If $x_n \to 0$ weakly in L_M and $x_n(t) \to y(t)$ in measure, then $y = 0$.*

Proof: Again, we may assume $x_n(t) \to y(t)$ μ-a.e. on G. Let $F = \{t \in G; y(t) \neq 0\}$. If $\mu F > 0$, then there exists $E \in \Sigma$ with $\mu E < \mu F$ and $x_n(t) \to y(t)$ uniformly on $G \setminus E$. Define $v(t) = \text{signy}(t)\chi_{F\setminus E}(t)$, then $v \in L_{M^*}$ and $\langle v, x_n\rangle = \int_{F\setminus E} v(t)x_n(t)\,d\mu \to \int_{F\setminus E} |y(t)|\,d\mu > 0$ contradicting the hypothesis $x_n \to 0$ weakly. □

We say that an interval $[a, b]$ is a structural affine interval of the N-function M, if M is affine on $[a, b]$ and it is neither affine on $[a - \varepsilon, b]$ nor on $[a, b + \varepsilon]$ for any $\varepsilon > 0$.

Theorem 1. *For any N-function M, L_M has weak sum property, therefore, it has weakly normal structure.*

Theorem 2. *The following are equivalent,*

(i) *L_M has sum property,*
(ii) *L_M has normal structure,*
(iii) *there exist $a > 0, C > 1$ such that for any structural affine interval $[u, v]$ of M with $u \ge a$, we have $v/u \le C$.*

Proof of Theorems 1 and 2: For any limit affine sequence (x_n) in L_M with $x_i \neq x_j$ whenever $i \neq j$, by [1], the "inf" in (1) is attainable for all $x \neq 0$. Therefore, for all $i \neq j$, we may find $k_{ij} > 0$ such that

$$
\|x_i - x_j\| = [1 + \varrho_M(k_{ij}(x_i - x_j))]/k_{ij}\,. \tag{2}
$$

First we show that there exists a subsequence N_1 of N such that for any $j \in N_1, \{k_{ij}\}_{i\in N_1}$ is bounded. Indeed, if $\{k_{ij}\}_i$ is bounded for all $j \in N$, then we let $N_1 = N$. Otherwise, there exist some $m \in N$ and a subsequence I of N such that

$k_{im} \to \infty$ as $i(\in I) \to \infty$. Hence, for any $\sigma > 0$, if we define $G_i = \{t \in G : |x_i(t) - x_m(t)| > \sigma\}$, then by (2), we have

$$\|x_i - x_m\| > \varrho_M(|x_i - x_m|k_{im})/k_{im} \geq [M(\sigma k_{im})/k_{im}]\mu G_i\,.$$

Since $\lambda(x_m) < \infty$ and $M(u)/u \to \infty$ as $u \to \infty$, we must have $\mu G_i \to 0$ as $i(\in I) \to \infty$. This shows that $\{x_i\}_{i\in I}$ converges to x_m in measure. We may assume that I does not contain m. We claim that $N_1 = I$ satisfies our requirement. In fact, if $k_{ij} \to \infty$ as $i(\in I_1) \to \infty$ for some $j \in I$ and some subsequence I, of I, then in the same way we can show that $x_i \to x_j$ in measure as $i(\in I_1) \to \infty$. This is impossible since $x_j \neq x_m$.

By the diagonal method, we can find a subsequence N_2 of N_1 such that $k_{ij} \to k_j < \infty$ as $i(\in N_2) \to \infty$ for each $j \in N_1$. We claim that $k_j \to \infty$ as $j(\in N_2) \to \infty$. In fact, if this is not true, then N_2 contains a subsequence N_3 such that $k_j \to k < \infty$ as $j(\in N_3) \to \infty$. By (1) and (2), for all $n, i, j \in N_3$, $n \neq i, j$,

$$\begin{aligned}
&\|x_n - x_i\| + \|x_n - x_j\| - \|2x_n - x_i - x_j\| \geq \\
&\geq [1 + \varrho_M(k_{ni}(x_n - x_i))]/k_{ni} + [1 + \varrho_M(k_{nj}(x_n - x_j))]/k_{nj} - \\
&- [1 + \varrho_M((2x_n - x_i - x_j)k_{ni}k_{nj}/(k_{ni} + k_{nj}))](k_{ni} + k_{nj})/k_{ni}k_{nj} = \\
&= \int_G [M((x_n(t) - x_i(t))k_{ni})/k_{ni} + M((x_n(t) - x_j(t))k_{nj})/k_{nj} - \\
&- M((2x_n(t) - x_i(t) - x_j(t))k_{ni}k_{nj}/(k_{ni} + k_{nj}))(k_{ni} + k_{nj})/k_{ni}k_{nj}]\, d\mu.
\end{aligned} \tag{3}$$

Denote the last integrand in (3) by $f_{nij}(t)$, then by the convexity of M, $f_{nij}(t) \geq 0$ for all $t \in G$. Since λ is affine on $\mathrm{conv}(x_k)$, let $n \to \infty$, by (3), $\int_G f_{nij}(t)\, d\mu \to 0$, therefore, $f_{nij}(t) \to 0$ in measure. By the diagonal method, we can choose a subsequence N_4 of N_3 such that $f_{nij}(t) \to 0$ μ-a.e. on G as $n(\in N_4) \to \infty$ for all $i, j \in N_3$.

For each $t \in G$, choose a subsequence $\{n_\tau = n_\tau(t)\}$ of N_4 such that

$$|v(t)| = \liminf_{n\in N_4} |x_n(t)|, \qquad \lim_\tau x_{n\tau}(t) = v(t), \tag{$*$}$$

then by Fatou's lemma, $|v(t)| < \infty$ μ-a.e. on G (one may prove this analogously as in Lemma 1). Let $\tau \to \infty$, by the continuity of M,

$$\begin{aligned}
0 &= \lim_\tau f_{n\tau ij}(t) = \\
&= M((v(t) - x_i(t))k_i)/k_i + M((v(t) - x_j(t))k_j)/k_j - \\
&\quad - M((2v(t) - x_i(t) - x_j(t))k_ik_j/(k_i + k_j))(k_i + k_j)/k_ik_j
\end{aligned} \tag{4}$$

μ-a.e. on G. Since for μ-a.e. $t \in G$, (4) holds for all $i, j \in N_3$, by replacing j by n_τ in (4) and taking $\tau \to \infty$, for each $t \in G$, we have

$$\begin{aligned}
&M((v(t) - x_i(t))k_i)/k_i = \\
&= M((v(t) - x_i(t))k_ik/(k_i + k))(k_i + k)/k_ik
\end{aligned} \tag{5}$$

μ-a.e. on G. Since $0 < k/(k_i + k) < 1$, and $u \neq 0, 0 < \alpha < 1$ implies $M(\alpha u) < \alpha M(u)$, for all $t \in G$ satisfying $v(t) \neq x_i(t)$, we have

$$M((v(t) - x_i(t))k_i k/(k_i + k))(k_i + k)/k_i k < M((v(t) - x_i(t))k_i)/k_i\,.$$

Since this inequality contradicts (5), we have $x_i(t) = v(t)$ μ-a.e. on G for all $i \in N_3$, which contradicts the assumption $x_i \neq x_j$ whenever $i \neq j$.

Now, we prove (iii) $\Rightarrow$ (i) in Theorem 2. If L_M does not have sum property, then there exists a growing limit affine sequence. Without loss of generality, we may assume that every two points in the sequence are different. By the above discussion, it contains a subsequence (x_n) satisfying $k_{ij} \to k_j < \infty$ as $i \to \infty$ and $k_j \to \infty$ as $j \to \infty$, where k_{ij} satisfies (2), $i, j \in N$. Since $M(u)/u \to \infty$ as $u \to \infty$, for the constant $a > 0$ in (iii), we can find $b > a$ such that $M(\frac{1}{2}(a+b)) < \frac{1}{2}[M(a)+M(b)]$. Since M is convex, by (iii)

$$M(\alpha u + (1-\alpha)v) \;<\; \alpha M(u) + (1-\alpha)M(v) \tag{6}$$

for all $0 < \alpha < 1$ and all $u \leq a, v \geq b$ or $u \geq a, v \geq Cu$. If we define $v(t)$ as in $(*)$, then by (4) and (6), for μ-a.e. $t \in G$, if $k_i|v(t) - x_i(t)| \leq a$, then $k_j|v(t) - x_j(t)| \leq b$; if $k_i|v(t) - x_i(t)| > a$, then $k_j|v(t) - x_j(t)| \leq Ck_i|v(t) - x_i(t)|$. Therefore, for μ-a.e. $t \in G$,

$$k_j|v(t) - x_j(t)| \leq \max\{b, Ck_i|v(t) - x_i(t)|\} := u_i(t). \tag{7}$$

By (2) and Fatou's lemma, we have

$$\lambda(x_j) \geq [1 + \varrho_M(k_j(v - x_j))]/k_j \geq \|v - x_j\|. \tag{8}$$

Thus, $v - x_j \in L_M$, therefore, $u_i \in L_M$. Since $\lambda > 0$, we have $\liminf \|v - x_j\| := \tau > 0$. It follows from (7) that

$$k_j = \|k_j(v - x_j)\|/\|v - x_j\| \leq \|u_i\|/\|v - x_j\|.$$

Let $j \to \infty$, we get a contradiction $\infty \leq \|u_i\|/\tau < \infty$.

Next, we turn to Theorem 1. If L_M does not have weak sum property, then by (3), there exists a weakly converging (to zero) limit affine sequence (x_n) with $\|x_n\| \to 1$ and $\lambda(x_n) \to 1$. By the first part of the proof, passing a subsequence if necessary, we may assume $k_{ij} \to k_j < \infty$ as $i \to \infty$ and $k_j \to \infty$ as $j \to \infty$, where k_{ij} satisfies (2). It follows from (8) that $x_j \to v$ in measure (similarly verified as in the first part of the proof). Therefore, by Lemma 2, $v = 0$. We may also assume $x_j \to 0$ μ-a.e. on G. We prove the theorem by showing $\lim \lambda(x_j) \geq 4/3$ contradicting the assumption $\lambda(x_j) \to 1$.

For each $j \in N$, we choose a set $G_j \in \Sigma$ such that x_j is bounded on G_j and

$$[1 + \varrho_M(k_j x_j \chi_{G_j})]/k_j \;>\; [1 + \varrho_M(k_j x_j)]/k_j - 1/k_j\,,$$

then by (1) and (8),

$$\begin{aligned}\lambda(x_j) &= [1+\varrho_M(k_jx_j)]/k_j = \\ &= [1+\varrho_M(k_jx_j\chi_{G_j})]/k_j + [1+\varrho_M(k_jx_j\chi_{G\setminus G_j})]/k_j - 1/k_j > \\ &> [1+\varrho_M(k_jx_j)]/k_j - 1/k_j + \|x_j\chi_{G\setminus G_j}\| - 1/k_j \geq \\ &\geq \|x_j\| + \|x_j\chi_{G\setminus G_j}\| - 2/k_j\,,\end{aligned}$$

i.e.

$$\|x_j\chi_{G\setminus G_j}\| \,<\, \lambda(x_j) - \|x_j\| + 2/k_j\,. \tag{9}$$

It follows that

$$\|x_j\chi_{G_j}\| \geq \|x_j\| - \|x_j\chi_{G\setminus G_j}\| \,>\, 2\|x_j\| - \lambda(x_j) - 2/k_j\,. \tag{10}$$

Since x_j is bounded on G_j, there exists $\delta=\delta(j)>0$ such that

$$\|x_j\chi_E\| \,<\, 1/k_j \quad \text{whenever} \quad E\subset G_j \ \text{ and } \ \mu E \,<\, \delta. \tag{11}$$

Since $x_i \to 0$ μ-a.e. on G, there exists $F\in\Sigma$ with $\mu F<\delta$ such that $x_i\to 0$ uniformly on $G\setminus F$. Hence, there exists $I=I(j)\in N$ such that for all $i>I$, we have

$$\|x_i\chi_{G\setminus F}\| \,<\, 1/k_j\,. \tag{12}$$

It follows that

$$\|x_i\chi_F\| \geq \|x_i\| - \|x_i\chi_{G\setminus F}\| \,>\, \|x_i\| - 1/k_j\,. \tag{13}$$

Hence, by (1), (2), (9)–(13),

$$\begin{aligned}&\|x_i-x_j\| = [1+\varrho_M(k_{ij}(x_i-x_j)\chi_{G\setminus(G_j\setminus F)})]/k_{ij} + \\ &+ [1+\varrho_M(k_{ij}(x_i-x_j)\chi_{G_j\setminus F})]/k_{ij} - 1/k_{ij} \geq \\ &\geq \|(x_i-x_j)\chi_{G\setminus(G_j\setminus F)}\| + \|(x_i-x_j)\chi_{G_j\setminus F}\| - 1/k_{ij} \geq \\ &\geq \|x_i\chi_{G\setminus(G_j\setminus F)}\| - \|x_j\chi_{G\setminus(G_j\setminus F)}\| + \\ &+ \|x_j\chi_{G_j\setminus F}\| - \|x_i\chi_{G_j\setminus F}\| - 1/k_{ij} = \\ &= \|x_i\chi_{G\setminus(G_j\setminus F)}\| - \|x_j\chi_{G\setminus G_j} + x_j\chi_{G_j\setminus F}\| + \\ &+ \|x_j\chi_{G_j} - x_j\chi_{G_j\setminus F}\| - \|x_i\chi_{G_j\setminus F}\| - 1/k_{ij} > \\ &> (\|x_i\| - 1/k_j) - (\lambda(x_j) - \|x_j\| + 2/k_j + 1/k_j) + \\ &+ (2\|x_j\| - \lambda(x_j) - 2/k_j - 1/k_j) - 1/k_j - 1/k_{ij} = \\ &= \|x_i\| + 3\|x_j\| - 2\lambda(x_j) - 8/k_j - 1/k_{ij}\,.\end{aligned}$$

Let $i \to \infty$, we have

$$\lambda(x_j) \ge 1 + 3\|x_j\| - 2\lambda(x_j) - 9/k_j \, .$$

Let $j \to \infty$, then $\lim \lambda(x_j) \ge 4/3$.

Finally, we prove (ii) $\Rightarrow$ (iii) in Theorem 2. If (iii) does not hold, then there exist the sequences $\{u_j\}, \{v_j\}$ such that $M(u_1)\mu G > 1, u_{j+1} > 2^j u_j, v_j > 2^j u_j$ and $p(u)$ is constant on $[u_j, v_j], j \in N$. By the first two assumptions, we can choose disjoint sets $g_j \in \Sigma$ such that $\mu G \setminus U_{j\in N} G_j > 0$ and

$$2^{-j} = u_j p(u_j)\mu G_j = [M(u_j) + N(p(u_j))]\mu G_j \tag{14}$$

(the last equality holds by the special case of Young's inequality). Hence, we can find u_0 large enough so that there is G_0 satisfying

$$\Sigma_{j\in N} N(p(u_j))\mu G_j + N(p(u_0))\mu G_0 = 1. \tag{15}$$

Define

$$\begin{aligned} &v = \Sigma_{j\ge 0} p(u_j)\chi_{G_j}, \\ &x_n = u_0\chi_{G_0} + \Sigma_{j\in N} v_j\chi_{G_j} + \Sigma_{j>n} u_j\chi_{G_j} \, , \end{aligned}$$

then by (15), $\varrho_N(v) = 1$, therefore, $v \in L_M^*$ and $\|v\| = 1$ (cf. [1]).

First we show that $x_n \in E_M$ for any $n \in N$. Given arbitrary $K > 1$, choose $J > n$ such that $2^j > K$, then $v_j > 2^j u_j > K u_j > u_j$ for all $j > J$. Therefore

$$\begin{aligned} \Sigma_{j>J} M(Ku_j)\mu G_j &= \Sigma_{j>J}[Ku_j p(Ku_j) - N(p(Ku_j))]\mu G_j < \\ &< \Sigma_{j>J} Ku_j p(Ku_j)\mu G_j = \Sigma_{j>J} Ku_j p(u_j)\mu G_j = K\Sigma_{j>J} 2^{-j} < \infty \, . \end{aligned}$$

This implies $\varrho_M(Kx_n) < \infty$. Since $K > 1$ is arbitrary, we have $x_n \in E_M$.

Let $k_n = \|x_n\|$ and $y_n = x_n/k_n$, then $y_n \in E_M$ and $\|y_n\| = 1$. By (1) and (15),

$$\begin{aligned} \|y_n\| \ge \langle v, y_n\rangle &= [u_0 p(u_0)\mu G_0 + \Sigma_{j\le n} v_j p(u_j)\mu G_j + \Sigma_{j>n} u_j p(u_j)\mu G_j]/k_n = \\ &= [\varrho_N(v) + \varrho M(k_n y_n)]/k_n \ge \|y_n\| = 1. \end{aligned}$$

Moreover, since

$$k_n = \|x_n\| \ge \langle v, x_n\rangle > \Sigma_{j\le n} v_j p(u_j)\mu G_j \ge \Sigma_{j\le n} 2^j u_j p(u_j)\mu G_j = n,$$

we have $k_n \to \infty$ as $n \to \infty$.

We complete the proof by showing $\lambda = 2$ on $\mathrm{conv}(y_n)$. Indeed, for any $y \in \mathrm{conv}(y_n)$, there exist $\lambda_i \ge 0, \Sigma_{i\le m}\lambda_i = 1$ such that $y = \Sigma_{i\le m}\lambda_i y_i$. Since $\langle v, y_n\rangle = 1$, we have $\langle v, y\rangle = \Sigma_{i\le m}\lambda_i\langle v, y_n\rangle = 1$. For any $\varepsilon > 0$, since $y \in E_M$, there exists $I > m$ such that $\|y\chi_F\| < \varepsilon$, where $F = U_{i>I} G_i$. In view of $x_n(t) \le \max\{v_I, u_0\}$

on $G \setminus F$ and $k_n \to \infty$ as $n \to \infty$, we can find $n_0 \in N$ such that $\|y_n \chi_{G\setminus F}\| < \varepsilon$ for all $n > n_0$. Define $v_0 = v\chi_{G\setminus F} - v\chi_F$, then $\|v_0\| = \|v\| = 1$ and for all $n > n_0$,

$$\begin{aligned}
2 &\geq \|y\| + \|y_n\| \geq \|y - y_n\| \geq \langle v_0, y - y_n \rangle = \\
&= \langle v_0, y\chi_{G\setminus F} \rangle + \langle v_0, y\chi_F \rangle - \langle v_0, y_n\chi_{G\setminus F} \rangle - \langle v_0, y_n\chi_F \rangle = \\
&= \langle v, y\chi_{G\setminus F} \rangle - \langle v, y\chi_F \rangle - \langle v, y_n\chi_{G\setminus F} \rangle + \langle v, y_n\chi_F \rangle = \\
&= \langle v, y \rangle - 2\langle v, y\chi_F \rangle - 2\langle v, y_n\chi_{G\setminus F} \rangle + \langle v, y_n \rangle > \\
&> 1 - 2\|y\chi_F\| - 2\|y_n\chi_{G\setminus F}\| + 1 > 2 - 4\varepsilon,
\end{aligned}$$

which shows that $\lambda(y) = 2$. □

Theorem 3. *L_M has isonormal structure.*

PROOF: If M is strictly convex, then the condition (iii) in Theorem 2 holds for all $u \neq 0$ and all $C \neq 1$. Therefore, L_M has normal structure in this case. By [1], for any Orlicz function M and any $\varepsilon > 0$, we can construct a strictly convex Orlicz function H such that

$$\|x\|_M \leq \|x\|_H \leq (1 + \varepsilon)\|x\|_M$$

for all $x \in L_M$, which shows that L_M is isomorphic to L_H, i.e. L_M has isonormal structure. □

References

[1] Wu Congxin, Wang Tingfu, Chen Shutao, Wang Yuwen, *Geometry of Orlicz spaces*, Harbin, 1986.

[2] Wu Congxin, Wang Tingfu, *Orlicz spaces and their applications*, Harbin, 1983.

[3] Landes T., *Permanence properties of normal structure*, Pacific J. Math. **110, No. 1** (1984), 125–143.

Harbin Normal University, Department of Mathematics, Harbin, China

Harbin Institute of Technology, Department of Mathematics, Harbin, China

(*Received* September 13, 1990, *revised* February 19, 1991)